4. Representation theory of the Poincaré group

$(L, a) \in \mathcal{P}$ $\qquad x' = Lx + a$

$(L', a')(L, a) = (LL', L'a + a')$ composition law

representation of $\mathcal{L}_+^\uparrow$ in a function space

$\rightarrow$ representation of $\mathcal{P}_+^\uparrow$ in this function space

$(L, a) \rightarrow T_{(L,a)} : \quad \Phi \mapsto \Phi' = T_{(L,a)} \Phi$

$$\Phi'(x) = D(L)\,\Phi(L^{-1}(x-a))$$

representation is <u>reducible</u>

linear homogeneous differential equations single out invariant subspaces:

$\mathcal{P}$ covariant field equ.

$$\left\{\begin{array}{ll} (\Box + m^2)\Phi(x) = 0 & \text{Klein Gordon equ.} \\ F^{\mu\nu}{}_{,\nu} = 0 = \widetilde{F}^{\mu\nu}{}_{,\nu} & \text{(free) Maxwell} \\ \Box A^\mu = 0, \ \partial_\mu A^\mu = 0 & \text{4potential in Lorenz' gauge} \\ (i\partial\!\!\!/ - m)\psi(x) = 0 & \text{Dirac equ.} \end{array}\right.$$

only $\mathcal{P}_+^\uparrow$ covariant $\begin{cases} i\sigma^\mu \partial_\mu \overline{\chi} = m\chi & \text{Majorana equ.} \\ i\bar{\sigma}^\mu \partial_\mu \overline{\chi} = 0 & \text{Weyl equ.} \end{cases}$

Lie algebra of the Poincaré group

$x' = Lx + a \qquad L \in \mathcal{L}$, a^μ is a four-vector

$(L', a')(L, a) = (L'L, L'a + a')$ composition law

$(L, a)^{-1} = (L^{-1}, -L^{-1}a)$ inverse to (L, a)

$(\mathbb{1}, 0)$ unit element

let $(L, a) \mapsto U(L, a)$ be a faithful representation in a space $\mathcal{H}$

infinitesimal transformation $[\,(L, a) \in \mathcal{P}_+^\uparrow\,]$

$$U(L, a) \simeq \mathbb{1}_{\mathcal{H}} - \frac{i}{2}\omega_{\alpha\beta} J^{\alpha\beta} + i a_\mu P^\mu$$

$J^{\alpha\beta}$ generators of Lorentz transformations

P^μ generators of translations

<u>remark</u>: factor i pulled out $\rightarrow$ $J^{\alpha\beta}$, P^μ are <u>hermitean</u> in the case of an <u>unitary</u> representation

we want to find the commutation relations of $J^{\alpha\beta}$, P^μ $\rightarrow$ to this end, we consider

$$U(L,0)^{-1}\,U(L',a')\,U(L,0) = U(L^{-1}L'L,\, L^{-1}a')$$

$$U(\mathbb{1},a)^{-1}\,U(L',a')\,U(\mathbb{1},a) = U(L',\, L'a + a' - a)$$

(L', a') <u>infinitesimal</u>:

$$U(L,0)^{-1}\left(\mathbb{1}_{\mathcal{H}} - \tfrac{i}{2}\,\omega'_{\alpha\beta}\, J^{\alpha\beta} + i\, a'_\mu P^\mu\right)U(L,0)$$

$$= U(L^{-1}L'L,\, L^{-1}a') = \mathbb{1}_{\mathcal{H}} - \tfrac{i}{2}\,\omega'_{\alpha\beta}\, L^\alpha{}_\mu L^\beta{}_\nu J^{\mu\nu} + i\, a'_\mu L^\mu{}_\nu P^\nu$$

$$\Rightarrow \boxed{U(L,0)^{-1}\, J^{\alpha\beta}\, U(L,0) = L^\alpha{}_\mu\, L^\beta{}_\nu\, J^{\mu\nu}}$$

$$\Rightarrow \boxed{U(L,0)^{-1}\, P^\mu\, U(L,0) = L^\mu{}_\nu\, P^\nu}$$

$$U(\mathbb{1},a)^{-1}\left(\mathbb{1}_{\mathcal{H}} - \frac{1}{2}\,\omega'_{\alpha\beta}\,J^{\alpha\beta} + i\,a'_\mu P^\mu\right)U(\mathbb{1},a) =$$

$$= \mathbb{1}_{\mathcal{H}} - \frac{i}{2}\,\omega'_{\alpha\beta}\left(J^{\alpha\beta} + a^\alpha P^\beta - a^\beta P^\alpha\right) + i\,a'_\mu P^\mu$$

$$\Rightarrow \quad \boxed{U(\mathbb{1},a)^{-1}\,J^{\alpha\beta}\,U(\mathbb{1},a) = J^{\alpha\beta} + a^\alpha P^\beta - a^\beta P^\alpha}$$

$$\Rightarrow \quad \boxed{U(\mathbb{1},a)^{-1}\,P^\mu\,U(\mathbb{1},a) = P^\mu}$$

next step: also (L,a) infinitesimal $\rightarrow$ Lie algebra relations of $\mathcal{P}_+^\uparrow$:

$$i\,[J^{\alpha\beta}, J^{\gamma\delta}] = g^{\alpha\gamma} J^{\beta\delta} - g^{\beta\gamma} J^{\alpha\delta} + g^{\alpha\delta} J^{\gamma\beta} - g^{\beta\delta} J^{\gamma\alpha}$$

$$i\,[J^{\alpha\beta}, P^\gamma] = g^{\gamma\alpha} P^\beta - g^{\gamma\beta} P^\alpha$$

$$[P^\alpha, P^\beta] = 0$$

<u>remark:</u> commutation relations may also be verified using the 5×5 matrix representation

$$(L,a) \mapsto \begin{pmatrix} L & a \\ 0 & 1 \end{pmatrix}$$

(ex.)

reducible, but <u>not</u> decomposable ($\mathcal{P}_+^\uparrow$ is <u>not</u> semisimple)

Casimir operators (invariant operators) for $\mathcal{P}_+^\uparrow$

(i.e. operators C with $[U(L,a), C] = 0$)

$J^{\mu\nu}$ is a tensor operator with respect to $\mathcal{L}_+^\uparrow$

P^μ —〃— vector —〃— —〃— —〃— —〃—

also translation invariance must be taken into account:

P^μ is translation invariant $\Rightarrow$ $[U(L,a), P_\mu P^\mu] = 0$

(ex.)

$M^2 := P_\mu P^\mu$, $[U(L,a), M^2] = 0$

in an irreducible representation: $M^2 = m^2 \mathbb{1}$

$J_{\mu\nu} J^{\mu\nu}$ and $\varepsilon_{\mu\nu\alpha\beta} \overset{J}{\cancel{M}}{}^{\alpha\beta} \overset{J}{\cancel{M}}{}^{\mu\nu}$ are $\mathcal{L}_+^\uparrow$ invariant

but not translation invariant (my convention: $\varepsilon_{0123} = +1$)

the Pauli-Lubanski vector

$$W_\mu := \frac{1}{2} \varepsilon_{\mu\alpha\beta\gamma} J^{\alpha\beta} P^\gamma$$

is a translationally invariant (pseudo) vector operator

- properties of the Pauli–Lubanski vector

$$W_\mu P^\mu = 0 \quad , \qquad [P_\mu, W_\nu] = 0$$

$$W^2 := W_\mu W^\mu \quad , \qquad [U(L,a), W^2] = 0$$

in an irred. repr. : $\quad W^2 = w^2 \mathbb{1}$

scalar field

$$x' = x + a \quad , \quad \phi'(x) = \phi(x-a) \approx \phi(x) - a^\mu \partial_\mu \phi(x)$$

$$= (\mathrm{id} \underbrace{- a^\mu \partial_\mu}_{i a^\mu P_\mu}) \phi(x)$$

$$\Rightarrow \quad P_\mu = i \partial_\mu \qquad \text{(in this representation)}$$

$$M^2 = P^\mu P_\mu = -\partial^\mu \partial_\mu = -\Box$$ operator of mass square

in an irreducible repr. of $\mathcal{P}_+^\uparrow$ we have $M^2 \phi = m^2 \phi$

$$\Rightarrow \quad (\Box + m^2)\phi = 0 \qquad \text{Klein–Gordon equ.}$$

infinitesimal Lorentz transformation

$$x' = L x = (\mathbb{1} + \omega) x$$

$$\phi'(x) = \phi(L^{-1} x) = \phi(x) - \omega^\mu{}_\nu \, x^\nu \partial_\mu \phi(x)$$

$$= \left[\mathbb{1} - \frac{1}{2} \omega_{\mu\nu} \left(x^\nu \partial^\mu - x^\mu \partial^\nu \right) \right] \phi(x)$$

$$\Rightarrow \quad i J^{\mu\nu} = x^\nu \partial^\mu - x^\mu \partial^\nu$$

$$\Rightarrow \quad J^{\mu\nu} = \underbrace{i \left(x^\mu \partial^\nu - x^\nu \partial^\mu \right)}_{=: L^{\mu\nu}}$$

generalization of the orbital angular momentum

<u>vector field</u>

$$A^{\mu\prime}(x) = \Big[\delta^\mu{}_\nu - \frac{i}{2} \omega_{\alpha\beta} \Big(\underbrace{i \delta^\mu{}_\nu \left(x^\alpha \partial^\beta - x^\beta \partial^\alpha \right)}_{(L^{\alpha\beta})^\mu{}_\nu}$$

$$+ \underbrace{i \left(g^{\mu\alpha} g^\beta{}_\nu - g^{\mu\beta} g^\alpha{}_\nu \right)}_{(S^{\alpha\beta})^\mu{}_\nu} \Big) \Big] A^\nu(x)$$

$$(J^{\alpha\beta})^\mu{}_\nu = (L^{\alpha\beta} + S^{\alpha\beta})^\mu{}_\nu$$

general form

$$J^{\mu\nu} = \underset{\text{orbital}}{L^{\mu\nu}} + \underset{\text{spin}}{S^{\mu\nu}}$$

part

$$L^{\mu\nu} = i\,(x^\mu \partial^\nu - \partial^\nu x^\mu)$$

Dirac field

$$\psi'(x) = \left(\mathbb{1} - \frac{i}{4}\,\omega_{\mu\nu}\,\sigma^{\mu\nu} - \frac{i}{2}\,\omega_{\mu\nu}\,L^{\mu\nu}\right)\psi(x)$$

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$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\,, \qquad S^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}$$

$$\Rightarrow \quad S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Weyl field

$D^{(\frac{1}{2},0)}$: $\quad S^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu\right)$

$D^{(0,\frac{1}{2})}$: $\quad S^{\mu\nu} = \frac{i}{4}\left(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu\right)$

$L^{\mu\nu}$ does not contribute to the Pauli-Lubanski vector $\rightarrow$ $W_\mu = \frac{1}{2} \varepsilon_{\mu\alpha\beta\gamma} S^{\alpha\beta} P^\gamma$ is related to the <u>spin</u>

for a <u>scalar</u> <u>field</u>: $W_\mu = 0$, $W^2 = 0$

<u>Dirac</u> <u>field</u>:

$$W_\mu = -\frac{1}{8} \varepsilon_{\mu\alpha\beta\gamma} [\gamma^\alpha, \gamma^\beta] \partial^\gamma$$

$$W^2 = +\frac{3}{4} \Box = -\frac{3}{4} M^2 \qquad \text{(ex.)}$$

$$= -\frac{1}{2}\left(\frac{1}{2}+1\right) M^2$$

<u>Weyl</u> <u>field</u>:

$$W^2 = \frac{3}{4} \Box = -\frac{3}{4} M^2 = -\frac{1}{2}\left(\frac{1}{2}+1\right) M^2 \qquad \text{(ex.)}$$

<u>vector</u> <u>field</u>:

$$(W^2)^\mu{}_\nu = 2\left(\delta^\mu{}_\nu \Box - \partial^\mu \partial_\nu\right)$$

for divergence free vector fields: $(W^2)^\mu{}_\nu = 2\, \delta^\mu{}_\nu \Box$

(ex.) $\qquad = -1(1+1)\delta^\mu{}_\nu M^2$

antisymmetric tensor field $T^{\mu\nu}(x)$: !!

$$W^2 = 2 \Box = -1(1+1) M^2 \qquad \text{(ex.)}$$

Irreducible unitary representations of the Poincaré group

(more precisely: $\tilde{\mathcal{P}}_+^\uparrow$) $U(L,a)$ unitary repr. in Hilbert space $\mathcal{H}$

all P^μ commute with each other ($[P^\mu, P^\nu] = 0$)

$\Rightarrow$ consider eigenvectors of P^μ:

$$P^\mu |p,\alpha\rangle = p^\mu |p,\alpha\rangle$$

$$U(\mathbb{1},a) |p,\alpha\rangle = e^{iP.a} |p,\alpha\rangle = e^{ip.a} |p,\alpha\rangle$$

consider the vector $U(L,0)|p,\alpha\rangle$:

$$P^\mu U(L,0) |p,\alpha\rangle = U(L,0) L^\mu{}_\nu P^\nu |p,\alpha\rangle$$

$$= U(L,0) L^\mu{}_\nu p^\nu |p,\alpha\rangle = L^\mu{}_\nu p^\nu U(L,0) |p,\alpha\rangle$$

$\Rightarrow$ $U(L,0)|p,\alpha\rangle$ is an eigenvector of P^μ with eigenvalue $(Lp)^\mu$

$$\Rightarrow \quad U(L,0)|p,\alpha\rangle = \sum_\beta |Lp,\beta\rangle \, Q_{\beta\alpha}(L,p)$$

remark: p given $\Rightarrow$ $(Lp)^2 = p^2$, $\operatorname{sign}(Lp)^0 = \operatorname{sign} p^0$ remain unchanged as $L \in \mathcal{L}_+^\uparrow$

→ classification of "orbits":

(a_+) $p^2 = m^2 > 0$, $\mathrm{sign}\, p^0 = +1$ (mass shell)

(a_-) $p^2 = m^2 > 0$, $\mathrm{sign}\, p^0 = -1$

(b_+) $p^2 = 0$, $\mathrm{sign}\, p^0 = +1$ (future light cone)

(b_-) $p^2 = 0$, $\mathrm{sign}\, p^0 = -1$ (past light cone)

(c) $p = 0$ (zero vector)

(d) $p^2 < 0$ (p spacelike)

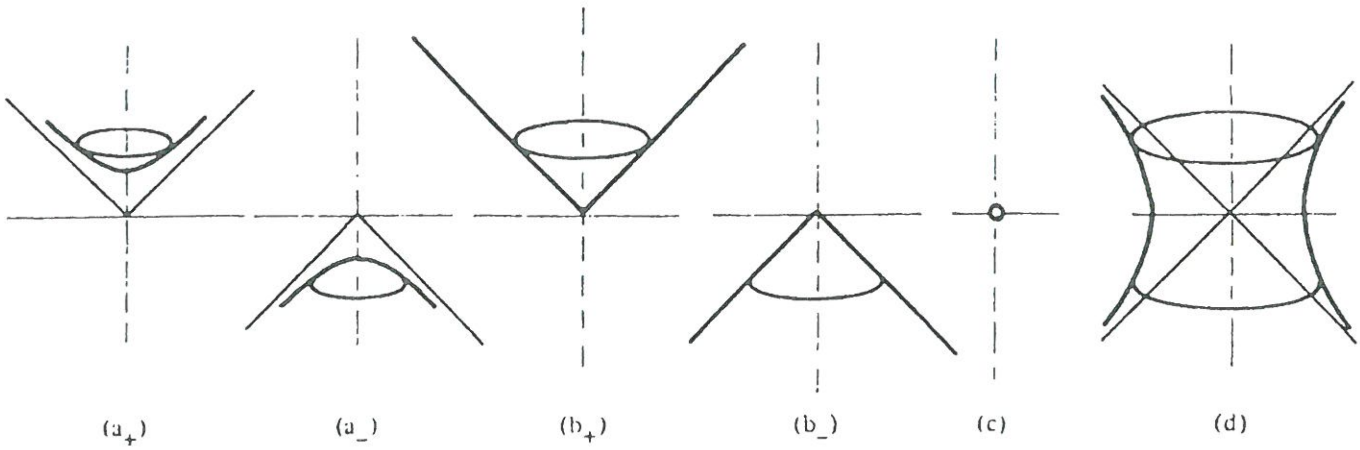


Fig. 9.1. Orbits of $\mathcal{L}_+^\uparrow$ in p-space

unitary irreducible representation of $\tilde{\mathcal{P}}_+^\uparrow$ has to fall into one of these classes

<u>next step</u>: classification of the possible Q in

$$U(L,0)\,|p,\alpha\rangle = \sum_{\beta} |Lp,\beta\rangle\, Q_{\beta\alpha}(L,p)$$

$$\overbrace{\underbrace{U(L',0)\,U(L,0)}_{U(L'L,0)}\,|p,\alpha\rangle}^{\sum_{\gamma} |L'Lp,\gamma\rangle\, Q_{\gamma\alpha}(L'L,p)} =$$

$$= U(L',0) \sum_{\beta} |Lp,\beta\rangle\, Q_{\beta\alpha}(L,p)$$

$$= \sum_{\gamma}\sum_{\beta} |L'Lp,\overset{\gamma}{\cancel{\beta}}\rangle\, Q_{\gamma\beta}(L',Lp)\, Q_{\beta\alpha}(L,p)$$

$$\Longrightarrow \qquad Q_{\gamma\alpha}(L'L,p) = \sum_{\beta} Q_{\gamma\beta}(L',Lp)\, Q_{\beta\alpha}(L,p)$$

looks <u>almost</u> like a representation property (Lp dependence is disturbing!)

<u>trick</u> (due to E. Wigner): choose an arbitrary four-vector $\bar{p}$ in the orbit; consider the <u>subgroup</u> of those elements $K \in \mathcal{L}_+^{\uparrow}$ with $K\bar{p} = \bar{p}$ ($\equiv \mathcal{K}_{\bar{p}}$ = <u>little group</u> for the <u>standard vector</u> $\bar{p}$)

examples:

(a) orbit $p^2 = m^2 > 0$, $\operatorname{sign} p^0 = +1$

choose $\bar{p} = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underset{\in SO(3)}{R} & \\ 0 & & & \end{pmatrix} \in \mathcal{K}_{\bar{p}}$

$\Rightarrow \mathcal{K}_{\bar{p}} \cong SO(3)$

(b) orbit $p^2 = 0$, $\operatorname{sign} p^0 = +1$

choose $\bar{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

infinitesimal Lorentz transformation:

$$t' = t + \vec{u} \cdot \vec{x}$$

$$\vec{x}' = \vec{x} + \underbrace{\vec{\alpha} \times \vec{x}}_{\begin{pmatrix} \alpha_y z - \alpha_z y \\ -\alpha_x z + \alpha_z x \\ \alpha_x y - \alpha_y z \end{pmatrix}} + \vec{u}\, t$$

in our case: $\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{p} = \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix}$

$$1 = 1 + u_z \quad \Rightarrow u_z = 0$$

$$0 = 0 + \alpha_y + u_x \quad \Rightarrow \quad u_x = -\alpha_y$$

$$0 = 0 - \alpha_x + u_y \quad \Rightarrow \quad u_y = \alpha_x$$

$1 = 1 + u_z$ no additional information (see first line)

$\Rightarrow$ 3 parameters $\alpha_x =: b_x \;,\; \alpha_y =: b_y \;,\; \alpha_z =: \alpha$

$$L = \mathbb{1} + \alpha_x M_x + \alpha_y M_y + \alpha_z M_z - \alpha_y N_x + \alpha_x N_y$$

$$= \mathbb{1} + \alpha_x \underbrace{(M_x + N_y)}_{=: T_x} + \alpha_y \underbrace{(M_y - N_x)}_{=: T_y} + \alpha_z \underbrace{M_z}_{=: M}$$

$$= \mathbb{1} + \vec{b} \cdot \vec{T} + \alpha M$$

$$\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$

commutation relations:

$$[T_x , T_y] = [M_x + N_y , M_y - N_x] = M_z - M_z = 0$$

$$[T_x, M] = [M_x + N_y, M_z] = -M_y + N_x = -(M_y - N_x) = -T_y$$

$$[T_y, M] = [M_y - N_x, M_z] = M_x + N_y = T_x$$

$\rightarrow$ commutation relations of the group $E(2)$ of translations and rotations in a euclidëan 2-plane:

$$\vec{x}\,' = D(\alpha)\,\vec{x} + \vec{b} \quad , \qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$D(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

group multiplication:

$$\vec{x}\,'' = D'\vec{x}\,' + \vec{b}\,' = D'(D\vec{x} + \vec{b}) + \vec{b}\,' =$$

$$= D'D\,\vec{x} + D'\vec{b} + \vec{b}\,'$$

$$(D', \vec{b}\,')\,(D, \vec{b}) = (D'D,\ D'\vec{b} + \vec{b}\,')$$

matrix representation:

$$(D, \vec{b}) \mapsto \begin{pmatrix} D & \vec{b} \\ 0 & 1 \end{pmatrix}$$

infinitesimal transformation:

$$\begin{pmatrix} 1 & -\alpha & b_x \\ \alpha & 1 & b_y \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_3 + \alpha \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{M} + b_x \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{T_x}$$

$$+ b_y \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{T_y}$$

indeed: $[T_x, T_y] = 0$, $[T_x, M] = -T_y$,

$[T_y, M] = T_x$ ✓

$U(K, 0)$ is a unitary representation of the little group $K_{\bar{p}}$ acting on the space spanned by the vectors $|\bar{p}, \alpha\rangle$ (this space is referred to as the <u>little</u> vector space)

$K \in K_{\bar{p}}$

we will show: this representation of the little group already determines the representation of the <u>whole</u> group! $\rightarrow$ classification problem is reduced to the one of finding the unitary irreducible representations of the little group!

→ we have to show: a <u>general</u> $Q(L,p)$ can be constructed from the <u>special</u> $Q(K,\bar{p})$

<u>proof</u>: for each p in the orbit under consideration we choose a transformation $\Lambda_p \in \mathcal{L}_+^\uparrow$ with

$$\Lambda_p \bar{p} = p \quad , \quad \Lambda_{\bar{p}} = \mathbb{1}$$

(Λ_p depending continuously on p)

then we factorize $L \in \mathcal{L}_+^\uparrow$ as

$$L = \Lambda_{Lp} \, K(L,p) \, \Lambda_p^{-1} ,$$

where $K(L,p)$ is <u>defined</u> by this equation:

$$K(L,p) := \Lambda_{Lp}^{-1} \, L \, \Lambda_p ,$$

where $K(L,p) \in \mathcal{K}_{\bar{p}}$:

$$K(L,p)\,\bar{p} = \Lambda_{Lp}^{-1} \, L \, \Lambda_p \bar{p} = \Lambda_{Lp}^{-1} \, L p = \bar{p} \;\checkmark$$

now we <u>define</u> the basis vectors $|p,\alpha\rangle$ by setting

$$|p,\alpha\rangle = U(\Lambda_p, 0)\,|\bar{p},\alpha\rangle$$ <u>Wigner basis</u>

$$\Rightarrow \quad U(L,0)\,|p,\alpha\rangle = U(\Lambda_{Lp}\,K(L,p)\,\Lambda_p^{-1}, 0)\,U(\Lambda_p,0)\,|\bar{p},\alpha\rangle$$

$$= U(\Lambda_{Lp},0)\,U(K(L,p),0)\,\underbrace{U(\Lambda_p,0)^{-1}\,U(\Lambda_p,0)}_{\mathbb{1}}\,|\bar{p},\alpha\rangle$$

$$= U(\Lambda_{Lp},0)\sum_\beta |\bar{p},\beta\rangle\, Q_{\beta\alpha}(K(L,p),\bar{p})$$

$$= \sum_\beta |Lp,\beta\rangle\,\underbrace{Q_{\beta\alpha}(K(L,p),\bar{p})}_{=\,Q_{\beta\alpha}(L,p)}$$

<u>remark</u>: the formula $Q_{\beta\alpha}(L,p) = Q_{\beta\alpha}(K(L,p),\bar{p})$ (valid in the Wigner basis!) can be used to check the relation

$$Q_{\gamma\alpha}(L'L,p) = \sum_\beta Q_{\gamma\beta}(L',Lp)\,Q_{\beta\alpha}(L,p)$$

(ex.)

Pauli-Lubanski vector:

$K \in \mathcal{K}_{\bar{p}}$ → infinitesimal form: $K^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$

with $\omega^\mu{}_\nu \bar{p}^\nu = 0$ ⇒ general solution (for $\bar{p} \neq 0$):

$$\omega_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} k^\rho \bar{p}^\sigma \,, \quad k^\rho \text{ arbitrary.}$$

$$U(K,0)\,|\bar{p},\alpha\rangle = \left(\mathbb{1} - \frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}\right)|\bar{p},\alpha\rangle$$

$$= \left(\mathbb{1} - \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma} k^\rho \bar{p}^\sigma J^{\mu\nu}\right)|\bar{p},\alpha\rangle$$

$$= \left(\mathbb{1} - \frac{i}{2} k^\rho \varepsilon_{\rho\mu\nu\sigma} J^{\mu\nu} \bar{p}^\sigma\right|\bar{p},\alpha\rangle$$

$$= \left(\mathbb{1} - i k^\rho \underbrace{\frac{1}{2}\varepsilon_{\rho\mu\nu\sigma} J^{\mu\nu} P^\sigma}_{W_\rho}\right)|\bar{p},\alpha\rangle$$

$$= \left(\mathbb{1} - i k^\rho W_\rho\right)|\bar{p},\alpha\rangle$$

→ Pauli-Lubanski vector generates the transformations of the little group in the "little vector space" spanned by $|\bar{p},\alpha\rangle$

<u>remark</u>: the relation $W^\mu P_\mu = 0$ allows to eliminate one of the components of the Pauli-Lubanski vector on the little vector space:

$$0 = W^\mu P_\mu \, |\bar{p}, \alpha\rangle = \bar{p}_\mu W^\mu \, |\bar{p}, \alpha\rangle$$

$\Rightarrow$ <u>number of parameters</u> in the little group is <u>only 3</u> (instead of 4)

commutation relations of the components of the Pauli-Lubanski vector:

$$[W_\mu, W_\nu] = -i\,\varepsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma \qquad \text{(ex.)}$$

this relation in the little vector space:

$$[W_\mu, W_\nu]\,|\bar{p}, \alpha\rangle = -i\,\varepsilon_{\mu\nu\rho\sigma} W^\rho \bar{p}^\sigma \,|\bar{p}, \alpha\rangle$$

<u>Case (a_+)</u>: $p^2 = m^2 > 0$, $\operatorname{sign} p^0 > 0$

$$\bar{p} = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ $\rightarrow$ little group $SO(3)$ $(SU(2))$ (see p. 134)

$$P^\mu W_\mu = 0 \Rightarrow \underbrace{m}_{\neq 0} W_0 |\bar{p}, \alpha\rangle = 0 \Rightarrow W_0 |\bar{p}, \alpha\rangle = 0$$

commutation relation $[W_\mu, W_\nu] |\bar{p}, \alpha\rangle = -i \varepsilon_{\mu\nu\rho\sigma} \bar{p}^\sigma W^\rho |\bar{p}, \alpha\rangle$

$$= -im\, \varepsilon_{\mu\nu\rho 0} W^\rho |\bar{p}, \alpha\rangle = +im\, \varepsilon_{0\mu\nu\rho} W^\rho |\bar{p}, \alpha\rangle$$

$$\Rightarrow \quad [W_0, W_k] |\bar{p}, \alpha\rangle = 0$$

$$[W_k, W_l] |\bar{p}, \alpha\rangle = im\, \varepsilon_{0kln} W^n |\bar{p}, \alpha\rangle$$

$\Rightarrow$ the operators $S^k := W^k / m$ satisfy on this subspace the commutation relations of ordinary angular momentum:

$$[S^k, S^l] |\bar{p}, \alpha\rangle = i\, \varepsilon^{kln} S^n |\bar{p}, \alpha\rangle$$

$$\varepsilon^{123} = +1$$

$\Rightarrow$ in an <u>irreducible</u> representation : $S^k S^k = s(s+1) \mathbb{1}$

$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$$\Rightarrow \quad W^2 |\bar{p}, \alpha\rangle = -W^k W^k |\bar{p}, \alpha\rangle = -m^2 s(s+1) |\bar{p}, \alpha\rangle$$

since generally in irreducible representations $W^2 = w^2 \mathbb{1}$

$$\Rightarrow \quad W^2 = -m^2 s(s+1) \mathbb{1} \qquad \text{(in irred. repr. of class (a))}$$

$\Rightarrow$ the vectors $|\bar{p},\alpha\rangle$ span a $(2s+1)$-dimensional space

$\rightarrow$ canonical basis $|\bar{p},\alpha\rangle = |\bar{p},\sigma\rangle$, $\sigma = -s, -s+1, \dots, +s$

$$\vec{S}^2 |\bar{p},\sigma\rangle = s(s+1)\,|\bar{p},\sigma\rangle$$

$$S^3 |\bar{p},\sigma\rangle = \sigma\,|\bar{p},\sigma\rangle, \quad \underbrace{S^1 \pm i S^2}_{S_\pm} |\bar{p},\sigma\rangle =$$

$$= \sqrt{s(s+1) \mp \sigma - \sigma^2}\;|\bar{p},\sigma \pm 1\rangle$$

$|\bar{p},\sigma\rangle$ = state vector of a <u>massive</u> <u>particle</u> at rest

$\vec{S}$ = <u>spin operator</u>, s = <u>spin</u> of the particle

in an irreducible presentation with $M^2 = m^2 \mathbb{1}$, $W^2 = -m^2 s(s+1)\,\mathbb{1}$, the complete set of basis vectors will be denoted by $\{|m,s;p,\sigma\rangle\}$, where $|m,s;p,\sigma\rangle = U(\Lambda_p, 0)\,|m,s;\bar{p},\sigma\rangle$ $(\Lambda_p \bar{p} = p;\ \Lambda_{\bar{p}} = \mathbb{1})$:

$$M^2 |m,s;p,\sigma\rangle = m^2 |m,s;p,\sigma\rangle \qquad M^2 = P_\mu P^\mu$$

$$W^2 |m,s;p,\sigma\rangle = -m^2 s(s+1)\,|m,s;p,\sigma\rangle$$

$$P^\mu |m,s;p,\sigma\rangle = p^\mu |m,s;p,\sigma\rangle \qquad (p^2 = m^2 > 0,\ p^0 > 0)$$

$$U(\mathbb{1}, a)\,|m, s; p, \sigma\rangle = e^{ip.a}\,|m, s; p, \sigma\rangle$$

$$U(L, 0)\,|m, s; p, \sigma\rangle = \sum_{\sigma'=-s}^{+s} |m, s; p, \sigma'\rangle\, D^{(s)}_{\sigma'\sigma}(K(L,p))$$

where $K(L,p) = \Lambda_{Lp}^{-1}\, L\, \Lambda_p$ (Wigner rotation for L,p)

<u>normalization</u> of the basis vectors:

$$\langle m, s; p', \sigma' | m, s; p, \sigma\rangle = \underbrace{(2\pi)^3\, 2p^0\, \delta^{(3)}(\vec{p}\,' - \vec{p})}_{=:\,\delta(p', p)}\, \delta_{\sigma\sigma'}$$

$p^0 = \sqrt{m^2 + \vec{p}^{\,2}}$ Lorentz invariant!

<u>completeness relation</u> of the one-particle states

$$\sum_{\sigma=-s}^{s} \int \underbrace{\frac{d^3p}{(2\pi)^3\, 2p^0}}_{d\mu(p)}\, |m, s; p, \sigma\rangle\langle m, s; p, \sigma| = \mathbb{1}$$

<u>remember</u>:

p^0 is an integration variable

$$\int d^4p\, \delta(p^2 - m^2)\, \Theta(p^0) \ldots =$$

$$= \int \frac{d^3p}{2p^0} \ldots$$

now $p^0 = \sqrt{m^2 + \vec{p}^{\,2}}$

Case (b): $p^2 = 0$, $p^0 > 0$

$$\bar{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \rightarrow \quad K_{\bar{p}} = E(2)$$

the elements of $E(2)$ are characterized by an angle α and a two-dimensional vector $\vec{b}$:

$$(D(\alpha), \vec{b})$$ (see p. 136)

multiplication rule $\quad (D(\alpha'), \vec{b}')(D(\alpha), \vec{b}) =$

$$= (\underbrace{D(\alpha')D(\alpha)}_{= D(\alpha'+\alpha)}, D(\alpha')\vec{b} + \vec{b}')$$

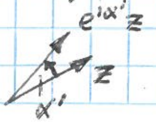
$(D(\alpha), \vec{b})$ corresponds to the $SL(2, \mathbb{C})$ element

$$A(\alpha, z) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ -ize^{-i\frac{\alpha}{2}} & e^{+i\frac{\alpha}{2}} \end{pmatrix}$$

with $z = b_x + i b_y$ (this corresponds in fact to $\widetilde{E(2)}$ with $0 \leq \alpha < 4\pi$)

$$A(\alpha', z') A(\alpha, z) = A(\alpha' + \alpha, e^{i\alpha'} z + z')$$

remark:



our goal: find the unitary irreps of $\widetilde{E(2)}$

$$[\vec{T}^2, M] = 0 \quad \Rightarrow \quad (i\vec{T})^2 = t^2 \mathbb{1}$$

$\Rightarrow$ orbits: (i) circles $t^2 > 0$ (no physical application)

(ii) zero point $t = 0$ $\Rightarrow$ $\vec{T} = \vec{0}$ (translations represented trivially)

(ii) $A(\alpha, z) \to e^{-i\lambda\alpha}$ $\quad (\lambda = 0, \pm\frac{1}{2}, \pm 1, \dots)$

(one-dimensional unitary irrep)

we want to determine the value of the invariant $W^2 = W_\mu W^\mu$ in the representations of $\mathcal{P}_+^\uparrow$ induced by (ii) of $\mathcal{K}_{\bar{p}}$:

one can see from $\omega_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} k^\rho \bar{p}^\sigma$ that

$$k^0 - k^3 = \alpha \,, \quad k^1 = -b_x = \mathrm{Re}\, z \,, \quad k^2 = -b_y = -\mathrm{Im}\, z$$

for (ii) we have $\quad W^1 |\bar{p}\rangle = W^2 |\bar{p}\dots\rangle = 0$

$$W_\mu P^\mu = 0 \quad \Rightarrow \quad 0 = W_\mu P^\mu |\bar{p}\dots\rangle = (W_0 + W_3)\, |\bar{p}\dots\rangle = 0$$

$$\Rightarrow \quad W_\mu W^\mu |\bar{p}\dots\rangle = 0$$

but $|p\dots\rangle = U(L,0)|\bar{p}\rangle$ and $[W_\mu W^\mu, U(L,0)] = 0$

$$\Rightarrow \quad W_\mu W^\mu |p\dots\rangle = 0 \quad , \text{i.e.} \quad W_\mu W^\mu = 0$$

W^μ, P^μ are hermitean operators with $[W^\mu, P^\nu] = 0$,

$W_\mu W^\mu = P_\mu P^\mu = 0$ satisfying on $|\bar{p}\rangle$:

$$P^1 |\bar{p}\rangle = P^2 |\bar{p}\rangle = 0 , \quad P^0 |\bar{p}\rangle = P^3 |\bar{p}\rangle = |\bar{p}\rangle$$

$$W^1 |\bar{p}\rangle = W^2 |\bar{p}\rangle = 0 , \quad W^0 |\bar{p}\rangle = W^3 |\bar{p}\rangle = c |\bar{p}\rangle$$

$$\Rightarrow \quad W^\mu = c P^\mu \quad \text{on } |\bar{p}\rangle$$

as a vector relation this is again valid for all $|p\rangle$:

$$|p\rangle = U(L,0) |\bar{p}\rangle \quad \Rightarrow \quad (W^\mu - c P^\mu) |p\rangle =$$

$$= (W^\mu - c P^\mu) \, U(L,0) |\bar{p}\rangle = U(L,0) \, L^\mu{}_\nu \underbrace{(W^\nu - c P^\nu) |\bar{p}\rangle}_{0}$$

determination of the constant of proportionality:

$$U(K(\alpha, z)) |\bar{p}\rangle \cong (\mathbb{1} - i k^\mu W_\mu) |\bar{p}\rangle =$$

$$= (\mathbb{1} - i k^0 W_0 - i k^3 W_3) |\bar{p}\rangle$$

$$= (\mathbb{1} - i \underbrace{(k^3 - k^0)}_{-\alpha} W_3 |\bar{p}\rangle = (\mathbb{1} - i \alpha W^3) |\bar{p}\rangle$$

on the other hand:

$$U(K(\alpha, z)) |\bar{p}\rangle = e^{-i\lambda\alpha} |\bar{p}\rangle \cong (\mathbb{1} - i \alpha \lambda) |\bar{p}\rangle$$

$$= (\mathbb{1} - i \alpha \lambda P^3) |\bar{p}\rangle \quad \Rightarrow \quad W^3 = \lambda P^3 \Rightarrow \underline{W^\mu = \lambda P^\mu}$$

physical meaning of the invariant λ: go to a specific inertial frame → consider the time component of $W^\mu = \lambda P^\mu$ and insert the definition of W_μ:

$$\lambda P_0 = W_0 = \frac{1}{2} \varepsilon_{0mnr} J^{mn} P^r = \vec{J}.\vec{P}$$

with $\quad J^r = \frac{1}{2} \varepsilon_{rmn} J^{mn}$

$$\lambda P_0 \, |p\rangle = \vec{J}.\vec{P} \, |p\rangle$$

$$\Rightarrow \quad \lambda p_0 \, |p\rangle = \vec{J}.\vec{p} \, |p\rangle$$

$$\Rightarrow \quad \lambda \, |p\rangle = \vec{J}.\frac{\vec{p}}{|\vec{p}|} \, |p\rangle \qquad (p^0 = |\vec{p}|)$$

$\Rightarrow \quad \lambda$ = projection of the total angular momentum onto the direction of motion

= helicity

$|\lambda|$ = "spin" of the massless particle (no rest frame!)

fix

$$M^2 |\lambda, p\rangle = W^2 |\lambda, p\rangle = 0$$

possible values

$$\lambda = 0, \pm\tfrac{1}{2}, \pm 1, \ldots$$

$$W^\mu |\lambda, p\rangle = \lambda P^\mu |\lambda, p\rangle = \lambda p^\mu |\lambda, p\rangle \qquad p^2 = 0,\ p_0 > 0$$

$$U(\mathbb{1}, a) |\lambda, p\rangle = e^{ip.a} |\lambda, p\rangle$$

$$U(L, 0) |\lambda, p\rangle = e^{-i\lambda\alpha(L,p)} |\lambda, Lp\rangle$$

where $\alpha(L,p)$ is the rotation angle about the 3-axis in the little-group element $K(L,p) = \Lambda_{Lp}^{-1} L \Lambda_p \in \mathcal{K}_{\bar{p}}$ according to

$$A(\alpha, z) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ -iz e^{-i\frac{\alpha}{2}} & e^{+i\frac{\alpha}{2}} \end{pmatrix}$$