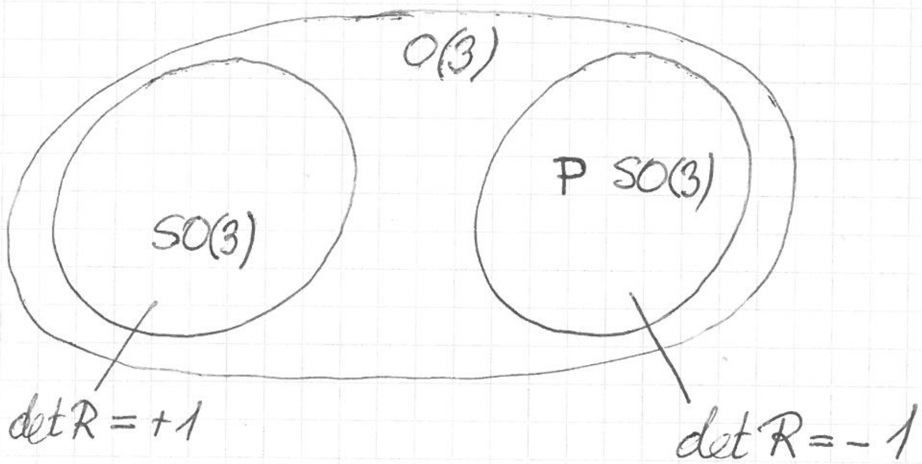
2. Representation theory of the rotation group

$R \in O(3)$ real 3×3 matrix with $R^T R = \mathbb{1} \Rightarrow \det R = \pm 1$

$R \in SO(3)$ — " — — " — — " — — " — and $\det R = +1$



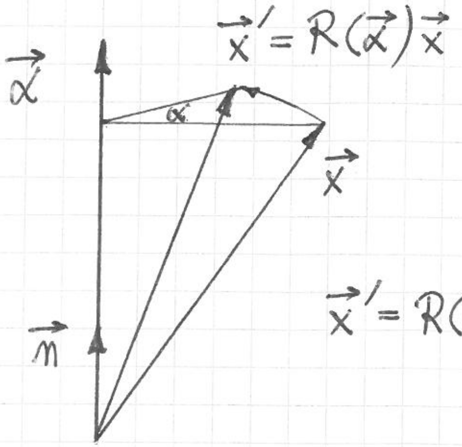
$P = -\mathbb{1}$

$SO(3)$ = proper rotations

R with $\det R = -1$ describes rotation + space reflection

SO(3)

more precisely: $SO(3, \mathbb{R})$



$\vec{x}\,' = R(\vec{\alpha})\vec{x}$ $\qquad \alpha = |\vec{\alpha}|$, $\vec{\alpha} = \alpha \vec{n}$, $|\vec{n}| = 1$

right-hand rule

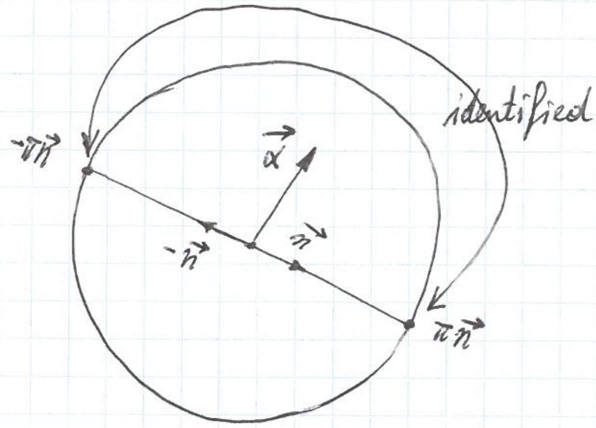
$$\vec{x}\,' = R(\vec{\alpha})\vec{x} = \vec{x}\cos\alpha + \vec{n}\,(\vec{n}\cdot\vec{x})(1-\cos\alpha) + \vec{n}\times\vec{x}\,\sin\alpha$$

$$R(\vec{\alpha})_{ij} = \delta_{ij}\cos\alpha + n_i n_j (1-\cos\alpha) - \varepsilon_{ijk} n_k \sin\alpha$$

$\mathrm{Tr}\, R(\vec{\alpha}) = 1 + 2\cos\alpha \quad , \quad n_i \sin\alpha = -\frac{1}{2}\, \varepsilon_{ijk} R_{jk}$

parameter range $\quad 0 \leq \alpha \leq \pi \quad , \vec{\alpha} = \pi\vec{n}$ and $-\pi\vec{n}$ identified

$|\vec{n}| = 1$



compact connected manifold (Lie group)

$R(\vec{\alpha}) \leftrightarrow$ points of a solid sphere $0 \leq |\vec{\alpha}| \leq \pi$ with identified antipodal surface points

Infinitesimal transformations

$$R(\vec{\varepsilon})\vec{x} = \vec{x} + \vec{\varepsilon} \times \vec{x} = (\mathbb{1} + \vec{\varepsilon} \cdot \vec{\Lambda})\vec{x}$$

$$(\vec{\varepsilon} \times \vec{x})_i = \epsilon_{ikj}\, \varepsilon_k\, x_j = -\varepsilon_k \epsilon_{kij}\, x_j$$

$\Rightarrow$ matrix representations of $\Lambda_1, \Lambda_2, \Lambda_3$ with respect to ONB $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$: $\quad (\Lambda_k)_{ij} = -\varepsilon_{kij}$

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

basis of the space of real antisymmetric 3×3 matrices remark: $R = \mathbb{1} + \Omega$

$$R^T R = \mathbb{1} \Rightarrow (\mathbb{1} + \Omega^T)(\mathbb{1} + \Omega) = \mathbb{1}$$

$$\Rightarrow \Omega^T = -\Omega$$

finite rotation $\vec{\alpha}$: $\quad R(\vec{\alpha}) = R\left(\frac{\vec{\alpha}}{N}\right)^N$

N large ($\frac{\alpha}{N}$ small): $\quad R(\vec{\alpha}) \simeq \left(\mathbb{1} + \frac{\vec{\alpha}.\vec{\Lambda}}{N}\right)^N$

$N \to \infty$: $\quad R(\vec{\alpha}) = e^{\vec{\alpha}.\vec{\Lambda}}$

we can generate any rotation from an infinitesimal one!

SO(3) is a nonabelian Lie group (noncommutative Lie group), i.e.

$$R(\vec{\alpha})R(\vec{\beta}) \overset{\text{in general}}{\neq} R(\vec{\beta})R(\vec{\alpha})$$

$$\boxed{R\left(R(\vec{\beta})\vec{\alpha}\right) = R(\vec{\beta})\,R(\vec{\alpha})\,R(\vec{\beta})^{-1}}$$

we show this relation first for infinitesimal $\vec{\alpha} = \vec{\varepsilon}$:

$$R(\vec{\beta})\,R(\vec{\varepsilon})\,R(\vec{\beta})^{-1}\vec{x} = R(\vec{\beta})\left[R(\vec{\beta})^{-1}\vec{x} + \vec{\varepsilon} \times R(\vec{\beta})^{-1}\vec{x}\right]$$

$$= \vec{x} + \left(R(\vec{\beta})\vec{\varepsilon}\right) \times \vec{x} = R\left(R(\vec{\beta})\vec{\varepsilon}\right)\vec{x} \checkmark$$

finite rotation angle $\vec{\alpha}$:

$$R(\vec{\beta})\,R(\vec{\alpha})\,R(\vec{\beta})^{-1} = R(\vec{\beta})\;R(\tfrac{\vec{\alpha}}{N})^{N}\,R(\vec{\beta})^{-1}$$

$$= [\underbrace{R(\vec{\beta})\,R(\tfrac{\vec{\alpha}}{N})\,R(\vec{\beta})^{-1}}_{R\left(R(\vec{\beta})\frac{\vec{\alpha}}{N}\right)}]^{N} \quad = R\left(R(\vec{\beta})\vec{\alpha}\right)$$

for N large enough

remark: if $\vec{\alpha} \parallel \vec{\beta} \Rightarrow R(\vec{\beta})\vec{\alpha} = \vec{\alpha} \Rightarrow [R(\vec{\alpha}), R(\vec{\beta})] = 0$

$\overbrace{\text{Lie algebra of } SO(3)}^{so(3):=}$ = (vector) space of all real antisymmetric 3×3 matrices$_{A,B,\dots}$ with the bilinear multiplication

$$A \circ B := [A,B] = -[B,A]$$

indeed : $[A,B]^T = [B^T, A^T] = -[A^T, B^T] = -[A,B]$

$\vec{\alpha}$ fixed $\quad e^{\tau\vec{\alpha}.\vec{\Lambda}}$ describes one-parameter ~~sou~~ subgroup of $SO(3)$ with generator $\vec{\alpha}.\vec{\Lambda} \in so(3)$

$$[\vec{m}.\vec{\Lambda}, \vec{n}.\vec{\Lambda}] = (\vec{m}\times\vec{n}).\vec{\Lambda}$$

$[\Lambda_i, \Lambda_j] = \varepsilon_{ijk}\Lambda_k$ fundamental commutation relations of $so(3)$

General definition of a Lie algebra $\mathcal{L}$

$\mathcal{L}$ = real vector space with a bilinear product $A \circ B$ where

1) $A, B \in \mathcal{L} \Rightarrow A \circ B \in \mathcal{L}$

2) $A \circ B = - B \circ A$ antisymmetry

3) $(A \circ B) \circ C + (B \circ C) \circ A + (C \circ A) \circ B = 0$ Jacobi identity

remark: with $A \circ B = [A, B]$ (commutator of matrices)

2), 3) automatically fulfilled

choose basis $\{X_A\}$

because of bilinearity, suffices to know the products $X_A \circ X_B$

$$X_A \circ X_B = C^D{}_{AB} \, X_D$$

$\uparrow$ structure constants (components of the structure constant tensor)

remark: structure constants have to satisfy

$$C^D{}_{AB} = - C^D{}_{BA} \, , \quad C^D{}_{AB} \, C^E{}_{CD} + C^D{}_{CA} \, C^E{}_{BD} + C^D{}_{BC} \, C^E{}_{AD} = 0$$

commutator relation of $so(3)$: $[\Lambda_i, \Lambda_j] = \varepsilon_{ijk} \Lambda_k$

$\Rightarrow$ structure constants of $so(3)$: ε_{ijk}

Lie algebra and representation of SO(3)

(V, D) representation D of $SO(3)$ in linear space V

$$\underbrace{D(g)}_{\in SO(3)} \in L(V) \,, \qquad D(g_1 g_2) = D(g_1)\, D(g_2)$$

$$D(e) = \mathbb{1}$$

$\det D(g) \neq 0$

$$D(g^{-1}) = D(g)^{-1}$$

let $g(\tau)$ be a one-parameter subgroup of $SO(3)$
$(g(0) = e)$

$$D(g(\tau)) = D(g(0)) + \tau \underbrace{\left.\frac{\partial D(g(\tau))}{\partial \tau}\right|_{\tau=0}}_{t} + O(\tau^2)$$

t = generator of the subgroup in the representation considered we consider $g(\tau) = R(\tau \vec{n})$, $\vec{n}$ fixed, $|\vec{n}| = 1$

$$\simeq 1\!\!1 + \tau \, \vec{n} \cdot \vec{\Lambda} + O(\tau^2)$$

$$D(g(\tau)) = D(R(\underbrace{\tau \vec{n}}_{\vec{\alpha}})) =$$

$$= 1\!\!1 + \left.\frac{\partial D(R(\vec{\alpha}))}{\partial \alpha_i}\right|_{\vec{\alpha}=0} n_i \, \tau + O(\tau^2)$$

chain rule

$$t = t_i n_i \quad , \quad t_i = \left.\frac{\partial D(R(\vec{\alpha}))}{\partial \alpha_i}\right|_{\vec{\alpha}=0}$$

t_i = generator of rotations with rotation axis $\vec{e}_i$

$\Rightarrow$ generators t form a real vector space spanned by t_1, t_2, t_3

<u>claim</u>: t_i fulfill the same commutation relations as Λ_i, i.e. $[t_i, t_j] = \varepsilon_{ijk} \, t_k$

<u>proof:</u> we know already the relation

$$R(\vec{\beta})\, R(\vec{\alpha})\, R(\vec{\beta})^{-1} = R(R(\vec{\beta})\vec{\alpha})$$

$$\Rightarrow \quad D(R(\vec{\beta}))\, D(R(\vec{\alpha}))\, D(R(\vec{\beta}))^{-1} = D(R(R(\vec{\beta})\vec{\alpha}))$$

$\vec{\alpha}$ infinitesimal:

$$D(R(\vec{\beta}))\, (\mathbb{1}_V + \vec{\alpha}.\vec{t}\,)\, D(R(\vec{\beta}))^{-1} = \mathbb{1}_V + (R(\vec{\beta})\vec{\alpha}).\vec{t}$$

$$\Rightarrow \quad D(R(\vec{\beta}))\, \vec{\alpha}.\vec{t}\, D(R(\vec{\beta}))^{-1} = (R(\vec{\beta})\vec{\alpha}).\vec{t}$$

now also $\vec{\beta}$ infinitesimal:

$$(\mathbb{1}_V + \vec{\beta}.\vec{t}\,)\, \vec{\alpha}.\vec{t}\, (\mathbb{1} - \vec{\beta}.\vec{t}\,) = (\vec{\alpha} + \vec{\beta} \times \vec{\alpha}).\vec{t}$$

$$\Rightarrow \quad [\vec{\beta}.\vec{t}, \vec{\alpha}.\vec{t}\,] = (\vec{\beta} \times \vec{\alpha}).\vec{t}$$

$$\beta_i\, \alpha_j\, [t_i, t_j] = \varepsilon_{kij}\, \beta_i\, \alpha_j\, t_k$$

$\vec{\alpha}, \vec{\beta}$ arbitrary $\Rightarrow \quad [t_i, t_j] = \varepsilon_{ijk}\, t_k$

commutation relations of $so(3)$ valid in <u>arbitrary</u> representation rewrite the decisive relation

$$D(R(\vec{\beta}))\,\vec{\alpha}.\vec{t}\,D(R(\vec{\beta}))^{-1} = (R(\vec{\beta})\vec{\alpha}).\vec{t}$$

with $S := R(\vec{\beta})^{-1}$ and $\vec{\alpha} = \vec{e}_i$

$$\Rightarrow \quad D(S)^{-1}\,t_i\,D(S) = (S^{-1}\vec{e}_i).\vec{t}$$

$$= (\vec{e}_j\,S^{-1}_{ji}).\vec{t}$$

$$\underset{\substack{\uparrow \\ S^T = S^{-1}, \\ \vec{e}_j.\vec{t} = t_j}}{=} S_{ij}\,t_j$$

$$D(S)^{-1}\,\vec{t}\,D(S) = S\vec{t}$$

generally, a triple $\vec{v}$ of operators $v_i \in L(V)$ satisfying

$$D(S)^{-1}\,\vec{v}\,D(S) = S\vec{v}$$

is called a <u>vector operator</u> on V

infinitesimal version: $D(S) = \mathbb{1}_V + \vec{\varepsilon}.\vec{t}$

$$(\mathbb{1}_V - \vec{\varepsilon}.\vec{t})\,\vec{v}\,(\mathbb{1} + \vec{\varepsilon}.\vec{t}) = \vec{v} + (\vec{\varepsilon} \times \vec{v})$$

$$\Rightarrow [\vec{v}, \vec{\varepsilon}.\vec{t}\,] = \vec{\varepsilon} \times \vec{v}$$

$\vec{v}^2 = v_i v_i$ is invariant under the representation:

$$D(S)^{-1}\vec{v}^2 D(S) = D(S)^{-1} v_i v_i D(S) =$$

$$= D(S^{-1})\, v_i \underbrace{D(S)\, D(S)^{-1}}_{\mathbb{1}_V} v_i D(S) =$$

$$= S_{ij} v_j\, S_{ik} v_k = v_j \underbrace{S^T_{ji} S_{ik}}_{=(S^T S)_{jk} = \delta_{jk}} v_k = \vec{v}^2$$

$$\vec{v}^2 D(S) = D(S)\vec{v}^2 \qquad \forall S \in SO(3)$$

$\vec{v}^2$ commutes with all operators $D(S)$ of the representation

in particular: $[\vec{t}^2, D(S)] = 0 \quad \forall S \in SO(3)$

infinitesimal version: $[\vec{v}^2, t_i] = 0 \;,\; i = 1,2,3$

More about representations

$\underset{\in G}{g} \to D(g)$ some representation of a group G in a linear space V

$\Rightarrow \; g \to (D(g)^{-1})^T$ is also a representation (contragredient representation)

$g \to S^{-1} D(g) S$, $S \in L(V)$, S nonsingular
is a representation $(\det S \neq 0)$

$D'(g) = S^{-1} D(g) S$ and $D(g)$ are
<u>equivalent</u> representations

<u>Kronecker product</u> (<u>tensor product</u>) of two representations
(V, D) and (V', D'): $(V \otimes V', D \otimes D')$
defined by $(D \otimes D')(g) := D(g) \otimes D'(g)$

<u>remark</u>: representation property follows from
$(A \otimes A')(B \otimes B') = AB \otimes A'B'$

$\{e_j\}$ basis of V with $D(g)\, e_j = e_i \, (D(g))^i{}_j$

$(D(g))^i{}_j$ = matrix representation of $D(g)$ with
respect to the basis $\{e_j\}$

$\{e'_\beta\}$ basis of V' with $D'(g) e'_\beta = e'_\alpha (D'(g))^\alpha{}_\beta$

$(D'(g))^\alpha{}_\beta$ = matrix representation of $D'(g)$ with
respect to basis $\{e'_\beta\}$

$\{ e_j \otimes e'_\beta \}$ basis of $V \otimes V'$

$$(D \otimes D')(g)\,(e_j \otimes e'_\beta) = (D(g) \otimes D'(g))(e_j \otimes e'_\beta)$$

$$= D(g) e_j \otimes D'(g) e'_\beta$$

$$= e_i \, (D(g))^i{}_j \otimes e'_\alpha \, (D'(g))^\alpha{}_\beta$$

$$= (e_i \otimes e'_\alpha) \underbrace{(D(g))^i{}_j \, (D'(g))^\alpha{}_\beta}_{((D \otimes D')(g))^{i\alpha}{}_{j\beta}}$$

action on tensor components:

$$T = T^{j\beta} e_j \otimes e'_\beta$$

$$(D \otimes D')(g)\, T = (e_i \otimes e'_\alpha)\, (D(g))^i{}_j \, (D'(g))^\alpha{}_\beta \, T^{j\beta}$$

<u>Direct sum</u> $(V \oplus V', D \oplus D')$ of two arbitrary representations (V, D) and (V', D') is defined by $(D \oplus D')(g) = D(g) \oplus D'(g)$,

i.e. $(D \oplus D')(g)\,(v \oplus v') = D(g) v \oplus D'(g) v'$

$$(D \oplus D')(g_1)\, (D \oplus D')(g_2) = (D(g_1) \oplus D'(g_1))\, (D(g_2) \oplus D'(g_2))$$

$$= D(g_1) D(g_2) \oplus D'(g_1) D'(g_2) = D(g_1 g_2) \oplus D'(g_1 g_2) \checkmark$$

matrix representation

$$\begin{bmatrix} (D(g))^i{}_j & \\ & (D'(g))^\alpha{}_\beta \end{bmatrix} \begin{bmatrix} v^j \\ v'^\beta \end{bmatrix} \begin{matrix} \}V \\ \}V' \end{matrix}$$

$V \oplus \{0'\}$ and $\{0\} \oplus V'$ are inv. subspaces under $D \oplus D'$

remark: distributive law $D \otimes (D' \oplus D'') =$

$= (D \otimes D') \oplus (D \otimes D'')$ can also be verified

definition: a representation (V, D) is called reducible if there exists a nontrivial invariant subspace V', i.e. $V' \neq \{0\}$, $V' \neq V$ with $D(g)V' \subset V' \ \forall g \in G$

a representation (V, D) is called irreducible if such a subspace does not exist

reducible representation (after a suitable equivalence transformation):

$$D(g) = \begin{bmatrix} D'(g) & A(g) \\ 0 & D''(g) \end{bmatrix} \begin{matrix} \}V' \\ \} \end{matrix}$$

subspace of vectors $\begin{bmatrix} v' \\ 0 \end{bmatrix}$ invariant in general: $A(g) \neq 0$ (representation not fully reducible)

completely reducible (fully reducible) representation

$(A(g) = 0 \ \forall g \in G)$:

$$D(g) = \left[\begin{array}{c|c} D'(g) & 0 \\ \hline 0 & D''(g) \end{array}\right]\begin{array}{l} \}V' \\ \}V'' \end{array} = D'(g) \oplus D''(g)$$

subspace of vector $\left[\begin{array}{c} v' \\ \hline 0'' \end{array}\right]$ $\left[\begin{array}{c} 0' \\ \hline v'' \end{array}\right]$ both invariant

definition: a representation is called completely reducible or fully reducible if it can be decomposed into a direct sum of irreducible representations

well known example from QM: decomposition of a Kronecker product of irreducible representations of SU(2)

→ Clebsch – Gordan decomposition

$$D^{(j_1)} \otimes D^{(j_2)} = D^{(j_1+j_2)} \oplus D^{(j_1+j_2-1)} \oplus \ldots \oplus D^{(|j_1-j_2|)}$$

Schur's lemma

two representations (V, D), (V', D') are called <u>equivalent</u>, $D \cong D'$, if there exists a bijective linear map $S: V \to V'$ such that

$$D(g) = S^{-1} D'(g) S \qquad \forall g \in G$$

$$\Updownarrow$$

$$S D(g) = D'(g) S$$

$$\begin{tikzcd} V \arrow[r, "D(g)"] \arrow[d, "S"'] & V \arrow[d] \\ V' \arrow[r, "D'(g)"'] & V' \end{tikzcd}$$

<u>commuting diagram</u>

now let $S: V \to V'$ be a linear map which is not necesseraly bijective but satisfies $S D(g) = D'(g) S$ (S <u>intertwines</u> the two representations)

$\Rightarrow$ <u>$SV \subset V'$</u> is an <u>invariant subspace</u> under D':

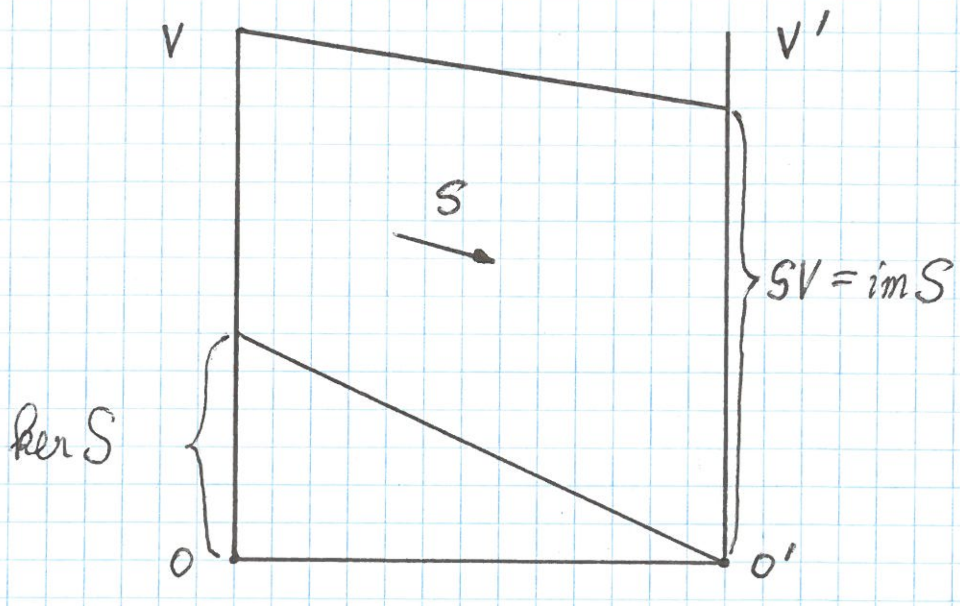
$$D'(g) S V = S D(g) V$$

$\Rightarrow$ <u>$\ker S$</u> $\subset V$ is an <u>invariant subspace under</u> D:

$\ker S = \{ v \in V \mid Sv = 0' \}$

$$S\, \underset{\in \ker S}{D(g)\, v} = D'(g) \underbrace{Sv}_{0'} = 0$$

$\Rightarrow \quad D(g)\, v \in \ker S$



<u>Schur I</u>

(V, D) and (V', D') two <u>irreducible</u> representations

$S : V \to V'$ <u>intertwiner</u> between D and D': $SD(g) = D'(g)S$

$\forall\, g \in G$

$\Rightarrow$ S either vanishes, or it is a bijection (in which case $D \cong D'$)

<u>proof</u>: (V', D') irreducible $\Rightarrow$ $SV = \mathrm{im}\, S = \{0'\}$ <u>or</u> V'

(V, D) irreducible $\Rightarrow$ $\ker S = \{0\}$ <u>or</u> V

$\Rightarrow$ two possibilities:

either $\ker S = V$ and $SV = \{0'\}$

$$\Updownarrow$$

$$S \equiv 0$$

or $\ker S = \{0\}$ and $SV = V'$

$$\Updownarrow$$

$$S \text{ is bijective}$$

Schur II

let (V, D) be a representation and $S \in L(V)$ with $[S, D(g)] = 0 \quad \forall g \in G$

$\Rightarrow$ if S possesses an eigenvalue s, then $S = s\mathbb{1}$ or else the representation is reducible (remark: in a finite-dim. complex vector space, the existence of an eigenvalue of S is always guaranteed)

proof: V_s = eigenspace of S belonging to the eigenvalue s

$v \in V_s$ (i.e. $Sv = sv$) $\Rightarrow$ $SD(g)v = D(g)Sv = sD(g)v$

$\Rightarrow$ $D(g)v \in V_s \quad \forall g \in G$ $\Rightarrow$ V_s is an invariant clearly $V_s \neq \{0\}$ $\Rightarrow$ two possibilities:

$V_s \subset V \quad \Rightarrow \quad D$ is reducible, $\quad S\big|_{V_s} = \alpha \mathbb{1}$

$V_s = V \quad \Rightarrow \quad D$ is irreducible, $\quad S = s\mathbb{1}$

remark: for a finite-dimensional irreducible representation (V, D) in a complex vector space V, $[D(g), S] = 0$ $\forall g \in G$ implies $S = s\mathbb{1}$

important special case: V a unitary vector space and D a unitary representation, i.e. $D(g)^\dagger = D(g)^{-1}$ $\forall g \in G$; W an invariant subspace $\Rightarrow$ $W^\perp$ is also an invariant subspace: $v \in W^\perp$, consider

$$\langle \underset{\in W}{w} \,|\, D(g) v \rangle = \langle D(g)^\dagger w \,|\, v \rangle \underset{\text{unitary repr.}}{=} \langle \underbrace{D(g^{-1}) w}_{\in W} \,|\, v \rangle$$

$$\underset{v \in W^\perp}{=} 0 \quad \Rightarrow \quad D(g) v \in W^\perp \ ✓ \quad \Rightarrow \text{ representation fully reducible}$$

Schur II in this case: $[D(g), S] = 0$

V (finite-dim.) complex vector space $\Rightarrow$ $\exists$ eigenvalue s_1 of S $\Rightarrow$ V_{s_1} is an invariant subspace with $S\big|_{V_s} = s\mathbb{1}$

$\Rightarrow \; V_{s_1}^{\perp}$ is also an invariant subspace

$\uparrow$

repr. D unitary

continue procedure with $S\big|_{V_{s_1}^{\perp}}$

after a certain number of steps $\rightarrow V = \overbrace{V_{s_1} \oplus V_{s_2} \oplus \ldots \oplus V_{s_n}}^{\underline{\text{orthogonal}} \text{ direct sum}}$

$V_{s_i} \perp V_{s_j}$ for $i \neq j \; (s_i \neq s_j)$, $\quad S\big|_{V_{s_j}} = s_j \mathbb{1}$

decomposition into invariant subspaces (V_{s_j} <u>not</u> necessarily irreducible)

<u>conversly:</u> $\quad V = V_1 \oplus V_2 \oplus \ldots \oplus V_N$ orthogonal direct sum of <u>irreducible</u> subspaces

if $[S, D(g)] = 0 \;\Rightarrow\; S v_i = \lambda_i v_i \,,\; v_i \in V_i$

($\lambda_i = \lambda_j$ for $i \neq j$ possible)

<u>example</u> from $SO(3)$: vector operator $\vec{v}$

D <u>unitary</u> <u>irreducible</u> representation

$[D(g), \vec{v}^{\,2}] = 0 \quad \forall g \in G \;\Rightarrow\; \vec{v}^{\,2} = \lambda \mathbb{1}$

Lie algebras of Lie groups

arbitrary Lie group G with some representation (V, D)

group elements g specified by n parameters $\alpha_1, \ldots, \alpha_n$

$g(\alpha_1, \ldots, \alpha_n) =: g(\vec{\alpha})$; convention : $g(\vec{0}) = e$

curve $\tau \to \vec{\alpha}(\tau)$ in parameter space $\to$ curve in the group manifold : $\tau \to g(\tau) = g(\vec{\alpha}(\tau))$

consider curves through the unit element, so that

$g(\vec{\alpha}(0)) = e$

in a representation (V, D) :

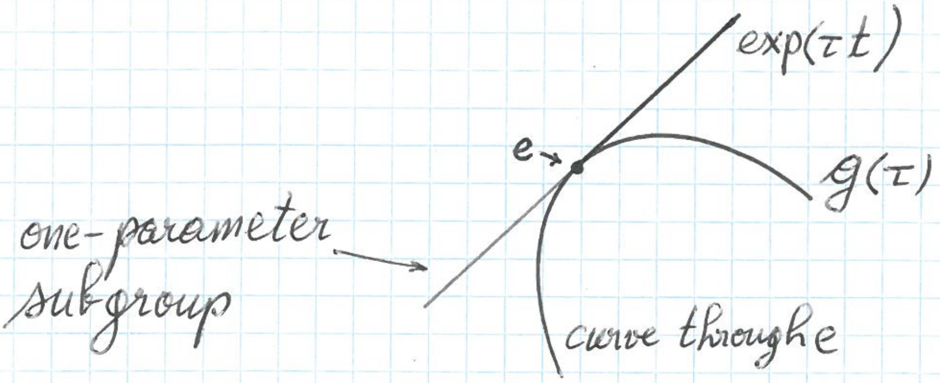
$$D(g(\tau)) = \mathbb{1}_V + \tau \left.\frac{\partial D(g(\tau))}{\partial \tau}\right|_{\tau=0} + \ldots$$

$$\left.\frac{\partial D(g(\tau))}{\partial \tau}\right|_{\tau=0} = \frac{\partial}{\partial \tau} D(g(\vec{\alpha}(\tau))) =$$

$$= \underbrace{\left.\frac{\partial D(g(\vec{\alpha}))}{\partial \alpha_a}\right|_{\vec{\alpha}=0}}_{\substack{=: t_a \\ a = 1, \ldots, n}} \left.\frac{\partial \alpha_a(\tau)}{\partial \tau}\right|_{\tau=0} =: t$$

t = generator of a one-parameter group (in the repr. considered)

$\rightarrow$ finite transformation $\exp(\tau t)$ with multiplication rule $\exp(\tau_1 t)\exp(\tau_2 t) =$
$= \exp((\tau_1+\tau_2)t)$



in <u>SO(3)</u>: one-parameter subgroup = rotation about <u>fixed</u> axis

given <u>two</u> curves $g_1(\tau)$ and $g_2(\tau)$ through e

$\rightarrow$ product $g_1(c_1\tau)\,g_2(c_2\tau)$, $c_{1,2} \in \mathbb{R}$ is also a curve through e

$$D(g_1(c_1\tau)\,g_2(c_2\tau)) = D(g_1(c_1\tau)) \cdot$$
$$\cdot D(g_2(c_2\tau)) \simeq (\mathbb{1}_V + c_1\tau t_1)(\mathbb{1}_V + c_2\tau t_2)$$
$$= \mathbb{1}_V + \tau(c_1 t_1 + c_2 t_2)$$

→ generators form a <u>real</u> vectorspace $\mathcal{L}_V$ spanned by $t_1, \dots, t_n$

in a <u>faithful</u> representation: $\dim \mathcal{L}_V = n$

Lie algebra structure ([,]):

consider the curve $h\, g(\tau)\, h^{-1}$ through e $(h \in G)$

$$D\left(h\, g(\tau)\, h^{-1}\right) = D(h)\, D(g(\tau))\, D(h^{-1})$$

$$\underset{\tau \text{ infinitesimal}}{\overset{\uparrow}{=}} D(h)\, (\mathbb{1}_V + \tau t)\, D(h)^{-1} = \mathbb{1}_V + \tau\, D(h)\, t\, D(h)^{-1},$$

i.e. $t \in \mathcal{L}_V \Rightarrow D(h)\, t\, D(h)^{-1} \in \mathcal{L}_V$

$h \to h(\tau)$, $D(h(\tau)) \simeq \mathbb{1}_V + \tau t'$

$$(\mathbb{1}_V + \tau t')\, t\, (\mathbb{1}_V - \tau t') = t + \tau [t', t] \in \mathcal{L}_V$$

$\Rightarrow [t, t'] \in \mathcal{L}_V$ ✓

<u>adjoint action</u>: $\underset{\in \mathcal{L}_V}{t} \to D(h)\, t\, D(h)^{-1}$ is a bijective linear map of $\mathcal{L}_V$ to itself $\forall\, h \in G$

curve $h(\tau)$ $\quad D(h(\tau)) \simeq \mathbb{1}_V + \tau t'$

$$\Rightarrow D(h(\tau))\, t\, D(h(\tau))^{-1} \simeq t + \tau \underbrace{[t', t]}_{=:\, ad_{t'}(t)} =$$

$$= (\mathbb{1}_{\mathcal{L}_V} + \tau\, ad_{t'})\, t$$

generator in this representation on $\mathcal{L}_V$

assume now that the representation (V, D) is <u>faithful</u> $\Rightarrow$ $Ad_h \underset{\in \mathcal{L}_V}{t} := D(h)\, t\, D(h)^{-1}$

$(\mathcal{L}_V, Ad)$: $\underset{\in G}{h} \to \underset{\in L(\mathcal{L}_V)}{Ad_h}$ <u>adjoint representation</u>

<u>example</u>: $SO(3, \mathbb{R})$ adjoint repr. = defining repr.

(remember $D(S)^{-1}\, t_i\, D(S) = S_{ij} t_j$, $S \in SO(3)$)

$$Ad_{h(\tau)}\, ad_t\, Ad_{h(\tau)}^{-1} \simeq ad_t + \tau\, ad_{[t', t]}$$

$$= ad_t + \tau\, [ad_{t'}, ad_t]$$

$\Rightarrow \quad [ad_{t'}, ad_t] = ad_{[t',t]}$ (follows also from Jacobi identity)

$t_1, \dots, t_n$ generators in faithful repr. (basis of $\mathcal{L}_V$)

$[t_a, t_b] = C_{ab}{}^d t_d \quad \Rightarrow \quad [ad_{t_a}, ad_{t_b}] = C_{ab}{}^d ad_{t_d}$

<u>matrix representation</u> of ad_{t_a} with respect to the basis $\{t_1, \dots, t_n\}$

$$C_{ab}{}^d t_d = [t_a, t_b] = ad_{t_a}(t_b) \underset{\substack{\uparrow \\ \text{def. of matrix repr.}}}{=} t_d \, (ad_{t_a})^d{}_b$$

$$\Rightarrow \quad (ad_{t_a})^d{}_b = C_{ab}{}^d$$

example: $SO(3) \qquad [\Lambda_i, \Lambda_j] = \varepsilon_{ijk} \Lambda_k$

$\Rightarrow \quad (T_a)^d{}_b = \varepsilon_{abd} = -\varepsilon_{adb}$ are representation matrices of the generators ✓

<u>some definitions:</u>

$\mathcal{L}$ a Lie algebra $[\, , \,]$

$\mathcal{L}$ is called <u>abelian</u> if $[X, Y] = 0 \quad \forall X, Y \in \mathcal{L}$

trivial example: one-dimensional subspace of $\mathcal{L}$

<u>subalgebra</u> $\mathcal{A}$ of $\mathcal{L}$: subspace $\mathcal{A}$ invariant under $[\,,\,]$, i.e. $[\mathcal{A},\mathcal{A}] \subseteq \mathcal{A}$

<u>ideal</u> $\mathcal{A}$ of $\mathcal{L}$: subalgebra $\mathcal{A}$ with $[X,\mathcal{A}] \subseteq \mathcal{A}$ $\forall\, X \in \mathcal{L}$ (trivial ideals: 0, $\mathcal{L}$)

<u>simple</u> Lie algebra $\mathcal{L}$: $\mathcal{L}$ possesses <u>no</u> nontrivial ideals and is <u>not</u> abelian (example: $so(3) \cong su(2)$)

<u>semisimple</u> Lie algebra $\mathcal{L}$: $\mathcal{L}$ is a direct sum of simple Lie algebras $\mathcal{L} = \mathcal{A} \oplus \mathcal{B}$, $[\mathcal{A},\mathcal{B}] = 0$

<u>examples</u>:

1) $su(2) \oplus su(2)$ is semisimple $\rightarrow$ Lie group $SU(2) \otimes SU(2)$

2) electroweak sector of the SM

$$\{\underbrace{T_1, T_2, T_3}_{\substack{\text{weak isospin} \\ su(2)}}, \underbrace{Y}_{\substack{\text{weak} \\ \text{hypercharge}}}\} \qquad [T_i, Y] = 0 \text{ for } i = 1,2,3$$

$\langle Y \rangle$ abelian $\rightarrow$ <u>not</u> semisimple

$\rightarrow$ Lie group $SU(2) \times U(1)$

Unitary irreducible representations of SO(3) and SU(2)

1. Every continuous representation of a compact Lie group in a Hilbert space is equivalent to a unitary representation.

2. Every continuous irreducible representation of a compact Lie group in a Hilbert space is finite-dimensional. Every continuous unitary representation of a compact Lie group is a direct orthogonal sum of irreducible (and thus finite-dimensional) sub-representations.

Classification of irreducible representations of SU(2) ($\to$ T2)

we can restrict ourselves to finitedimensional unitary repr.

$\to$ finite-dim. Hilbert space $\mathcal{H}$

$e^{\vec{\alpha}.\vec{t}}$ unitary $\Rightarrow$ t_k anti-hermitean

$\to$ introduce hermitean generators $\quad J_k = i\,t_k$

$$[J_k, J_l] = i\,\varepsilon_{klm} J_m$$

$$[\vec{J}^2, J_k] = 0 \quad \Rightarrow \quad \vec{J}^2 = \lambda\,\mathbb{1} \text{ in an}$$

<u>irreducible</u> repr.

$$J_\pm = J_1 \pm i J_2 = J_\mp^\dagger \quad , \quad J_3$$

$$[J_+, J_-] = 2J_3 \quad , \quad [J_3, J_\pm] = \pm J_\pm$$

$$\vec{J}^2 = J_\pm J_\mp \mp J_3 + J_3^2$$

$\Rightarrow \quad \lambda = j(j+1)$ possible values of j: $0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

$2j+1$-dim. irred. repr. $D^{(j)}$

basis $|j\rangle, |j-1\rangle, \dots, |-j\rangle$

$$J_+ |j\rangle = 0 \quad , \quad J_- |-j\rangle = 0$$

$$J_- |m\rangle = \sqrt{j(j+1) - m(m-1)}\; |m-1\rangle$$

$$J_+ |m\rangle = \sqrt{j(j+1) - m(m+1)}\; |m+1\rangle$$

<u>remark:</u> $e^{-i\vec{\alpha}\vec{J}}$ is a representation of $SU(2)$ for $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ but <u>not</u> of $SO(3)$

consider the special case $\vec{\alpha} = 2\pi \vec{e}_3$:

$$e^{-2\pi i J_3} = \operatorname{diag}\left(e^{-2\pi i j}, \dots, e^{2\pi i j}\right) = \begin{cases} \mathbb{1}_{2j+1} & 2j \text{ even} \\ -\mathbb{1}_{2j+1} & 2j \text{ odd} \end{cases}$$

we also know from T2: <u>Clebsch-Gordan decomposition</u> of a product of two irreps of $SU(2)$:

$$D^{(j_1)} \otimes D^{(j_2)} = D^{(j_1+j_2)} \oplus D^{(j_1+j_2-1)} \oplus \dots \oplus D^{(|j_1-j_2|)}$$

<u>example</u>: tensor of rank 2 $\quad T_{ij} \xrightarrow{R \in SO(3)} R_{ik} R_{jl} T_{kl}$

~~invariant~~ irreducible subspaces? $\quad D^{(1)} \otimes D^{(1)} = \underbrace{D^{(2)}}_{5\text{-dim.}} \oplus \underbrace{D^{(1)}}_{3\text{-dim.}} \oplus \underbrace{D^{(0)}}_{1\text{-dim.}}$

$$\operatorname{Tr} T = T_{ii} \xrightarrow{SO(3)} \underbrace{R_{ik} R_{il}}_{(R^T R)_{kl} = \delta_{kl}} T_{kl} = \operatorname{Tr} T \quad \text{inv.} \rightarrow D^{(0)}$$

antisymmetric tensor $\quad T_{ij} = -T_{ji} \quad$ also inv. under $SO(3)$

$$T_{ij} \rightarrow R_{ik} R_{jl} T_{kl} = -R_{ik} R_{jl} T_{lk} = -R_{jl} R_{ik} T_{lk}$$

$\rightarrow$ 3-dim. irrep. $D^{(3)}$

traceless, symmetric tensor $\quad T_{ij} = T_{ji}, \ \operatorname{Tr} T = 0$

$6 - 1 = 5$-dim. irrep.

decomposition of a general tensor of rank 2 into irreducible components:

$$T_{ij} = \frac{1}{3} \mathrm{Tr}\, T \, \delta_{ij}$$
$$+ \frac{1}{2} \left(T_{ij} - T_{ji} \right)$$
$$+ \frac{1}{2} \left(T_{ij} + T_{ji} \right) - \frac{1}{3} \mathrm{Tr}\, T \, \delta_{ij}$$

we define the projection operators

$$(P_0)_{ij\,kl} = \frac{1}{3} \delta_{ij} \delta_{kl}$$ subspace of multiples of δ_{ij}

$$(P_A)_{ij\,kl} = \frac{1}{2} \left(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \right)$$ antisymm.

$$(P_S)_{ij\,kl} = \frac{1}{2} \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)$$ symm.

$P_0 \rightarrow D^{(0)}$

$P_A \rightarrow D^{(3)}$

$P_S - P_0 \rightarrow D^{(5)}$

$$P_A + P_S = \underbrace{\mathbb{1} \otimes \mathbb{1}}_{\text{unit operator on tensor space}}$$

$$R(\vec{\alpha}) \otimes R(\vec{\alpha}) \simeq (\mathbb{1} + \vec{\alpha}.\vec{\Lambda}) \otimes (\mathbb{1} + \vec{\alpha}.\vec{\Lambda})$$

$$= \mathbb{1} \otimes \mathbb{1} + \vec{\alpha}.\underbrace{(\mathbb{1} \otimes \vec{\Lambda} + \vec{\Lambda} \otimes \mathbb{1})}_{\vec{t} = -i\vec{J}}$$

$$\vec{t}^{\,2} = (\mathbb{1} \otimes \Lambda_m + \Lambda_m \otimes \mathbb{1})(\mathbb{1} \otimes \Lambda_m + \Lambda_m \otimes \mathbb{1})$$

$$= \mathbb{1} \otimes \underbrace{\vec{\Lambda}^2}_{-1(1+1)\mathbb{1}} + \vec{\Lambda}^2 \otimes \mathbb{1} + 2\Lambda_m \otimes \Lambda_m$$

$$= -4\, \mathbb{1} \otimes \mathbb{1} + 2\Lambda_m \otimes \Lambda_m$$

remember: $(\Lambda_m)_{ik} = -\varepsilon_{mik}$

$$\Rightarrow (\Lambda_m \otimes \Lambda_m)_{ij\,kl} = \varepsilon_{mik}\,\varepsilon_{mjl} =$$

$$= \delta_{ij}\,\delta_{kl} - \delta_{il}\,\delta_{kj}$$

$$= (3P_0 + P_A - P_S)_{ij\,kl}$$

$$\Lambda_m \otimes \Lambda_m = 3P_0 + P_A - P_S$$

$$\Rightarrow \vec{J}^2 = -\vec{t}^{\,2} = +4\,\underbrace{\mathbb{1} \otimes \mathbb{1}}_{P_S + P_A} - 2(3P_0 + P_A - P_S)$$

$$\vec{J}^2 = 2P_A + 6(P_S - P_0) \quad \text{spectral representation}$$

$$= \underset{\substack{\uparrow \\ j=0}}{0 \cdot P_0} + \underset{\substack{\uparrow \\ j=1}}{1(1+1)} \cdot P_A + \underset{\substack{\uparrow \\ j=2}}{2(2+1)} \cdot (P_S - P_0)$$

$$\mathbb{1} \otimes \mathbb{1} = P_0 + P_A + (P_S - P_0), \quad P_0 P_A = 0, \; P_0(P_S - P_0) = 0$$

$$P_A(P_S - P_0) = 0$$

$$\underline{3} \times \underline{3} = \underline{1} + \underline{3} + \underline{5}$$

(pairwise orthogonal)

SU(2), spinors

$j = \frac{1}{2}$ 2-dim. repr. $\vec{J} = \frac{\vec{\sigma}}{2}$

<u>cannot</u> be obtained by reducing tensor repr. (as all repr. with half integer weights)

$$U(\vec{\alpha}) := e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} = \mathbb{1}_2 \cos\frac{\alpha}{2} - i\,\vec{n}\cdot\vec{\sigma}\sin\frac{\alpha}{2}, \quad \vec{\alpha} = \alpha\vec{n}, \; \vec{n}^2 = 1$$

$U(\vec{\alpha})$ is clearly unitary and $\det U(\vec{\alpha}) = e^{\mathrm{Tr}\left(-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}\right)} = 1$

(unimodular) $\Rightarrow U(\vec{\alpha}) \in SU(2)$ (all elements of SU(2) can be written in this form)

we already know from T2:

$U(\vec{\alpha})$ is <u>not</u> a representation of ~~[illegible]~~ SO(3) : $U(2\pi\vec{n}) = -\mathbb{1}_2$

→ set of matrices $U(\vec{\alpha})$ forms a group only if the $SO(3)$-domain $0 \leq |\vec{\alpha}| \leq \pi$ is extended to $0 \leq |\vec{\alpha}| \leq 2\pi$ ~~is extended to $0 \leq |\vec{\alpha}| \leq 2\pi$~~

→ set of rotations gets <u>doubly covered</u>:

$U(-(2\pi - \alpha)\vec{n}) = -U(\alpha\vec{n})$

in $SU(2)$, <u>no</u> identification $R(\pi\vec{n}) = R(-\pi\vec{n})$:

$U(-\pi\vec{n}) = -U(\pi\vec{n})$

$R(\vec{\alpha}) \to \pm U(\vec{\alpha})$ two-valued "representation" of $SO(3)$

(analogy to complex analysis: $w = \sqrt{z}$ → proper function only on a Riemann surface covering the complex plane <u>twice</u> → function on a <u>covering space</u>

<u>groups:</u> representation of a <u>covering group</u>)

<u>SU(2) as a manifold:</u>

let $\vec{\alpha}$ range over $0 \leq |\vec{\alpha}| \leq 2\pi$ → $U(\vec{\alpha})$ ranges over the whole group $SU(2)$

write $U \in SU(2)$ in the form

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad , \quad |a|^2 + |b|^2 = 1 \quad , \quad a, b \in \mathbb{C}$$

$$(\mathrm{Re}\,a)^2 + (\mathrm{Im}\,a)^2 + (\mathrm{Re}\,b)^2 + (\mathrm{Im}\,b)^2 = 1$$

$\Rightarrow$ $SU(2)$ = 3-sphere S_3 (unit sphere in $\mathbb{R}^4$)

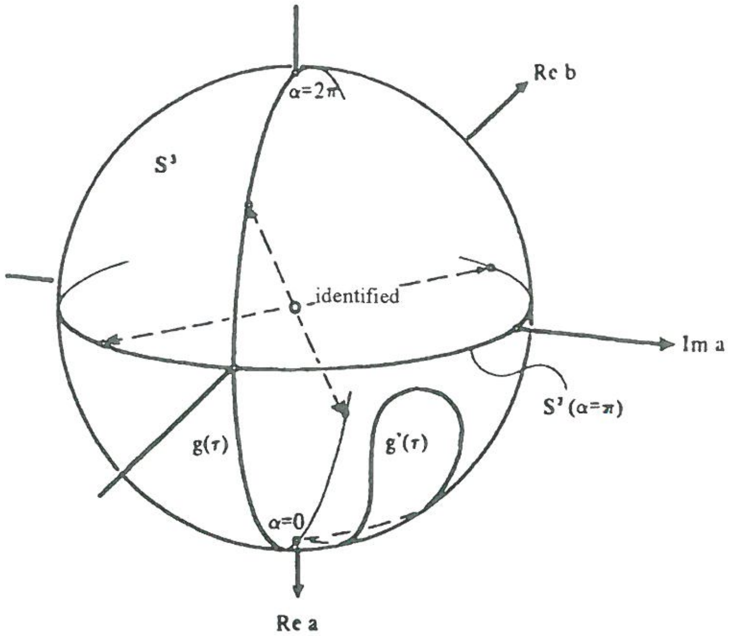


Fig. 7.4. SU(2)= S_3 and SO(3). The coordinate Im b has been omitted

$SU(2)$ is <u>simply</u> connected

$SO(3)$ is <u>doubly</u> connected

<u>relation between SO(3) and SU(2)</u>

$$\vec{x} \in \mathbb{R}^3 \longrightarrow X = \vec{x}.\vec{\sigma} = \begin{pmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{pmatrix}$$

$$X^\dagger = X, \quad \mathrm{Tr}\, X = 0$$

X given $\rightarrow$ $\vec{x} = \frac{1}{2} \operatorname{Tr}(X\vec{\sigma})$

$X^2 = \vec{x}^{\,2} \mathbb{1}_2$, $\det X = -\vec{x}^{\,2}$

$U \in SU(2)$

$X' = UXU^\dagger$ also hermitean and traceless

mapping $X \rightarrow UXU^\dagger = X'$ defines a linear transformation $\vec{x} \rightarrow R\vec{x}$

$\vec{x} = \frac{1}{2} \operatorname{Tr}(X\vec{\sigma}) \longrightarrow \vec{x}\,' = \frac{1}{2} \operatorname{Tr}(X'\vec{\sigma}) = \frac{1}{2} \operatorname{Tr}(UXU^\dagger\vec{\sigma})$

$\vec{x}\,'^2 \mathbb{1}_2 = X'^2 = UXU^\dagger UXU^\dagger = UX^2U^\dagger = \vec{x}^{\,2}\mathbb{1}_2$

alternatively:

$-\vec{x}\,'^2 = \det X' = \det UXU^\dagger = \det X = -\vec{x}^{\,2}$

$\Rightarrow$ transformation R is <u>orthogonal</u>

$U = \mathbb{1}_2 \rightarrow R = \mathbb{1}_3$, $SU(2)$ connected $\Rightarrow R \in SO(3)$

$x_i' = R_{ij}\, x_j = \frac{1}{2} \operatorname{Tr}(UXU^\dagger \sigma_i) = \frac{1}{2} \operatorname{Tr}(\sigma_i U \sigma_j x_j U^\dagger)$

$\Rightarrow R_{ij} = \frac{1}{2} \operatorname{Tr}(\sigma_i U \sigma_j U^\dagger)$

<u>remark:</u> $U(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}$

$$\frac{1}{2}\,\mathrm{Tr}\left(\sigma_i\, U(\vec{\alpha})\,\sigma_j\, U(\vec{\alpha})^\dagger\right) = \left(R(\vec{\alpha})\right)_{ij} \qquad \text{(ex.)}$$

<u>remark:</u> the mapping $\phi: SU(2) \to SO(3)$

$$U \to R_U$$

with $(R_U)_{ij} = \frac{1}{2}\,\mathrm{Tr}\left(\sigma_i\, U\, \sigma_j\, U^\dagger\right)$ is <u>surjective</u>,

i.e. $\phi(SU(2)) = SO(3)$

ϕ is a <u>group homomorphism</u>, which means

$\phi(U_1 U_2) = \phi(U_1)\,\phi(U_2)$ (ex.)

<u>$\ker\phi$</u> $:= \{U \in SU(2) \mid \phi(U) = \mathbb{1}_3 \in SO(3)\} =$

$= \{\mathbb{1}_2, -\mathbb{1}_2\}$

<u>proof:</u> assume $\phi(U) = \mathbb{1}_3$, i.e. $U\vec{x}\cdot\vec{\sigma}\,U^\dagger = \vec{x}\cdot\vec{\sigma} \;\; \forall \vec{x} \in \mathbb{R}^3$

$\Rightarrow [U, \vec{x}\cdot\vec{\sigma}] = 0 \;\; \forall \vec{x} \in \mathbb{R}^3 \Rightarrow U = c\,\mathbb{1}_2 \overset{U \in SU(2)}{\Rightarrow} U = \pm\mathbb{1}_2$

<u>proposition:</u> $\forall R(\vec{\alpha}) \in SO(3)$ $\exists!$ <u>two</u> elements in $SU(2)$

with $\phi(U) = R(\vec{\alpha})$, namely $U = \pm U(\vec{\alpha})$

<u>proof:</u> $U\,\vec{x}\cdot\vec{\sigma}\,U^\dagger = \underbrace{\left(R(\vec{\alpha})\vec{x}\right)}_{=:\vec{x}'}\cdot\vec{\sigma}$

ansatz: $U = U_1\, U(\vec{\alpha})$

$\Rightarrow \quad U_1\, \vec{x}\,' \cdot \vec{\sigma}\, U_1^\dagger = \vec{x}\,' \cdot \vec{\sigma} \quad \forall \vec{x}\,' \Rightarrow U_1 = \pm \mathbb{1}_2$

$$\boxed{\begin{array}{l} \{U(\vec{\alpha}), -U(\vec{\alpha})\} \longleftrightarrow R(\vec{\alpha}) \\[2ex] SU(2)/\{\mathbb{1}_2, -\mathbb{1}_2\} = SU(2)/\mathbb{Z}_2 \cong SO(3) \end{array}}$$

factor (quotient) group "SU(2) mod(ulo) $\mathbb{Z}_2$"

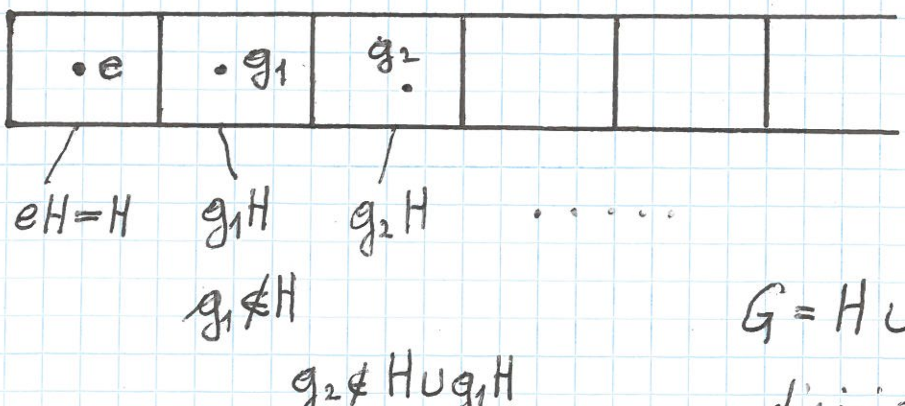
some explanations concerning the notation:

group G, subgroup H

$gH := \{gh \mid h \in H\}$ left coset of H $(g \in G)$

Hg = right coset of H

decomposition of G into (left) cosets with respect to the subgroup H:



$g_i H$ = coset

$G = H \cup g_1 H \cup g_2 H \cup \ldots$

disjoint union

$G/H = \{H, g_1H, g_2H, \dots\}$ set of all (left) cosets

"G mod H"

<u>if</u> $gH = Hg \ \forall g \in G$ (left cosets = right cosets)

then H is called <u>normal</u> or <u>invariant</u> subgroup

in this case G/H is also a <u>group</u> (<u>factor group</u> or <u>quotient group</u>) with the group multiplication rule $(g_1H)(g_2H) = (g_1g_2)H$ (the definition is <u>independent</u> of the choice of the representative g_i)

<u>unit</u> element: $eH = H$

element <u>inverse</u> to gH: $g^{-1}H$

<u>in the case of SU(2):</u>

$G = SU(2)$, $H = \{\mathbb{1}_2, -\mathbb{1}_2\}$ is an invariant subgroup of $SU(2)$, because

$$\underset{\in G}{U}\, H = U\{\mathbb{1}, -\mathbb{1}\} = \{U, -U\} = \{\mathbb{1}, -\mathbb{1}\}U$$

$$G/H = SU(2)/\underbrace{\{\mathbb{1}, -\mathbb{1}\}}_{\cong Z_2} = \{\{U, -U\} \mid U \in SU(2)\} = SO(3)$$

remark: $U \in SU(2)$ can be expressed in terms of $R \in SO(3)$ by the relation

$$U = \pm \frac{\mathbb{1} + R_{ij}\,\sigma_i \sigma_j}{2\sqrt{1+\mathrm{Tr}\,R}}$$

spinors

= vectors of the two-dimensional representation space

possible basis: $\chi_+, \chi_- \in \mathbb{C}^2$, $\sigma_3 \chi_\pm = \pm \chi_\pm$

$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$D^{(1/2)} \otimes D^{(1/2)} = D^{(1)} \oplus D^{(0)}$$

one-dimensional invariant subspace corresponding to trivial representation $D^{(0)}$ (singlet) spanned by

$$(\chi_+ \otimes \chi_- - \chi_- \otimes \chi_+)/\sqrt{2}$$

(antisymmetric part of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$)

<u>three-dimensional</u> invariant subspace corresponding to repr. $D^{(1)}$ (<u>triplet</u>) spanned by

$$\chi_+ \otimes \chi_+$$

$$(\chi_+ \otimes \chi_- + \chi_- \otimes \chi_+)/\sqrt{2}$$

$$\chi_- \otimes \chi_-$$

(<u>symmetric</u> part of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$)

$$\underbrace{D^{(1/2)} \otimes D^{(1/2)} \otimes \ldots \otimes D^{(1/2)}}_{p \text{ terms}} = D^{(\frac{p}{2})} \oplus \ldots$$

(<u>example</u>: $D^{(1/2)} \otimes D^{(1/2)} \otimes D^{(1/2)} = D^{(3/2)} \oplus D^{(1/2)} \oplus D^{(1/2)}$)

invariant subspace corresponding to $D^{(\frac{p}{2})}$ = <u>totally symmetric</u> part of the p-fold tensor product of $\mathbb{C}^2$:

$$\mathbb{C}^2 \otimes_s \mathbb{C}^2 \otimes_s \ldots \otimes_s \mathbb{C}^2$$

spanned by

$$\left.\begin{array}{c}
\chi_+ \otimes \chi_+ \otimes \dots \otimes \chi_+ \\[2ex]
(\chi_- \otimes \chi_+ \otimes \dots \otimes \chi_+ + \chi_+ \otimes \chi_- \otimes \dots \otimes \chi_+ + \dots \\
+ \chi_+ \otimes \chi_+ \otimes \dots \otimes \chi_-)/\sqrt{p} \\
\vdots \\
\chi_- \otimes \chi_- \otimes \dots \otimes \chi_-
\end{array}\right\} \begin{array}{l} p+1 \\ \text{terms} \end{array}$$

indeed: $2\left(\frac{p}{2}\right)+1 = p+1$ ✓

Representations on function spaces

scalar field $\Phi'(\vec{x}') = \Phi(\vec{x})$, $\vec{x}' = R\vec{x}$ pass. transf.

or

$\Phi'(\vec{x}) = \Phi(R^{-1}\vec{x})$ active transf.

representation space $= L^2(\mathbb{R}^3) =$ Hilbert space of square-integrable functions in $\mathbb{R}^3$ with the scalar product

$$\langle \Phi | \Psi \rangle = \int_{\mathbb{R}^3} d^3x \; \Phi^*(\vec{x})\, \Psi(\vec{x})$$

representation $D(R)$ on $L^2(\mathbb{R}^3)$ $\quad (R \in SO(3))$

defined by

$$\big(D(R)\,\psi\big)(\vec{x}) = \psi(R^{-1}\vec{x}) \,,$$

$D(R)$ is unitary:

$$\langle D(R)\Phi \,|\, D(R)\Psi \rangle = \langle \Phi | \Psi \rangle$$

<u>infinitesimal transformation</u>:

$$\big(D(R(\vec{\varepsilon}))\,\psi\big)(\vec{x}) = \Psi\big(R(\vec{\varepsilon})^{-1}\vec{x}\big) =$$

$$= \psi(\vec{x} - \vec{\varepsilon} \times \vec{x}) = \psi(\vec{x}) - (\vec{\varepsilon} \times \vec{x}) . \vec{\nabla}\psi(\vec{x})$$

$$= \psi(\vec{x}) - \vec{\varepsilon} . (\vec{x} \times \vec{\nabla})\,\psi(\vec{x})$$

$$= \psi(\vec{x}) - i\,\vec{\varepsilon} . (\vec{L}\,\psi)(\vec{x})$$

with $\vec{L} = \vec{x} \times \frac{1}{i} \vec{\nabla}$

generator in this function space = differential operator = orbital angular momentum operator

$\vec{L}$ does not involve $r = |\vec{x}|$, but only θ, φ

$\rightarrow$ consider $L^2(S_2)$ with functions $\psi(\theta, \varphi)$ and the scalar product

$$\langle \phi | \psi \rangle_{S_2} = \int_{\theta=0}^{\pi} d\theta \sin\theta \int_{0}^{2\pi} d\varphi \; \phi^*(\theta, \varphi) \, \psi(\theta, \varphi)$$

$$L_3 = \frac{1}{i} \frac{\partial}{\partial \varphi} \quad , \quad L_\pm = e^{\pm i\varphi} \left(\pm \frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \varphi} \right)$$

$$\Rightarrow \vec{L}^2 = - \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right)$$

decomposition of $L^2(S_2)$ into irreducible subspaces:

$$L^2(S_2) = \bigoplus_{l=0}^{\infty} H_l(S_2)$$

$H_\ell(S_2)$ spanned by the spherical harmonics

$Y_{\ell m}(\Theta,\varphi) \quad m=\ell,\ell-1,\dots,-\ell$

they can be obtained from

$$Y_{\ell\ell}(\Theta,\varphi) = \frac{(-1)^\ell}{\sqrt{4\pi}} \sqrt{\frac{1\cdot3\cdot5\cdots(2\ell+1)}{2\cdot4\cdot6\cdots2\ell}}\,(\sin\Theta)^\ell\, e^{i\ell\varphi}$$

$$Y_{\ell,m-1}(\Theta,\varphi) = \frac{L_-}{\sqrt{\ell(\ell+1)-m(m-1)}}\, Y_{\ell m}(\Theta,\varphi)$$

$$m=\ell,\ell-1,\dots,-\ell+1$$

Description of particles with spin

vector, tensor, spinor fields

vector field: $\quad \vec{v}\,'(\vec{x}) = R\,\vec{v}\,(R^{-1}\vec{x})$

in general: (finite-dim.) repr. (V,D) of $SU(2)$

$\rightarrow$ field of type D : $v(\vec{x}) \in V$

$$v'(\vec{x}) = D(g)\, v\,(R(g)^{-1}\vec{x}),$$

$$g \in SU(2)$$

<u>example</u>: $D =$ irred. repr. $D^{(s)}$ (<u>spin s</u>)

scalar product $\langle v|w\rangle = \int_{\mathbb{R}^3} d^3x \sum_{\sigma=-s}^{+s} v_\sigma(\vec{x})^* w_\sigma(\vec{x})$

infinitesimal transformation:

$$v'(\vec{x}) = (\mathbb{1} - i\vec{\varepsilon}.\vec{S} - i\vec{\varepsilon}.\vec{L})\, v(\vec{x})$$

spin orbital angular momentum

$[L_m, S_n] = 0 \quad \forall\, m,n$

$[L_m, L_n] = i\,\varepsilon_{mnr} L_r \quad , \quad [S_m, S_n] = i\,\varepsilon_{mnr} S_r$

total angular momentum $\vec{J} = \vec{L} + \vec{S}$

mathematical notation:

$\vec{J} = \mathbb{1}_V \otimes \vec{L} + \vec{S} \otimes \mathbb{1}_{L^2(\mathbb{R}^3)}$ acting on $V \otimes L^2(\mathbb{R}^3)$

$$e^{-i\vec{\alpha}.J} = \underbrace{e^{-i\vec{\alpha}.\vec{S}}}_{D(g(\vec{\alpha}))} \otimes \underbrace{e^{-i\vec{\alpha}.\vec{L}}}_{e^{-i\vec{\alpha}.(\vec{x}\times\frac{1}{i}\vec{\nabla})}}$$

for $D = D^{(s)}$ we have :

$$D^{(s)} \otimes \bigoplus_{\ell=0}^{\infty} D^{(\ell)} = \bigoplus_{\ell=0}^{\infty} \left(D^{(s+\ell)} \oplus \dots \oplus D^{(|\ell-s|)} \right)$$

$$
\begin{aligned}
= \; & D^{(s)} \\
& \oplus \left(D^{(s+1)} \oplus D^{(s)} \oplus \dots \oplus D^{(|1-s|)} \right) \\
& \oplus \left(D^{(s+2)} \oplus D^{(s+1)} \oplus D^{(s)} \oplus \dots \oplus D^{(|2-s|)} \right) \\
& \oplus \left(D^{(s+3)} \oplus D^{(s+2)} \oplus D^{(s+1)} \oplus D^{(s)} \oplus \dots \oplus D^{(|3-s|)} \right) \\
& \oplus \dots
\end{aligned}
$$

degenerate !

$$\vec{J} = \vec{L} + \vec{S}$$

$\vec{J}^{\,2}$, J_3, $\vec{L}^{\,2}$, $\vec{S}^{\,2}$ are pairwise commuting operators

(note that $\vec{L}^{\,2}$, $\vec{S}^{\,2}$ are squares of vector operators)

choice $|j, m, \ell, s\rangle$ removes degeneracy of j (states from which of the products $D^{(s)} \otimes D^{(\ell)}$ the $D^{(j)}$ under consideration stems)

$$\vec{J}^{\,2} |j, m, \ell, s\rangle = j(j+1) |j, m, \ell, s\rangle$$

$$J_3 |j,m,\ell,s\rangle = m\,|j,m,\ell,s\rangle$$

$$\vec{L}^2 |j,m,\ell,s\rangle = \ell(\ell+1)\,|j,m,\ell,s\rangle$$

$$\vec{S}^2 |j,m,\ell,s\rangle = s(s+1)\,|j,m,\ell,s\rangle$$

in fact superfluous, as <u>only</u> $D^{(s)}$ present

on the other hand: $D^{(s)} \otimes D^{(\ell)}$

$$|s,\sigma\rangle \otimes |\ell,\lambda\rangle \equiv |s,\sigma;\ell,\lambda\rangle$$

$$v_\sigma \otimes Y_{\ell\lambda}$$

$$|j,m,\ell,s\rangle = \sum_{\lambda,\sigma} |s,\sigma;\ell,\lambda\rangle\, c_{\sigma\lambda} \qquad (j = \ell+s, \dots, |\ell-s|)$$

$$c_{\sigma\lambda} = \langle s,\sigma;\ell,\lambda | j,m,\ell,s\rangle$$

<u>Clebsch-Gordan coefficients</u>

$$J_3 = L_3 + S_3 \Rightarrow m = \lambda + \sigma$$

$$|j,m,\ell,s\rangle = \sum_\lambda \underbrace{|s,m-\lambda;\ell,\lambda\rangle}_{v_{m-\lambda}\otimes Y_{\ell\lambda}} \langle s,m-\lambda;\ell,\lambda | j,m,\ell,s\rangle$$

<u>example</u>: vector field $(s=1)$

$$D^{(1)} \otimes D^{(\ell)} = D^{(\ell+1)} \oplus D^{(\ell)} \oplus D^{(\ell-1)} \qquad \ell \neq 0$$

$$= D^{(1)} \qquad \ell = 0$$

$|j, m, \ell, 1\rangle =$ <u>vector spherical harmonics</u> $= \vec{Y}_{j\ell m}(\Theta, \varphi)$

$j = \ell+1, \ell, \ell-1 \quad (\ell \neq 0)$

$= 1 \quad (\ell = 0)$

$$\vec{Y}_{j\ell m}(\Theta,\varphi) = \sum_{\lambda} \vec{e}_{\underbrace{m-\lambda}_{\sigma}} Y_{\ell\lambda}(\Theta,\varphi) \langle 1, m-\lambda; \ell, \lambda | j, m, \ell, 1\rangle$$

$\vec{e}_\sigma \quad (\sigma = -1, 0, +1)$ are the <u>canonical</u> basis vectors of $D^{(1)}$:

$$\vec{e}_{\pm} = -(\pm \vec{e}_1 + i\vec{e}_2)/\sqrt{2}, \quad \vec{e}_0 = \vec{e}_3$$

remember: $\vec{J} = i\vec{\Lambda}$

$$\Lambda_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

vector spherical harmonics form a complete orthonormal system for vector fields $\vec{v}(\Theta,\varphi)$ defined on the unit sphere i.e., every $\vec{v}(\theta,\varphi)$ has a unique expansion

$$\vec{v}(\theta,\varphi) = \sum_{j=0}^{\infty} \vec{v}_j(\theta,\varphi)$$

$$\vec{v}_j(\theta,\varphi) = \vec{v}_{01}(\theta,\varphi) + \sum_{\substack{l=j-1 \\ j\geq 1}}^{j+1} \vec{v}_{jl}(\theta,\varphi)$$

$$\vec{v}_{jl}(\theta,\varphi) = \sum_{m=-j}^{j} c_{jlm} \vec{Y}_{jlm}(\theta,\varphi)$$

$$c_{jlm} = \int_{S_2} d\Omega \; \vec{Y}^{*}_{jlm}(\theta,\varphi)\,\vec{v}(\theta,\varphi)$$