3. Representation theory of the Lorentz group

Lie algebra and representations of $\mathcal{L}_+^\uparrow$

infinitesimal Lorentz transformation:

$$t' = t + \vec{u}\cdot\vec{x}$$

$$\vec{x}' = \vec{x} + \underbrace{\vec{\alpha}\times\vec{x}}_{\text{rotation}} + \underbrace{\vec{u}\,t}_{\text{boost}}$$

$$x'^\mu = x^\mu + \omega^\mu{}_\nu\, x^\nu$$

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0 & -u_x & -u_y & -u_z \\ u_x & 0 & \alpha_z & -\alpha_y \\ u_y & -\alpha_z & 0 & \alpha_x \\ u_z & \alpha_y & -\alpha_x & 0 \end{pmatrix}$$

$$x' = \overbrace{\left(\mathbb{1} + \vec{\alpha}\cdot\vec{M} + \vec{u}\cdot\vec{N}\right)}^{L(\vec{\alpha},\vec{u})} x$$

$$M_i = \begin{bmatrix} 0 & \vec{0}^T \\ \vec{0} & \Lambda_i \end{bmatrix}, \qquad N_i = \begin{bmatrix} 0 & \vec{e}_i^{\,T} \\ \vec{e}_i & 0 \end{bmatrix}$$

commutation relations of the Lie algebra $\mathcal{L} = so(1,3)$

$$[M_i, M_j] = \varepsilon_{ijk} M_k$$

$$[N_i, N_j] = -\varepsilon_{ijk} M_k$$

$$[N_i, M_j] = \varepsilon_{ijk} N_k \qquad \rightarrow \vec{N} \text{ is a vector operator under rotations}$$

complex linear combinations $\vec{M}^{\pm} = \frac{1}{2}(\vec{M} \pm i\vec{N})$

satisfy the commutation relations

$$[M_i^{\pm}, M_j^{\pm}] = \varepsilon_{ijk} M_k^{\pm} \quad , \quad [M_i^{+}, M_j^{-}] = 0$$

complexified Lie algebra $\mathcal{L}^c = \mathcal{L}^+ \oplus \mathcal{L}^-$

$\mathcal{L}^{\pm}$ have commutation relations of $so(3)$!

$\rightarrow$ complex irreducible representations of $\mathcal{L} = so(1,3)$ are of the form

$$D^{(j,j')} := D^{(j)} \otimes D^{(j')}$$

dimension of the product representation: $(2j+1)(2j'+1)$

Casimir operators $(\vec{M}^{\pm})^2 = (\vec{M}^2 - \vec{N}^2 \pm 2i\,\vec{M}.\vec{N})/4$

in irrep: $(\vec{M}^{+})^2 = -j(j+1)\mathbb{1}$, $(\vec{M}^{-})^2 = -j'(j'+1)\mathbb{1}$

$\Rightarrow \quad \vec{M}^2 - \vec{N}^2 = -2[j(j+1) + j'(j'+1)]$, $\vec{M}.\vec{N} = i[j(j+1) - j'(j'+1)]$

<u>infinitesimal</u> Lorentz transformation

$$L(\vec{\alpha},\vec{u}) = \mathbb{1} + \vec{\alpha}.\vec{M} + \vec{u}.\vec{N} = \mathbb{1} + \vec{\alpha}.(\vec{M}^+ + \vec{M}^-) - i\vec{u}(\vec{M}^+ - \vec{M}^-)$$
$$= \mathbb{1} + (\vec{\alpha} - i\vec{u}).\vec{M}^+ + (\vec{\alpha}+i\vec{u}).\vec{M}^-$$

$$L(\vec{\alpha},\vec{u}) \to D^{(j,j')}(\vec{\alpha},\vec{u}) = D^{(j)}(\vec{\alpha}-i\vec{u}) \otimes D^{(j')}(\vec{\alpha}+i\vec{u})$$

<u>finite</u> Lorentz transformation

$$L(\vec{\alpha},\vec{0}) = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & R(\vec{\alpha}) \end{pmatrix}$$

$$L(\vec{\alpha},\vec{v}) := \underbrace{L(\vec{\alpha},\vec{0})}_{\text{pure rotation}} \underbrace{L(0,\vec{v})}_{\text{pure boost}}$$

$$L(\vec{0},\vec{v}) = \begin{pmatrix} \gamma & \gamma\vec{v}^T \\ \gamma\vec{v} & \mathbb{1}_3 + \frac{\gamma^2}{1+\gamma}\vec{v}\vec{v}^T \end{pmatrix}$$

$$L(\vec{\alpha},\vec{v}) \neq \exp[\cancel{\mathbb{1}+} (\vec{\alpha}-i\vec{v}).\vec{M}^+ + (\vec{\alpha}+i\vec{v}).\vec{M}^-]$$

<u>reasons</u>:

(i) rotations and boosts do <u>not</u> commute (except for $\vec{\alpha} \parallel \vec{v}$) : $L(\vec{\alpha},\vec{0})\,L(\vec{0},\vec{v}) = L(\vec{0},R(\vec{\alpha})\vec{v})\,L(\vec{\alpha},\vec{0})$

(ii) for a given direction of the velocity $\vec{v}$, its length $|\vec{v}| = v$ is <u>not</u> an additive parameter (the <u>rapidity</u> $u = \operatorname{artanh} v$ is additive instead)

$$e^{\vec{u}.\vec{N}} = e^{u\,\vec{n}.\vec{N}} = \begin{pmatrix} \cosh u & \sinh u\, \vec{n}^T \\ \sinh u\, \vec{n} & 1\!\!1_3 + (\cosh u - 1)\, \vec{n}\vec{n}^T \end{pmatrix}$$

$|\vec{n}| = 1 \;,\; u \geq 0$

$$= L(\vec{0}, \vec{v})$$

$$\Rightarrow \quad \vec{v} = \vec{n} \tanh u$$

$\Rightarrow \quad L(\vec{\alpha}, \vec{v}) := L(\vec{\alpha}, \vec{0})\, L(\vec{0}, \vec{v})$ represented by

$$D^{(j,j')}(\vec{\alpha}, \vec{v}) = D^{(j)}(\vec{\alpha})\, D^{(j)}(-i\vec{u}) \otimes D^{(j')}(\vec{\alpha})\, D^{(j')}(i\vec{u})$$

with $\vec{u} = \operatorname{artanh}|\vec{v}| \cdot \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\vec{n}}$

The spinor representation

$j = \frac{1}{2} \;,\; j' = 0$

$$e^{-i(\vec{\alpha} - i\vec{u})\cdot\frac{\vec{\sigma}}{2}} \in SL(2,\mathbb{C})$$

remember: $D^{(\frac{1}{2},0)}(\vec{\alpha},\vec{v}) = e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} e^{-\vec{u}\cdot\frac{\vec{\sigma}}{2}} \neq e^{-i(\vec{\alpha}-i\vec{u})\cdot\frac{\vec{\sigma}}{2}}$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} \operatorname{artanh}|\vec{v}|$$

we know: $SO(3) \cong SU(2)/_{\mathbb{Z}_2}$

generalization: $\mathcal{L}_+^\uparrow \cong SL(2,\mathbb{C})/_{\mathbb{Z}_2}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \quad (a,b,c,d \in \mathbb{C})$

$\det A = ad - bc = 1 \;\rightarrow\;$ 2 real conditions

$\Rightarrow$ $8-2=6$ real parameters $\hat{=}$ 6 real parameters of $\mathcal{L}_+^\uparrow$

we define:

$\sigma^0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, i.e. $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$

real four-vector x_μ $\leftrightarrow$ <u>hermitean</u> matrix $x_\mu \sigma^\mu =: X$

$$X = x_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix}$$

$A \in SL(2, \mathbb{C})$

$X \to X' = A X A^\dagger$ <u>also</u> hermitean

$X' = x'_\mu \sigma^\mu$

$$\det X' \underset{\det A = 1}{\overset{\uparrow}{=}} \det X \quad \Rightarrow \quad x'_\mu x^{\mu\prime} = x_\mu x^\mu$$

$\Rightarrow$ x and x' are related by a Lorentz transformation

define: $\bar{\sigma}^\mu := (\mathbb{1}, -\vec{\sigma})$

$$x^\mu = \frac{1}{2} \operatorname{Tr}(X \bar{\sigma}^\mu)$$

$$\frac{1}{2} \operatorname{Tr}(X \bar{\sigma}^\mu) = \frac{1}{2} \operatorname{Tr}(x_\nu \sigma^\nu \bar{\sigma}^\mu) = \frac{x_\nu}{2} \underbrace{\operatorname{Tr}(\sigma^\nu \bar{\sigma}^\mu)}_{2 g^{\mu\nu}}$$

$$= x^\mu$$

$$x'^\mu = L^\mu{}_\nu x^\nu = \frac{1}{2} \mathrm{Tr}\,(X' \bar{\sigma}^\mu) =$$

$$= \frac{1}{2} \mathrm{Tr}\,(A X A^\dagger \bar{\sigma}^\mu) = \frac{1}{2} \mathrm{Tr}\,(A x^\nu \sigma_\nu A^\dagger \bar{\sigma}^\mu)$$

$$= \frac{1}{2} \mathrm{Tr}\,(\bar{\sigma}^\mu A \sigma_\nu A^\dagger)\, x^\nu$$

$$\Rightarrow \quad L^\mu{}_\nu = \frac{1}{2} \mathrm{Tr}\,(\bar{\sigma}^\mu A \sigma_\nu A^\dagger)$$

$\pm A \in SL(2,\mathbb{C})$ lead to the same L

(analogously to $\pm U \in SU(2) \rightarrow R_U \in SO(3)$)

$$L^0{}_0 = \frac{1}{2} \mathrm{Tr}\,(A A^\dagger) > 0 \quad \Rightarrow \quad L \in \mathcal{L}^\uparrow$$

<u>$\det L = +1$</u> can also be shown:

$\det L$ is basis-independent $\Rightarrow$ it is also equal to the determinant of the mapping $X \rightarrow A X A^\dagger$

$$\Rightarrow \det L = \det (A \otimes A^*) = \det [(A \otimes \mathbb{1})(\mathbb{1} \otimes A^*)]$$

$$= (\det A)^2 (\det A^*)^2 = 1$$

$\Rightarrow \quad L \in \mathcal{L}_+^\uparrow$ (L is a proper orthochronous Lorentz transformation) example: pure velocity transformation (boost)

$$A = e^{-\vec{u}\cdot\frac{\vec{\sigma}}{2}} = \cosh\frac{u}{2} - \vec{n}\cdot\vec{\sigma}\,\sinh\frac{u}{2}\,, \quad \vec{u} = u\vec{n},\ |\vec{n}| = 1$$

associated $L \in \mathcal{L}_+^\uparrow$:

$$(L^\mu{}_\nu) = \begin{pmatrix} \cosh u & \vec{n}^T \sinh u \\ \vec{n}\sinh u & \mathbb{1}_3 + (\cosh u - 1)\,\vec{n}\vec{n}^T \end{pmatrix}$$

$A \in SL(2,\mathbb{C}) \Rightarrow A, A^*, (A^T)^{-1}, (A^\dagger)^{-1}$ are representations of $SL(2,\mathbb{C})$

A and A^* ($(A^T)^{-1}$ and $(A^\dagger)^{-1}$, respectively) are <u>not</u> equivalent, i.e. $\nexists$ nonsingular matrix S with $A^* = SAS^{-1}$, but A and $(A^T)^{-1}$ (A^* and $(A^\dagger)^{-1}$) <u>are</u> equivalent.

<u>remark</u>: $U \in SU(2) \Rightarrow U^*$ is equivalent to U,

as $\sigma^2\vec{\sigma}\,\sigma^2 = -\vec{\sigma}^* = -\vec{\sigma}^T \qquad (\sigma^2\sigma^2 = \mathbb{1}_2)$

$$\Rightarrow \sigma_2 \underbrace{e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}}_{U} \sigma_2 = \underbrace{e^{+i\vec{\alpha}\cdot\frac{\vec{\sigma}^*}{2}}}_{U^*} = \underbrace{e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}^T}{2}}}_{(U^T)^{-1}}$$

this is <u>not</u> true for generic elements $A = e^{-i(\vec{\alpha}-i\vec{u})\cdot\frac{\vec{\sigma}}{2}}$

for $\underline{\vec{u} \neq 0}$

$$\sigma^2 A \sigma^2 = \sigma^2 \underbrace{e^{-i(\vec{\alpha}-i\vec{u})\cdot\frac{\vec{\sigma}}{2}}}_{\in D^{(\frac{1}{2},0)}} \sigma^2 = \underbrace{e^{i(\vec{\alpha}-i\vec{u})\cdot\frac{\vec{\sigma}^*}{2}}}_{\neq A^* \text{ for } \vec{u}\neq 0}$$

$$= e^{i(\vec{\alpha}-i\vec{u})\cdot\frac{\vec{\sigma}^T}{2}} = (A^T)^{-1}$$

$$A^* = e^{i(\vec{\alpha}+i\vec{u})\cdot\frac{\vec{\sigma}^*}{2}}$$

$$\sigma^2 A^* \sigma^2 = \sigma^2 e^{i(\vec{\alpha}+i\vec{u})\cdot\frac{\vec{\sigma}^*}{2}} \sigma^2 = \underbrace{e^{-i(\vec{\alpha}+i\vec{u})\cdot\frac{\vec{\sigma}}{2}}}_{\in D^{(0,\frac{1}{2})}}$$

$$= (A^\dagger)^{-1}$$

Spinor algebra

transformation properties of spinors

$$\chi'_\alpha = A_\alpha{}^\beta \chi_\beta \qquad \bar{\chi}'_{\dot{\alpha}} = A^*{}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\chi}_{\dot{\beta}}$$

$$\chi'^\alpha = A^{-1}{}_\beta{}^\alpha \chi^\beta \qquad \bar{\chi}'^{\dot{\alpha}} = (A^*)^{-1}{}_{\dot{\beta}}{}^{\dot{\alpha}} \bar{\chi}^{\dot{\beta}}$$

$\alpha, \beta = 1,2$ $\qquad$ $\dot{\alpha}, \dot{\beta} = 1,2$

Lorentz scalars:

$$\varphi^{\alpha}\psi_{\alpha} =: \varphi\psi \quad , \qquad \overline{\varphi}_{\dot\alpha}\,\overline{\chi}^{\dot\alpha} =: \overline{\varphi}\,\overline{\chi}$$

invariant tensors: $\varepsilon^{\alpha\beta}$, $\varepsilon_{\alpha\beta}$

$$\varepsilon_{21} = \varepsilon^{12} = +1 \quad , \qquad \varepsilon_{12} = \varepsilon^{21} = -1 .$$

diagonal elements $= 0$

$$\varepsilon_{\alpha\beta} = A_{\alpha}{}^{\gamma} A_{\beta}{}^{\delta}\, \varepsilon_{\gamma\delta}$$

$$\varepsilon^{\alpha\beta} = \varepsilon^{\gamma\delta} A_{\gamma}{}^{\alpha} A_{\delta}{}^{\beta} \Leftrightarrow \varepsilon^{\alpha\beta} = \varepsilon^{\gamma\delta} (A^{-1})_{\gamma}{}^{\alpha} (A^{-1})_{\delta}{}^{\beta}$$

the meaning of these equations is simply that $A \sim (A^T)^{-1}$:

$$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma^2, \quad \varepsilon^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2$$

$$\varepsilon = A\varepsilon A^T \Leftrightarrow (A^T)^{-1} = \varepsilon^{-1} A \varepsilon$$

$$\Updownarrow$$

$$\varepsilon^{-1} = A^T \varepsilon^{-1} A$$

$$\Updownarrow$$

$$(A^T)^{-1} \varepsilon^{-1} A^{-1} = \varepsilon^{-1}$$

spinors with upper and lower indices are related by the ε - tensor:

$$\chi^\alpha = \varepsilon^{\alpha\beta}\chi_\beta \quad , \quad \chi_\alpha = \varepsilon_{\alpha\beta}\chi^\beta$$

<u>remarks:</u> (i) $\varepsilon_{\alpha\beta}\,\varepsilon^{\beta\gamma} = \delta_\alpha{}^\gamma$

(ii) $\varepsilon_{\dot\alpha\dot\beta}$ and $\varepsilon^{\dot\alpha\dot\beta}$ are defined analogously

(iii) spinors are regarded as <u>anticommuting</u> objects (Grassmann variables)
(reason: in QFT, spinors will be promoted to operators satisfying <u>anti</u>commutation relations!)

$$\Rightarrow \quad \varphi\chi = \chi\varphi \quad , \quad \bar\varphi\bar\chi = \bar\chi\bar\varphi$$

<u>proof:</u> $\varphi\chi = \varphi^\alpha\chi_\alpha = -\chi_\alpha\varphi^\alpha =$

$$= -\varepsilon_{\alpha\beta}\chi^\beta\varepsilon^{\alpha\gamma}\varphi_\gamma = \chi^\beta\underbrace{\varepsilon_{\beta\alpha}\,\varepsilon^{\alpha\gamma}}_{\delta_\beta{}^\gamma}\varphi_\gamma$$

$$= \chi^\beta\varphi_\beta = \chi\varphi$$

the index structure of σ^μ can be read off from $x_\mu \sigma^\mu \xrightarrow{A \in SL(2,\mathbb{C})} A x_\mu \sigma^\mu A^\dagger$:

$$x_\mu \, A_\alpha{}^\beta (\sigma^\mu)_{\beta\dot\beta} \, A^*{}_{\dot\alpha}{}^{\dot\beta} = x'_\mu (\sigma^\mu)_{\alpha\dot\alpha}$$

$$\rightarrow \quad (\sigma^\mu)_{\alpha\dot\alpha}$$

index structure of $\bar\sigma^\mu$ can be inferred from the formula

$$L^\mu{}_\nu = \frac{1}{2} \operatorname{Tr}(\bar\sigma^\mu A \sigma_\nu A^\dagger) =$$

$$= \frac{1}{2} (\bar\sigma^\mu)^{\dot\alpha\alpha} A_\alpha{}^\beta (\sigma_\nu)_{\beta\dot\beta} \, A^*{}_{\dot\alpha}{}^{\dot\beta}$$

$$\rightarrow \quad (\bar\sigma^\mu)^{\dot\alpha\alpha}$$

<u>remark:</u> $(\bar\sigma^\mu)^{\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta} \varepsilon^{\alpha\beta} \sigma^\mu_{\beta\dot\beta}$

<u>Spinor fields</u>

spinor $\chi_\alpha \longrightarrow$ spinor field $\chi_\alpha(x)$ (Weyl spinor field)

$\varphi^\alpha(x) \, \sigma^\mu_{\alpha\dot\alpha} \, \partial_\mu \bar\chi^{\dot\alpha}(x) =: \varphi(x) \sigma^\mu \partial_\mu \bar\chi(x)$ is a scalar field

<u>remark</u>: $\varphi \sigma^\mu \bar{\chi} = - \bar{\chi} \bar{\sigma}^\mu \varphi$

<u>proof</u>: $\varphi \sigma^\mu \bar{\chi} = \varphi^\beta \sigma^\mu_{\beta\dot\beta} \bar{\chi}^{\dot\beta} = \varepsilon^{\beta\alpha} \varphi_\alpha \sigma^\mu_{\beta\dot\beta} \varepsilon^{\dot\beta\dot\alpha} \bar{\chi}_{\dot\alpha}$

$$= \varphi_\alpha \varepsilon^{\alpha\beta} \varepsilon^{\dot\alpha\dot\beta} \sigma^\mu_{\beta\dot\beta} \bar{\chi}_{\dot\alpha} = \varphi_\alpha (\bar{\sigma}^\mu)^{\dot\alpha\alpha} \bar{\chi}_{\dot\alpha} =$$

$$= - \bar{\chi}_{\dot\alpha} (\bar{\sigma}^\mu)^{\dot\alpha\alpha} \varphi_\alpha = - \bar{\chi} \bar{\sigma}^\mu \varphi$$

construction of a Lorentz-invariant action for a <u>free field</u> using <u>one</u> Weyl spinor field $\chi(x)$:

$$S_{\text{Majorana}} = \int d^4x \; \frac{1}{2} \Big[i \chi \sigma^\mu \partial_\mu \bar{\chi} + i \bar{\chi} \bar{\sigma}^\mu \partial_\mu \chi - m (\chi\chi + \bar{\chi}\bar{\chi}) \Big]$$

<u>remarks</u>: (i) $\bar{\chi}_{\dot\alpha} = \chi^\dagger_\alpha$

(ii) $\int d^4x \; \bar{\chi} \bar{\sigma}^\mu \partial_\mu \chi \overset{\text{partial integration}}{=} - \int d^4x \; \partial_\mu \bar{\chi} \bar{\sigma}^\mu \chi$

$$= \int d^4x \; \chi \sigma^\mu \partial_\mu \bar{\chi}$$

(iii) anticommuting fields are essential for the construction of the Lagrange density (otherwise $\chi\chi = 0$)

$\Rightarrow$ field equation $\quad i\sigma^\mu \partial_\mu \bar{\chi} - m\chi = 0.$ <u>Majorana</u>

$i\bar{\sigma}^\mu \partial_\mu \chi - m\bar{\chi} = 0$ <u>equation</u>

($m = 0 \rightarrow$ Weyl equation)

the mass term in S_{Majorana} is <u>not</u> invariant under U(1) transformations $\chi \rightarrow e^{i\alpha}\chi$ $\Rightarrow$ Majorana equation can only describe a <u>neutral</u> field $\rightarrow$ charged particles (like the electron) cannot be described by a <u>single</u> Weyl spinor $\rightarrow$ possible field equation for a <u>massive neutrino</u> (S_{Majorana} also P-violating $\rightarrow$ see later)

Dirac field

charged spin $\frac{1}{2}$ fermion with P-invariant interaction described by <u>two</u> Weyl spinors φ, χ

$$S_{\text{Dirac}} = \int d^4x \, [\, i\varphi\sigma^\mu\partial_\mu\bar{\varphi} + i\bar{\chi}\sigma^\mu\partial_\mu\chi - m(\varphi\chi + \bar{\varphi}\bar{\chi})\,]$$

$$= \int d^4x \; [i \varphi \sigma^\mu \partial_\mu \bar{\varphi} + i \chi \sigma^\mu \partial_\mu \bar{\chi} -$$
$$- m (\varphi \chi + \bar{\varphi} \bar{\chi})]$$

S_{Dirac} is invariant under the $U(1)$ transformation

$\varphi \to e^{i\alpha} \varphi$, $\chi \to e^{-i\alpha} \chi$

Dirac equation:

index form:

$$i \sigma^\mu \partial_\mu \bar{\varphi} - m \chi = 0 \qquad i \, (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu \bar{\varphi}^{\dot{\alpha}} - m \chi_\alpha = 0$$

$$i \bar{\sigma}^\mu \partial_\mu \chi - m \bar{\varphi} = 0 \qquad i \, (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \partial_\mu \chi_\alpha - m \bar{\varphi}^{\dot{\alpha}} = 0$$

$\rightarrow$ Dirac spinor $\quad \psi = \begin{pmatrix} \chi_\alpha \\ \bar{\varphi}^{\dot{\alpha}} \end{pmatrix}$

$\rightarrow$ Dirac equation assumes the form

$$(i \gamma^\mu \partial_\mu - m) \psi(x) = 0 \, ,$$

where $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ (γ matrices in the "Weyl basis")

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

<u>anticommutation</u> relations of the γ-matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1}_4$$

<u>ex.</u>:

$$(\sigma^\mu \bar\sigma^\nu + \sigma^\nu \bar\sigma^\mu)_\alpha{}^\beta = 2 g^{\mu\nu} \delta_\alpha{}^\beta$$

$$(\bar\sigma^\mu \sigma^\nu + \bar\sigma^\nu \sigma^\mu)^{\dot\alpha}{}_{\dot\beta} = 2 g^{\mu\nu} \delta^{\dot\alpha}{}_{\dot\beta}$$

$$\Rightarrow \quad \{\gamma^\mu, \gamma^\nu\} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0 \end{pmatrix} + (\mu \leftrightarrow \nu)$$

$$= \begin{pmatrix} \sigma^\mu \bar\sigma^\nu + \sigma^\nu \bar\sigma^\mu & 0 \\ & \bar\sigma^\mu \sigma^\nu + \bar\sigma^\nu \sigma^\mu \end{pmatrix} =$$

$$= 2 g^{\mu\nu} \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$

<u>remarks</u>: (i) many text books start with the Dirac equation in its four-dimensional form (often with some dubious arguments like "taking the square root" of the Klein-Gordon equation)

(ii) a (basis) transformation $\psi = S\tilde{\psi}$ with a non-singular 4×4 matrix S does <u>not</u> change the physics:

$$S^{-1}\Big|\; 0 = (i\gamma^\mu \partial_\mu - m)\psi = (i\gamma^\mu \partial_\mu - m) S\tilde{\psi}$$

$$0 = (i \underbrace{S^{-1}\gamma^\mu S}_{\tilde{\gamma}^\mu} \partial_\mu - m)\tilde{\psi}$$

the transformed matrices $\tilde{\gamma}^\mu = S^{-1}\gamma^\mu S$ still fullfil the anticommutation relations $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$

(usually $S^\dagger = S^{-1} \Rightarrow \tilde{\gamma}^0$ hermitean, $\tilde{\gamma}^i$ antihermitean)

Lagrange density written with 4-spinor ψ:

$$\mathcal{L}_{\text{Dirac}} = i\,\varphi^\beta (\sigma^\mu)_{\beta\dot{\alpha}}\, \partial_\mu \bar{\varphi}^{\dot{\alpha}} + i\,\bar{\chi}_{\dot{\beta}} (\bar{\sigma}^\mu)^{\dot{\beta}\alpha}\, \partial_\mu \chi_\alpha$$

$$- m\,(\varphi^\alpha \chi_\alpha + \bar{\chi}_{\dot{\alpha}} \bar{\varphi}^{\dot{\alpha}})$$

$$= i\,(\varphi^\beta, \bar{\chi}_{\dot{\beta}}) \underbrace{\begin{pmatrix} 0 & (\sigma^\mu)_{\beta\dot{\alpha}} \\ (\bar{\sigma}^\mu)^{\dot{\beta}\alpha} & 0 \end{pmatrix}}_{\gamma^\mu} \partial_\mu \underbrace{\begin{pmatrix} \chi_\alpha \\ \bar{\varphi}^{\dot{\alpha}} \end{pmatrix}}_{\psi}$$

$$- m\,(\varphi^\alpha, \bar{\chi}_{\dot{\alpha}}) \begin{pmatrix} \chi_\alpha \\ \bar{\varphi}^{\dot{\alpha}} \end{pmatrix}$$

$$= i\,\overline{\psi}\gamma^\mu\partial_\mu\psi - m\,\overline{\psi}\psi$$

with $\overline{\psi} = (\varphi^\beta, \overline{\chi}_{\dot\beta}) = \underbrace{(\overline{\chi}_{\dot\alpha}, \varphi^\alpha)}_{\psi^\dagger} \underbrace{\begin{pmatrix} 0 & \delta^{\dot\alpha}{}_{\dot\beta} \\ \delta_\alpha{}^\beta & 0 \end{pmatrix}}_{=:\beta}$

$$\overline{\psi} = \psi^\dagger\beta$$

<u>remarks:</u> (i) $\beta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ agrees numerically with $\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \overline{\sigma}^0 & 0 \end{pmatrix}$, but the index structure differs (usually no distinction made)

(ii) $\gamma_\mu{}^\dagger = \beta\gamma_\mu\beta$ (ex.)

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observations:

(i) $(i\gamma^\nu\partial_\nu + m)\,(i\gamma^\mu\partial_\mu - m)\psi = 0$

$$(-\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - m^2)\psi = 0$$

$$\Rightarrow \quad \Big(\tfrac{1}{2}\underbrace{\{\gamma^\mu, \gamma^\nu\}}_{2g^{\mu\nu}}\partial_\mu\partial_\nu + m^2\Big)\psi = 0$$

$$\Rightarrow \quad (\Box + m^2)\psi = 0$$

→ each of the four spinor components fulfills the Klein–Gordon equation (expression for relativistic energy–momentum relation in space-time!)

(ii) $j^\mu = \bar{\psi}\gamma^\mu\psi$ is a conserved current
(candidate for electromagnetic 4-current density)

$$\partial_\mu j^\mu = \bar{\psi}\underbrace{\gamma^\mu\partial_\mu\psi}_{-im\psi} + \underbrace{(\partial_\mu\bar{\psi})\gamma^\mu}_{im\bar{\psi}}\psi = 0$$

j^μ is the Noether current of the $U(1)$ gauge symmetry $\psi \to e^{i\alpha}\psi$ $\left(\Rightarrow \bar{\psi} \to e^{-i\alpha}\bar{\psi}\right)$
of the Lagrange density $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$
$(\not{a} := a_\mu\gamma^\mu)$

How does a Dirac field transform under $SL(2,\mathbb{C})$?

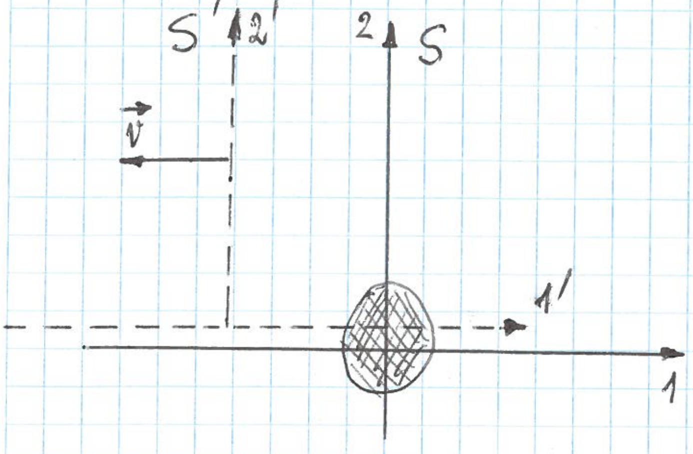
$$\psi(x) = \begin{pmatrix} \chi_\alpha(x) \\ \bar{\varphi}^{\dot{\alpha}}(x) \end{pmatrix}$$

$$A \in SL(2,\mathbb{C}) \longrightarrow L \in \mathcal{L}_+^\uparrow$$

$$x^{\mu\prime} = L^\mu{}_\nu x^\nu$$

$$\chi'(x') = A\,\chi(x) = A\,\chi(L^{-1}x') \quad \text{(passive interpretation)}$$

or $\chi(x) \rightarrow \chi'(x) = A\,\chi(L^{-1}x)$ (active interpretation)



$\chi(x)$ describes field in S

$\chi'(x') = A\,\chi(L^{-1}x)$

describes field seen from S' (passive interpr.)

e.g. observer in S sees a field being essentially concentrated around the origin of system S

$\rightarrow$ observer in S' sees a field moving with velocity v in the direction of the positive $1'$-axis

<u>active interpretation</u> (Lorentz boost): I stay in the reference frame S and I observe a field $\chi'(x)$ moving with velocity v in the direction of the positive 1-axis with respect to a reference field $\chi(x)$ "at rest". Of course, because of the principle of relativity, I see the same field as the observer in S' in the previous case. $\Rightarrow$ $\chi'(x) = A\,\chi(L^{-1}x)$

$$\psi(x) = \begin{pmatrix} \chi_\alpha(x) \\ \bar{\varphi}^{\dot{\alpha}}(x) \end{pmatrix} \longrightarrow S\psi(L^{-1}x) = \begin{pmatrix} A_\alpha{}^\beta & 0 \\ 0 & (A^*)^{-1}{}_{\dot{\beta}}{}^{\dot{\alpha}} \end{pmatrix} \begin{pmatrix} \chi_\beta(L^{-1}x) \\ \bar{\varphi}^{\dot{\beta}}(L^{-1}x) \end{pmatrix}$$

$$S = \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix}$$

<u>reducible</u> representation of $SL(2,\mathbb{C})$

explicit parametrization:

$$A = e^{-i(\vec{\alpha} - i\vec{u})\cdot\frac{\vec{\sigma}}{2}}$$

$$A^\dagger = e^{+i(\vec{\alpha} + i\vec{u})\cdot\frac{\vec{\sigma}}{2}} \Rightarrow (A^\dagger)^{-1} = e^{-i(\vec{\alpha} + i\vec{u})\cdot\frac{\vec{\sigma}}{2}}$$

$$S = \begin{pmatrix} e^{-i(\vec{\alpha}-i\vec{u})\cdot\frac{\vec{\sigma}}{2}} & 0 \\ 0 & e^{-i(\vec{\alpha}+i\vec{u})\cdot\frac{\vec{\sigma}}{2}} \end{pmatrix}$$

$$= \exp\begin{pmatrix} -i(\vec{\alpha}-i\vec{u})\cdot\frac{\vec{\sigma}}{2} & 0 \\ 0 & -i(\vec{\alpha}+i\vec{u})\cdot\frac{\vec{\sigma}}{2} \end{pmatrix}$$

$$= e^{-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}} ,$$

$$\sigma_{\mu\nu} := \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

$$\omega^{\mu\nu} = \begin{pmatrix} 0 & -u_x & -u_y & -u_z \\ u_x & 0 & \alpha_z & -\alpha_y \\ u_y & -\alpha_z & 0 & \alpha_x \\ u_z & \alpha_y & -\alpha_x & 0 \end{pmatrix} \qquad (\text{ex.})$$

Solutions of the free Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0$$

$$(\Box + m^2)\,\psi_r(x) = 0 \qquad \text{for } r = 1, \dots, 4$$

$\Rightarrow$ general solution is a linear combination of $u(\vec{p})\,e^{-ipx}$ and $v(\vec{p})\,e^{+ipx}$

$p^0 = \sqrt{\vec{p}^{\,2} + m^2}$, $u(\vec{p})$ and $v(\vec{p})$ are 4-spinors

a) solutions with positive frequency

$$\psi(x) = u(\vec{p})\,e^{-ipx} \quad \Rightarrow \quad (\underbrace{\gamma^\mu p_\mu}_{\not p} - m)\,u(\vec{p}) = 0$$

start with the solution for $\vec{p} = 0$ (rest frame)

$$m\,(\gamma^0 - \mathbb{1}_4)\,u(\vec{0}) = 0$$

$$\begin{pmatrix} -\mathbb{1}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & -\mathbb{1}_2 \end{pmatrix} u(\vec{0}) = 0 \quad \Rightarrow \quad u(\vec{0}) = \overset{\text{convention}}{\sqrt{m}} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

ξ is an arbitrary two-dim. vector

solution for <u>arbitrary</u> $\vec{p}$: Lorentz boost on

$$u(\vec{0})\, e^{-imx^0} \quad\rightarrow\quad u(\vec{p})\, e^{-ipx} = S\, u(\vec{0})\, e^{-im(L^{-1}x)^0}$$

$$S = \begin{pmatrix} e^{-\vec{u}\cdot\frac{\vec{\sigma}}{2}} & 0 \\ 0 & e^{+\vec{u}\cdot\frac{\vec{\sigma}}{2}} \end{pmatrix}$$ with a pertinent $\vec{u}$

we consider first the <u>special case</u> of a boost in 3-direction (generalization to an arbitrary space direction will turn out to be trivial):

$$S = \begin{pmatrix} e^{-u\frac{\sigma^3}{2}} & \\ & e^{+u\frac{\sigma^3}{2}} \end{pmatrix} \Rightarrow L = \begin{pmatrix} \cosh u & 0 & 0 & \sinh u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh u & 0 & 0 & \cosh u \end{pmatrix}$$

$$\Rightarrow L^{-1} = \begin{pmatrix} \cosh u & 0 & 0 & -\sinh u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh u & 0 & 0 & \cosh u \end{pmatrix}$$

$$(L^{-1}x)^0 = \cosh u\; x^0 - \sinh u\; x^3$$

$$u(\vec{0})\, e^{-imx^0} \rightarrow \underbrace{\begin{pmatrix} e^{-u\frac{\sigma^3}{2}} & 0 \\ 0 & e^{+u\frac{\sigma^3}{2}} \end{pmatrix} \overbrace{\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}}^{u(\vec{0})}}_{u(\vec{p})} \underbrace{e^{-im(\cosh u\, x^0 - \sinh u\, x^3)}}_{e^{-ipx}}$$

$$p^0 = m \cosh u = \frac{m}{2}\left(e^{u} + e^{-u}\right)$$

$$p^1 = p^2 = 0$$

$$p^3 = m \sinh u = \frac{m}{2}\left(e^{u} - e^{-u}\right)$$

$$\Rightarrow \quad m e^{u} = p^0 + p^3 \quad , \quad m e^{-u} = p^0 - p^3$$

$$u(\vec{p}) = \begin{pmatrix} e^{-u/2} & 0 & 0 & 0 \\ 0 & e^{+u/2} & 0 & 0 \\ 0 & 0 & e^{+u/2} & 0 \\ 0 & 0 & 0 & e^{-u/2} \end{pmatrix} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} =$$

$$= \begin{pmatrix} \sqrt{p^0 - p^3} & 0 & 0 & 0 \\ 0 & \sqrt{p^0 + p^3} & 0 & 0 \\ 0 & 0 & \sqrt{p^0 + p^3} & 0 \\ 0 & 0 & 0 & \sqrt{p^0 - p^3} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{p^\mu \sigma_\mu} & 0 \\ 0 & \sqrt{p^\mu \bar{\sigma}} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = \begin{pmatrix} \sqrt{p.\sigma}\, \xi \\ \sqrt{p.\bar{\sigma}}\, \xi \end{pmatrix}$$

generalisation is now obvious

<u>summary:</u> $\forall$ 3-vector $\vec{p}$ $\exists!$ two linear independent solutions of the form $u(\vec{p},s)\,e^{-ipx}$,

$$u(\vec{p},s) = \begin{pmatrix} \sqrt{p.\sigma}\,\xi_s \\ \sqrt{p.\bar{\sigma}}\,\xi_s \end{pmatrix}, \qquad s=\pm$$

possible choice $\quad \xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

normalization $\quad \xi_r^\dagger \xi_s = \delta_{rs}$

$$\Rightarrow \quad \bar{u}(\vec{p},r)\,u(\vec{p},s) = 2m\,\delta_{rs} \qquad \text{(ex.)}$$

b) solutions with <u>negative</u> frequency:

$$\psi(x) = v(\vec{p},s)\,e^{+ipx}$$

$$v(\vec{p},s) = \begin{pmatrix} \sqrt{p.\sigma}\,\eta_s \\ -\sqrt{p.\bar{\sigma}}\,\eta_s \end{pmatrix} \qquad s=\pm$$

$$\eta_r^\dagger \eta_s = \delta_{rs}$$

$$\Rightarrow \quad \bar{v}(\vec{p},r)\,v(\vec{p},s) = -2m\,\delta_{rs}$$

Relation between spinors and vectors

$$V_{\alpha\dot{\alpha}} := \sigma^{\mu}{}_{\alpha\dot{\alpha}} V_{\mu}$$

$$V^{\mu} = \frac{1}{2} (\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}}$$

proof:

$$\frac{1}{2} (\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}} = \frac{1}{2} (\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} \sigma^{\nu}{}_{\alpha\dot{\alpha}} V_{\nu}$$

$$= \underbrace{\frac{1}{2} \mathrm{Tr}(\bar{\sigma}^{\mu}\sigma^{\nu})}_{g^{\mu\nu}} V_{\nu} = V^{\mu}$$

Decomposition of a tensor product of two irreps of $\mathcal{L}_+^{\uparrow}$

$$D^{(j_1,j_2)} \otimes D^{(j_1',j_2')} = D^{(j_1+j_1',\, j_2+j_2')} \oplus D^{(j_1+j_1'-1,\, j_2+j_2')}$$

$$\oplus \dots \oplus D^{(|j_1-j_1'|,\, |j_2-j_2'|)}$$