1. Symmetries in classical mechanics and classical field theory

Principle of least action

$$L(\underbrace{q_1, \dots, q_f}_{\text{generalized coordinates}}, \underbrace{\dot{q}_1, \dots, \dot{q}_f}_{\text{generalized velocities}}, t) \equiv L(q, \dot{q}, t)$$

f ... number of <u>dof</u>

$$S = \int_{t_1}^{t_2} dt \, L(q, \dot{q}, t)$$ <u>action</u> (integral)

$\delta S = 0$ with $\delta q(t_1) = \delta q(t_2) = 0$ $\Rightarrow$ $$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$ <u>equations of motion</u>

$i = 1, \dots, f$ (Euler-Lagrange)

<u>remark</u>: $L \to L' = L + \frac{d}{dt} f(q, t)$ does <u>not</u> change EOM

reason: $$\delta \int_{t_1}^{t_2} dt \, \frac{d}{dt} f(q, t) = \delta \left[f(q(t_2), t_2) - f(q(t_1), t_1) \right] = 0$$

L and L' are physically <u>equivalent</u>

closed subsystems A, B
no interaction with each other

$$L \longrightarrow L_A + L_B \qquad \underline{\text{additivity}}$$

→ no <u>individual</u> rescaling of Lagrange functions of closed subsystems (only rescaling for all subsystems with the <u>same</u> factor)

<u>point particle in inertial frame</u> $\quad \vec{x}, \; \dot{\vec{x}} \equiv \vec{v}$

<u>inertial frame</u> <u>characterized</u> by the following <u>symmetry</u> properties:

<u>time homogeneous</u>

<u>space homogeneous and isotropic</u>

$$\Rightarrow \; L = L(\vec{v}^{\,2}) \qquad \text{EOM} \quad \frac{d}{dt}\frac{\partial L(\vec{v}^{\,2})}{\partial v_i} = 0 \Rightarrow v_i = \text{const.}$$

<u>nonrelativistic mechanics</u> → coordinates of inertial frames related by <u>Galilei transformations</u>

<u>universal</u> time $\quad t' = t \qquad$ (special case: no shift)

$\vec{x}' = \vec{x} + \vec{u}\, t \qquad$ (special case: pure velocity transformation)

Galileian principle of relativity

→ L and L′ equivalent (equal up to total time derivative)

$$L' = L(\vec{v}'^2) = L((\vec{v}+\vec{u})^2) = L(\vec{v}^2 + 2\vec{v}\cdot\vec{u} + \vec{u}^2)$$

consider infinitesimal $\vec{u}$:

$$\rightarrow \quad L(\vec{v}'^2) = L(\vec{v}^2) + \frac{\partial L}{\partial \vec{v}^2}\, 2\vec{u}\cdot\underbrace{\vec{v}}_{\equiv \frac{d\vec{x}}{dt}} + \mathcal{O}(\vec{u}^2)$$

already total time derivative

$$\Rightarrow \quad \frac{\partial L}{\partial \vec{v}^2} = \text{const.} \quad \Rightarrow \quad L(\vec{v}^2) = \frac{m}{2}\vec{v}^2$$

constant m is called "mass"

EOM $\quad \frac{d}{dt}\underbrace{\frac{\partial L}{\partial v_i}}_{p_i = m v_i} = \underbrace{\frac{\partial L}{\partial x_i}}_{0} \qquad i = 1,2,3$

$p_i = m v_i$ momentum $\qquad \vec{p} = m\vec{v}$

$\dot{p}_i = 0 \;\Rightarrow\; p_i = \text{const.} \;\Rightarrow\; v_i = \text{const.}$ (Newton I)

finite Galilei transformation:

$$L' = \frac{m}{2}\vec{v}'^2 = \frac{m}{2}(\vec{v}+\vec{u})^2 = \frac{m}{2}\vec{v}^2 + \underbrace{m\,\vec{v}\cdot\vec{u} + \frac{m}{2}\vec{u}^2}_{\frac{d}{dt}\left(m\,\vec{x}\cdot\vec{u} + \frac{m}{2}\vec{u}^2 t\right)}$$

system of noninteracting particles $\quad L = \sum_a \frac{m_a}{2}\vec{v}_a^2$ (additivity)

<u>general case</u> $L = L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$ <u>generalized momenta</u> $(i = 1, \dots, f)$

<u>if</u> L <u>independent</u> of q_i (<u>symmetry!</u>) $\Rightarrow \frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} = 0$

$\Rightarrow$ $p_i = \text{const.}$

<u>example</u>: Lagrange function of free particle (in inertial frame) written in <u>cylindrical coordinates</u>

$$L = \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\varphi}^2 + \dot{z}^2 \right)$$

$$\frac{\partial L}{\partial \varphi} = 0 \quad \Rightarrow \quad p_\varphi \equiv \underbrace{L_z}_{\text{component of angular momentum in } z\text{-direction}} = m r^2 \dot{\varphi} = \text{const.}$$

<u>rotational symmetry</u>

<u>Time translation symmetry</u>

<u>if</u> $\frac{\partial L}{\partial t} = 0$ (no <u>explicit</u> time-dependence) $L = L(q, \dot{q})$

$$\Rightarrow \frac{d}{dt} L(q, \dot{q}) = \sum_{i=1}^{f} \left(\underbrace{\frac{\partial L}{\partial q_i}}_{\text{EOM } \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) = \frac{d}{dt} \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i$$

$$\Rightarrow \quad \frac{d}{dt}\left(\sum_{i=1}^{f} \underbrace{\frac{\partial L}{\partial \dot q_i}}_{p_i} \dot q_i - L\right) = 0$$

$$\Rightarrow \quad \underset{\substack{\uparrow \\ \text{called "energy"}}}{E} := \sum_{i=1}^{f} \frac{\partial L}{\partial \dot q_i}\dot q_i - L = \text{const.}$$

example: point particle in inertial frame

$$E = m\vec v\cdot\vec v - \frac{m\vec v^{\,2}}{2} = \frac{m\vec v^{\,2}}{2}$$ nonrelativistic kinetic energy

Galilean invariant system of two point particles (closed system)

$$L = \frac{m_1\vec v_1^{\,2}}{2} + \frac{m_2\vec v_2^{\,2}}{2} - U(|\vec x_1 - \vec x_2|) \qquad \text{EOM} \quad m_a\dot{\vec v}_a = \underbrace{-\frac{\partial U}{\partial \vec x_a}}_{\vec F_a}, \quad a = 1,2$$

$$0 = \frac{\partial L}{\partial t} \Rightarrow E = \frac{m_1\vec v_1^{\,2}}{2} + \frac{m_2\vec v_2^{\,2}}{2} + U(|\vec x_1 - \vec x_2|)$$ total energy conserved

space homogeneous

L invariant under $\vec x_a \to \vec x_a + \vec\varepsilon$, $a = 1,2$

$$\delta L = \sum_a \frac{\partial L}{\partial \vec x_a}\cdot\vec\varepsilon = \vec\varepsilon\cdot\sum_a \frac{\partial L}{\partial \vec x_a} = 0 \overset{\vec\varepsilon \text{ arbitrary}}{\Rightarrow} \sum_a \overbrace{\frac{\partial L}{\partial \vec x_a}}^{\vec F_a} = 0 \quad \text{(Newton III)}$$

$$\Rightarrow \sum_a \frac{d}{dt}\frac{\partial L}{\partial \vec v_a} = \sum_a \frac{\partial L}{\partial \vec x_a} = 0 \quad \Rightarrow \quad \frac{d}{dt}\underbrace{\sum_a \frac{\partial L}{\partial \vec x_a}}_{\vec P} = 0$$

$\vec P$ total momentum

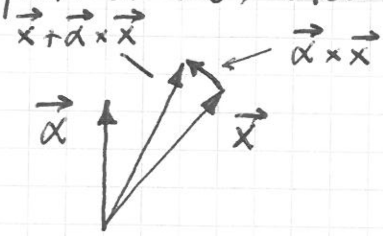
here: $\vec P = m_1\vec v_1 + m_2\vec v_2$

<u>space isotropic</u>

L invariant under $\quad \vec{x}_a \to \vec{x}_a + \overbrace{\vec{\alpha} \times \vec{x}_a}^{\delta \vec{x}_a}$

$\Rightarrow \quad \vec{v}_a \to \vec{v}_a + \underbrace{\vec{\alpha} + \vec{v}_a}_{\delta \vec{v}_a}$

infinitesimal rotation



$$\delta L = \sum_a \left(\underbrace{\frac{\partial L}{\partial \vec{x}_a}}_{\text{EOM} \ \dot{\vec{p}}_a} \cdot \delta \vec{x}_a + \underbrace{\frac{\partial L}{\partial \vec{v}_a}}_{\vec{p}_a} \cdot \delta \vec{v}_a \right) =$$

$$= \sum_a \left[\dot{\vec{p}}_a \cdot (\vec{\alpha} \times \vec{x}_a) + \vec{p}_a \cdot (\vec{\alpha} \times \vec{v}_a) \right]$$

$$= \vec{\alpha} \cdot \sum_a \left(\vec{x}_a \times \dot{\vec{p}}_a + \dot{\vec{x}}_a \times \vec{p}_a \right)$$

$$= \vec{\alpha} \cdot \frac{d}{dt} \sum_a (\vec{x}_a \times \vec{p}_a) \quad \overset{\vec{\alpha} \text{ arbitrary}}{\Longrightarrow} \quad \frac{d}{dt} \underbrace{\sum_a (\vec{x}_a \times \vec{p}_a)}_{\vec{L}} = 0$$

<u>total angular momentum conserved</u>

$$\vec{L} = \sum_a (\vec{x}_a \times \vec{p}_a)$$

nonrelativistic: $\quad \vec{L} = \sum_a m_a (\vec{x}_a \times \cancel{p} \vec{v}_a)$

Relativistic mechanics

invariant $ds^2 = c^2 dt^2 - d\vec{x}^2$ proper time: $d\tau = ds/c$

$$\rightarrow \quad S = -\alpha \int_1^2 ds = -\alpha c \int_{t_1}^{t_2} dt \sqrt{1 - \frac{\vec{v}^2}{c^2}}$$

$$\rightarrow \quad L = -\alpha c \sqrt{1 - \frac{\vec{v}^2}{c^2}} \xrightarrow[|\vec{v}| \ll c]{} -\alpha c \left(1 - \frac{\vec{v}^2}{2c^2} + \ldots\right)$$

$$= -\alpha c + \frac{\alpha}{c} \frac{\vec{v}^2}{2} + \ldots$$

$$\Rightarrow \quad \frac{\alpha}{c} = m$$

$$\Rightarrow \quad S = -mc \int ds = -mc^2 \int dt \sqrt{1 - \frac{\vec{v}^2}{c^2}}$$

$$L = -mc^2 \sqrt{1 - \frac{\vec{v}^2}{c^2}}$$

rel. momentum $\vec{p} = \dfrac{\partial L}{\partial \vec{v}} = \dfrac{m\vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}$

rel. energy $E = \vec{p} \cdot \vec{v} - L = \dfrac{mc^2}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}$

$$\xrightarrow[|\vec{v}| \ll c]{} \underbrace{mc^2}_{\text{rest energy}} + \frac{m\vec{v}^2}{2} + \ldots$$

rel. energy-momentum relation → Hamilton function

$$E = \sqrt{m^2 c^4 + \vec{p}^{\,2} c^2} \xrightarrow[|\vec{p}| \ll mc]{} mc^2 + \frac{\vec{p}^{\,2}}{2m} + \ldots$$

massless particle: $E = |\vec{p}|\, c$

velocity: $\vec{v} = \dfrac{\vec{p}\, c^2}{E} \xrightarrow[m=0]{} \vec{v} = \dfrac{\vec{p}}{|\vec{p}|}\, c$

$|\vec{v}| = c$ for massless particle

angular momentum: $\vec{L} = \vec{x} \times \vec{p}$

Noether theorem

general form of a symmetry transformation

$$t \to t' , \quad q(t) \to q'(t')$$

EOM (form) invariant

sufficient condition for invariance:

$$S[q'(t')] = S[q(t)]$$

$$\Leftrightarrow \quad \int_{t_1'}^{t_2'} dt' \, L(q'(t'), \dot{q}'(t'), t') = \int_{t_1}^{t_2} dt \, L(q(t), \dot{q}(t), t)$$

<u>classical</u> physics : <u>only</u> continuous transformations lead to conserved quantities

relevant structure already contained in <u>infinitesimal transformations</u>

→ <u>Lie groups</u>

$$t \to t' = t + \alpha \tau(q, \dot{q}, t) + \mathcal{O}(\alpha^2)$$

$$q(t) \to q'(t') = q(t) + \alpha \rho(q, \dot{q}, t) + \mathcal{O}(\alpha^2)$$

<u>action invariant</u> :

$$0 = S[q'(t')] - S[q(t)]$$

$$= \int_{t_1'}^{t_2'} dt' \, L(q'(t'), \dot{q}'(t'), t') - S[q(t)]$$

$$= \int_{t_1}^{t_2} dt \left(1 + \alpha \frac{d\tau}{dt}\right) L\Big[q(t) + \alpha \rho(q, \dot{q}, t), \left(\dot{q}(t) + \alpha \frac{d\rho}{dt}\right)\left(1 - \alpha \frac{d\tau}{dt}\right), t + \alpha\tau\Big] - S[q(t)]$$

$$= \int_{t_1}^{t_2} dt \left(1 + \alpha \frac{d\tau}{dt}\right) L\left[q + \alpha\rho, \dot{q} + \alpha \frac{d\rho}{dt} - \dot{q}(t)\, \alpha \frac{d\tau}{dt}, t + \alpha\tau\right]$$

$$- S[q(t)]$$

$$= \alpha \int_{t_1}^{t_2} dt \, \Big[\sum_{i=1}^{f} \overbrace{\frac{\partial L}{\partial q_i}}^{\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}} \rho_i + \sum_{i=1}^{f} \frac{\partial L}{\partial \dot q_i} \left(\frac{d\rho_i}{dt} - \dot q_i \frac{d\tau}{dt} \right)$$

$$+ \frac{\partial L}{\partial t} \tau + L \frac{d\tau}{dt}$$

$$\Rightarrow \frac{d}{dt} \sum_{i=1}^{f} \frac{\partial L}{\partial \dot q_i} \rho_i + \left(L - \sum_{i=1}^{f} \frac{\partial L}{\partial \dot q_i} \dot q_i \right) \frac{d\tau}{dt} + \frac{\partial L}{\partial t} \tau = 0$$

$$\frac{d}{dt} \left(L - \sum_{i=1}^{f} \frac{\partial L}{\partial \dot q_i} \dot q_i \right) = \frac{\partial L}{\partial t} + \sum_{i=1}^{f} \Big(\cancel{\frac{\partial L}{\partial q_i} \dot q_i} + \cancel{\frac{\partial L}{\partial \dot q_i} \ddot q_i}$$

$$\cancel{- \frac{\partial L}{\partial q_i} \dot q_i} \cancel{- \frac{\partial L}{\partial \dot q_i} \ddot q_i} \Big) = \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt} \left[\sum_{i=1}^{f} \frac{\partial L}{\partial \dot q_i} \rho_i + \left(L - \sum_{i=1}^{f} \frac{\partial L}{\partial \dot q_i} \dot q_i \right) \tau \right] = 0$$

action invariant under continuous transformation

$$t \to t' = t + \alpha \tau(q, \dot q, t) + \mathcal{O}(\alpha^2)$$

$$q(t) \to q'(t') = q(t) + \alpha \rho(q, \dot q, t) + \mathcal{O}(\alpha^2)$$

$\rightarrow$ <u>constant of motion</u>

$$\sum_{i=1}^{f} \frac{\partial L}{\partial \dot q_i} \rho_i + \left(L - \sum_{i=1}^{f} \frac{\partial L}{\partial \dot q_i} \dot q_i \right) \tau = \text{const.}$$

generalized Noether theorem

L only determined up to total derivative!

$$\left(1+\alpha\frac{d\tau}{dt}\right) L\left[q(t)+\alpha\,\rho(q,\dot q,t),\ \left(\dot q(t)+\alpha\frac{d\rho}{dt}\right)\times\right.$$
$$\left.\times\left(1-\alpha\frac{d\tau}{dt}\right),\ t+\alpha\tau\right]$$

$$= L(q(t),\dot q(t),t) + \alpha\frac{d}{dt} f(q(t),t)$$

$$\Rightarrow \sum_{i=1}^{f}\frac{\partial L}{\partial \dot q_i}\rho_i + \left(L-\sum_{i=1}^{f}\frac{\partial L}{\partial \dot q_i}\dot q_i\right)\tau - f = \text{const.}$$

consider Galilei-invariant Lagrangian function

$$L(\vec x_1,\dots,\vec x_N,\dot{\vec x}_1,\dots,\dot{\vec x}_N) = \underbrace{\sum_a \frac{m_a \dot{\vec x}_a^{\,2}}{2}}_{T} - U(|\vec x_a-\vec x_b|)$$

Galilei group (symmetry group of nonrelativistic mechanics)

Galilei transformation $\vec x_a{}' = R\vec x_a + \vec u t + \vec\xi$

$$t' = t + t_0$$

$R\in SO(3,\mathbb{R})$ $\quad R^T R = \mathbb{1}$

$\det R = 1$

$\vec u \in \mathbb{R}^3$ $\qquad R, \vec u, \vec\xi, t_0 \quad 3+3+3+1 = 10$ parameters

$\vec\xi \in \mathbb{R}^3$ $\qquad \Rightarrow$ 10 constants of motion

(i) invariance under spatial translations

$\vec{\xi} \to 3$ parameters

arbitrary constant vector

$$\vec{x}_a(t) \to \vec{x}_a{}'(t) = \vec{x}_a(t) + \underbrace{\alpha \vec{d}}_{\vec{\xi}}$$

$$t' = t$$

$$\dot{\vec{x}}_a(t) = \dot{\vec{x}}_a{}'(t) \quad \Rightarrow \quad T \text{ invariant}$$

U depends only on $|\vec{x}_a - \vec{x}_b|$ $\Rightarrow$ U invariant

$$\vec{\rho}_a = \vec{d}, \quad \tau = 0$$

$$\Rightarrow \quad \sum_{a=1}^{N} \vec{p}_a \cdot \vec{d} = \text{const}$$

$\vec{d}$ arbitrary $\Rightarrow$ total momentum $\sum_{a=1}^{N} \vec{p}_a =: \vec{P}$ conserved

(ii) invariance under time translation ($t_0 \to 1$ parameter)

$$t' = t + \alpha$$

$$\vec{x}_a{}'(t') = \vec{x}_a{}'(t+\alpha) = \vec{x}_a(t)$$

$$\tau = 1, \quad \vec{\rho}_a = 0$$

$$\frac{\partial U}{\partial t} = 0 \quad \Rightarrow \quad L \text{ invariant}$$

$$\overset{\text{Noether}}{\Rightarrow} \quad \underbrace{L}_{T-U} - \underbrace{\sum_{a=1}^{N} m_a \dot{\vec{x}}_a^{\,2}}_{2T} = -(T+U) = -E \text{ conserved}$$

<u>energy conservation</u>

(iii) invariance under spatial rotations ($R \to 3$ parameters)

$$t' = t \quad \Rightarrow \quad \tau = 0$$

$$\vec{x}_a{}' = \vec{x}_a + \alpha \underbrace{\vec{n} \times \vec{x}_a}_{\vec{\rho}_a} \qquad , \quad |\vec{n}| = 1$$

$$\dot{\vec{x}}_a{}' = R \dot{\vec{x}}_a \quad \Rightarrow \quad \dot{\vec{x}}_a{}' \cdot \dot{\vec{x}}_a{}' = \dot{\vec{x}}_a \cdot \dot{\vec{x}}_a \quad \text{Because of } R^T R = \mathbb{1}$$

$\Rightarrow T$ invariant

$|\vec{x}_a - \vec{x}_b|$ invariant under rotations $\Rightarrow U$ invariant

$$\overset{\text{Noether}}{\Rightarrow} \quad \sum_a \vec{p}_a \cdot (\vec{n} \times \vec{x}_a) = \vec{n} \cdot \underbrace{\sum_a \vec{x}_a \times \vec{p}_a}_{\vec{L}} \quad \text{conserved}$$

$\vec{n}$ arbitrary unit vector $\Rightarrow$ total angular momentum $\vec{L}$ conserved

(iv) pure velocity transformation (boost)

($\vec{u} \rightarrow$ 3 parameters)

$$t' = t \Rightarrow \tau = 0$$

$$\vec{x}_a{}'(t) = \vec{x}_a(t) + \underbrace{\alpha \vec{n}}_{\vec{u}} t \qquad , \ |\vec{n}| = 1$$

$$\Rightarrow \quad \vec{\rho}_a = \vec{n} t$$

$|\vec{x}_a - \vec{x}_b|$ invariant $\Rightarrow$ U invariant

$$T' = \sum_{a=1}^{N} \frac{m_a \dot{\vec{x}}_a{}'^2}{2} = \sum_{a=1}^{N} \frac{m_a}{2} \left(\dot{\vec{x}}_a + \alpha \vec{n}\right)^2$$

$$= T + \alpha \underbrace{\sum_{a=1}^{N} m_a \dot{\vec{x}}_a \cdot \vec{n}}_{\frac{d}{dt} \underbrace{\vec{n} \cdot \sum_{a=1}^{N} m_a \vec{x}_a}_{f(\vec{x}_1, \dots, \vec{x}_N)}} + \mathcal{O}(\alpha^2)$$

T invariant up to <u>total derivative</u>

generalized Noether theorem $\Rightarrow$

$$\sum_{a=1}^{N} \vec{p}_a \cdot \vec{n}\, t - \vec{n} \cdot \sum_{a=1}^{N} m_a \vec{x}_a = \vec{n} \sum_{a=1}^{N} \left(\vec{p}_a t - m_a \vec{x}_a\right) = \text{const.}$$

CM $\quad \vec{R}(t) = \frac{1}{M} \sum_{a=1}^{N} m_a \vec{x}_a(t) , \quad M = \sum_{a=1}^{N} m_a$

$$\Rightarrow \quad \vec{R}(t) = \vec{R}_0 + \frac{\vec{P}}{M} t$$

movement of CM (CM theorem)

remark: (spatial) translation invariance $\Rightarrow$ U invariant under velocity transf.

$\Rightarrow$ conservation of total momentum implies CM theorem

$$\dot{\vec{R}}(t) = \frac{1}{M} \sum_{a=1}^{N} m_a \dot{\vec{x}}_a(t) = \frac{\overbrace{\vec{P}}^{\text{conserved}}}{M} = \text{const.}$$

$\overset{\text{integration}}{\Rightarrow} \quad \vec{R}(t) = \vec{R}_0 + \frac{\vec{P}}{M} t$

(Relativistic) field theory

$\downarrow$

action Lorentz invariant

$x^0 = ct , \ \vec{x} \quad \rightarrow \quad x^\mu = (x^0, \vec{x})$

from now on: $c = 1$

$$S = \int \underbrace{d^4x}_{\text{Lorentz scalar}} \overbrace{\mathcal{L}(\phi_a(x), \phi_{a,\mu}(x), x)}^{\text{Lagrangian density (Lagrangian)}}$$

$\mathcal{L}$ Lorentz invariant $\Rightarrow$ S Lorentz invariant

(inverse not necessary, but in (almost) all relevant cases: $\mathcal{L}$ inv.)

Principle of least action for fields

fields $\phi(x_1^0, \vec{x})$ and $\phi(x_2^0, \vec{x})$ fixed

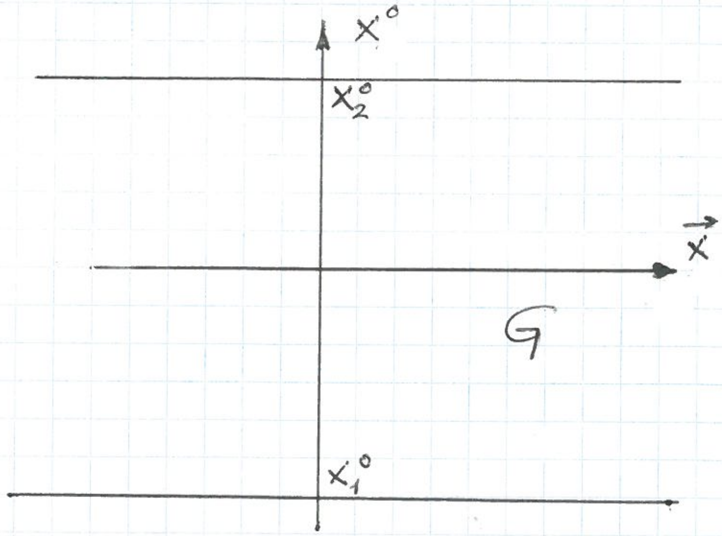
for $x_1^0 \leq x^0 \leq x_2^0$: choose $\phi(x)$ such that $S \to \min.$

assumption: $\lim\limits_{|\vec{x}| \to \infty} \phi(x^0, \vec{x}) = 0$

field variation $\phi(x) \to \phi(x) + \eta(x)$

with $\eta(x_1^0, \vec{x}) = \eta(x_2^0, \vec{x}) = 0$, $\lim\limits_{|\vec{x}| \to \infty} \eta(x^0, \vec{x}) = 0$

$\delta S = 0$ (S extremal)



$$\delta S = \int_{x_1^0}^{x_2^0} dx^0 \int d^3x \, [\mathcal{L}(\phi + \eta, \phi_{,\mu} + \eta_{,\mu}, x) -$$

$$- \mathcal{L}(\phi, \phi_{,\mu}, x)]$$

(index a suppressed)

$$= \int_{x_1^0}^{x_2^0} dx^0 \int d^3x \, \Big[\frac{\partial \mathcal{L}}{\partial \phi} \eta + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \eta_{,\mu}}_{\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \eta\right) - \left(\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}\right) \eta} + \mathcal{O}(\eta^2) \Big]$$

$$= \int_{x_1^0}^{x_2^0} dx^0 \int d^3x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) \eta + \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \eta \right) \right]$$

$$\int_{x_1^0}^{x_2^0} dx^0 \int_{\mathbb{R}^3} d^3x \, \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \eta \right) =$$

$$= \underbrace{\int_{\mathbb{R}^3} d^3x \, \frac{\partial \mathcal{L}}{\partial \phi_{,0}} \eta \Bigg|_{x_1^0}^{x_2^0}}_{0} + \underbrace{\int_{x_1^0}^{x_2^0} dx^0 \int_{\partial \mathbb{R}^3} df_i \left(\frac{\partial \mathcal{L}}{\partial \phi_{,i}} \eta \right)}_{0}$$

because of $\eta(x_{1,2}^0, \vec{x}) = 0$ because of $\eta(x^0, \vec{x}) \xrightarrow[|\vec{x}| \to \infty]{} 0$

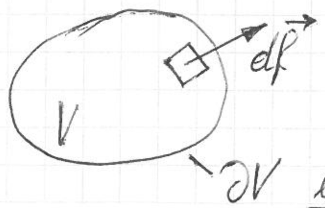
η arbitrary in G $\Rightarrow$ <u>Euler – Lagrange field equation</u>

$$\boxed{\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = \frac{\partial \mathcal{L}}{\partial \phi}}$$

<u>remark:</u> remember Gauss theorem in $d = 3$

$$\int_V d^3x \, \operatorname{div} \vec{A} = \int_{\partial V} d\vec{f} \cdot \vec{A}$$

surface $\vec{x}(u,v)$



oriented surface element $d\vec{f} = \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \, du \, dv$

∂V <u>boundary of V</u> $\quad df_i = \varepsilon_{ijk} \frac{\partial x_j}{\partial u} \frac{\partial x_k}{\partial v} \, du \, dv$

theorem of Gauss in $d=4$

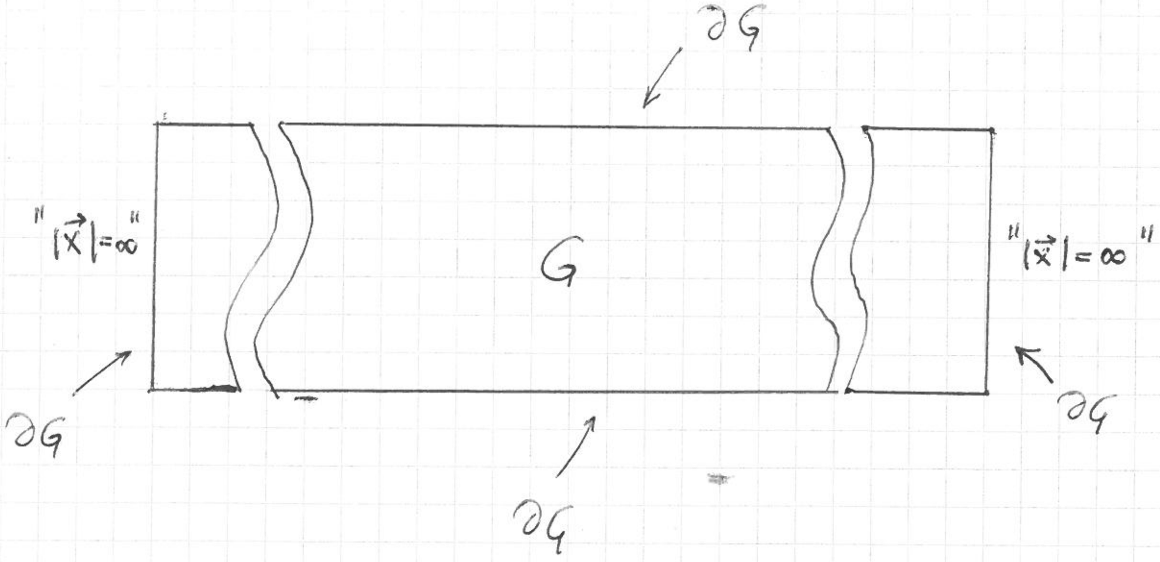
$$\int_G d^4x\, \partial_\mu A^\mu = \int_{\partial G} d\sigma_\mu A^\mu$$

3-dim. hypersurface in $d=4$: $\quad x(u,v,w)$

$\rightarrow$ oriented surface element $\quad d\sigma_\mu = \varepsilon_{\mu\nu\rho\sigma} \frac{\partial x^\nu}{\partial u} \frac{\partial x^\rho}{\partial v} \frac{dx^\sigma}{dw} \cdot du\, dv\, dw$

$\varepsilon_{\mu\nu\rho\sigma}$ totally antisymmetric, convention: $\varepsilon_{0123} = +1$

in our case:



$$\eta\big|_{\partial G} = 0$$

some field equations:

(a) Klein-Gordon equation (free scalar field)

$$S = \int d^4x \; \frac{1}{2} \left(\partial^\mu \phi \, \partial_\mu \phi - m^2 \phi^2 \right) \quad , \quad \phi^* = \phi$$

$$\Rightarrow \left(\Box + m^2 \right) \phi = 0 , \qquad \Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta$$

$\uparrow$ d'Alembert operator

(b) Massive vector field (Proca equations)

real vector field $A_\mu(x)$

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \quad , \quad m \neq 0 \; !!!$$

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \right)$$

$$\Rightarrow \left(\Box + m^2 \right) A_\mu = 0 \quad , \quad \partial_\mu A^\mu = 0$$

(c) Massless vector field (electrodynamics)

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad , \quad \partial_\mu j^\mu = 0$$

$\Rightarrow$ field equation $\quad \partial_\nu F^{\nu\mu} = j^\mu \quad$ (Maxwell)

Conservation laws in relativistic field theory

$$S = \int_{\mathcal{D}} d^4x \, \mathcal{L}(\phi_a, \phi_{a,\mu}, x)$$

$\mathcal{D}$ ... 4-dimensional domain in Minkowski space with boundary $\partial\mathcal{D}$

change of S under infinitesimal change of ϕ_a as well as $\mathcal{D}$?

for every x $\quad \phi_a(x)$ changed by $\delta\phi_a(x)$

x shifted by Δx (displacement vector)

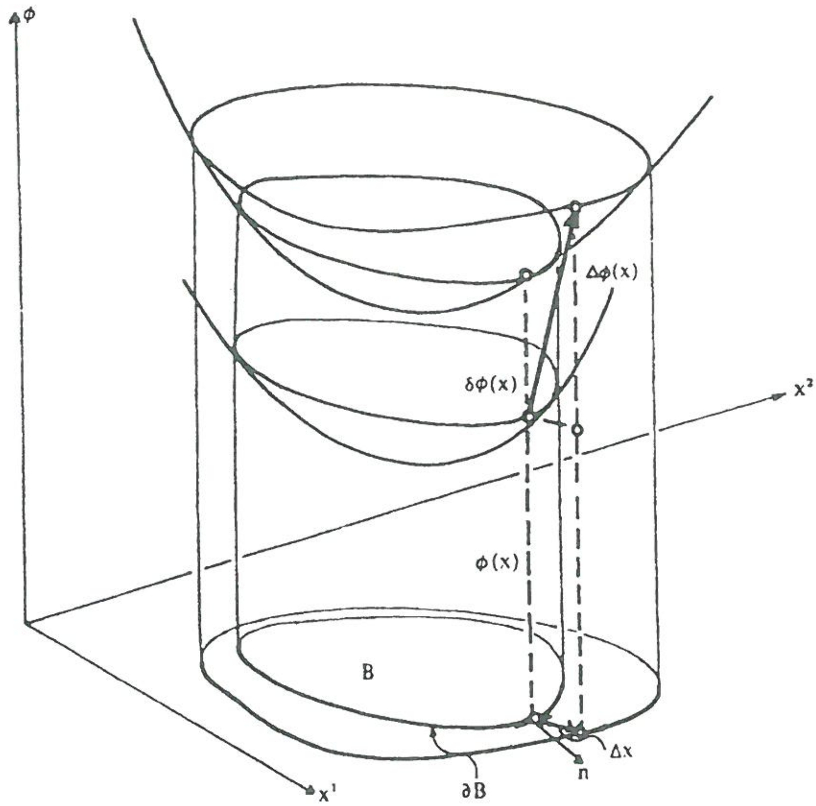


Fig. 10.1. The graph of $\phi_a = \phi_a(x)$ and its variation in (x, ϕ)-space

$$\Delta S = \int_{\mathcal{D}} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \delta \phi_{a,\mu}}_{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \delta \phi_a \right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \right) \delta \phi_a} \right]$$

summation over a!

$$+ \int_{\partial \mathcal{D}} d\sigma_\mu \, \mathcal{L} \, \Delta x^\mu$$

$$\overset{\text{Gauß}}{=} \int_{\mathcal{D}} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \right] \delta \phi_a$$

$$+ \int_{\partial \mathcal{D}} d\sigma_\mu \left[\frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \underbrace{\delta \phi_a}_{\Delta \phi_a - \phi_{a,\nu} \Delta x^\nu} + \mathcal{L} \, \Delta x^\mu \right]$$

"vertical" variation $\delta\phi_a(x)$

"skew" variation $\Delta\phi_a(x) = (\phi_a + \delta\phi_a)(x+\Delta x) - \phi_a(x)$

$$= \delta\phi_a(x) + \phi_{a,\mu}(x)\,\Delta x^\mu$$

$$\Delta S = \int_{\mathcal{D}} d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\phi_{a,\mu}}\right]\delta\phi_a$$

$$+ \int_{\partial\mathcal{D}} d\sigma_\mu \left[\frac{\partial\mathcal{L}}{\partial\phi_{a,\mu}}\Delta\phi_a - \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_{a,\mu}}\phi_{a,\nu} - \mathcal{L}\,\delta^\mu_{\ \nu}\right)}_{\Theta^\mu_{\ \nu}}\Delta x^\nu\right]$$

we recover the Euler-Lagrange field equations

$\partial\mathcal{D}$ and values on $\partial\mathcal{D}$ kept fixed $\quad(\Delta x^\mu = 0,\ \delta\phi_a = 0$ on $\partial\mathcal{D})$

$\Rightarrow \Delta S = 0$ implies $\quad \dfrac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \dfrac{\partial\mathcal{L}}{\partial\phi_{a,\mu}} = 0$

(<u>no</u> influence of special choice of $\mathcal{D}$)

→ $\mathcal{L}$ defines an <u>action principle</u> (<u>variation principle</u>)

on the other hand: $\mathcal{D}$ specified + given boundary values for ϕ_a on $\partial\mathcal{D}$ → variational <u>problem</u>

transformation $\quad (x, \phi) \to (x', \phi')$

field equations form-invariant *if* action integral invariant

$$x^\mu \to x'^\mu = X^\mu(x, \phi)$$

$$\phi_a \to \phi_a' = \Phi_a(x, \phi)$$

$$\phi_a'(x') = \Phi_a\left(x(x'), \phi(x(x'))\right)$$

$$x \in \mathcal{D} \to x' \in \mathcal{D}'$$

$$S' = \int_{\mathcal{D}'} d^4x' \, \mathcal{L}\left(x', \phi'(x'), \frac{\partial \phi'(x')}{\partial x'^\mu}\right) =$$

$$= \int_{\mathcal{D}} d^4x \, \mathcal{L}'\left(x, \phi(x), \frac{\partial \phi(x)}{\partial x^\mu}\right)$$

$\mathcal{L}'$ *defined* by this equation

if $\mathcal{L} \equiv \mathcal{L}' \Rightarrow S = S'$

one-parameter group of transformations

$$x'^\mu = X^\mu(x, \phi; \alpha)$$

$$\phi_a' = \Phi_a(x, \phi; \alpha)$$

$\alpha = 0 \leftrightarrow$ identity

consequences of $S' = S$ for <u>infinitesimal</u> α:

$$x'^{\mu} = X^{\mu}(x, \phi; \alpha) = \underbrace{X^{\mu}(x, \phi; 0)}_{x^{\mu}} + \underbrace{\left.\frac{\partial X^{\mu}}{\partial \alpha}\right|_{\alpha=0}}_{\Delta x^{\mu}} \alpha + O(\alpha^2)$$

$$\Delta \phi_a = \phi_a'(x') - \phi_a(x) = \Phi_a(x, \phi; \alpha) - \phi_a(x)$$

$$= \underbrace{\Phi_a(x, \phi; 0)}_{\phi_a(x)} + \left.\frac{\partial \Phi_a}{\partial \alpha}\right|_{\alpha=0} \alpha - \phi_a(x) + O(\alpha^2)$$

$$\Rightarrow \Delta \phi_a = \left.\frac{\partial \Phi_a}{\partial \alpha}\right|_{\alpha=0} \alpha$$ skew variation!

assumption: ϕ satisfies field equations

$$\Rightarrow \int_{\partial D} d\sigma_{\mu} \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \Delta \phi_a - \Theta^{\mu}{}_{\nu} \Delta x^{\nu}\right]}_{j^{\mu} \alpha} = 0$$

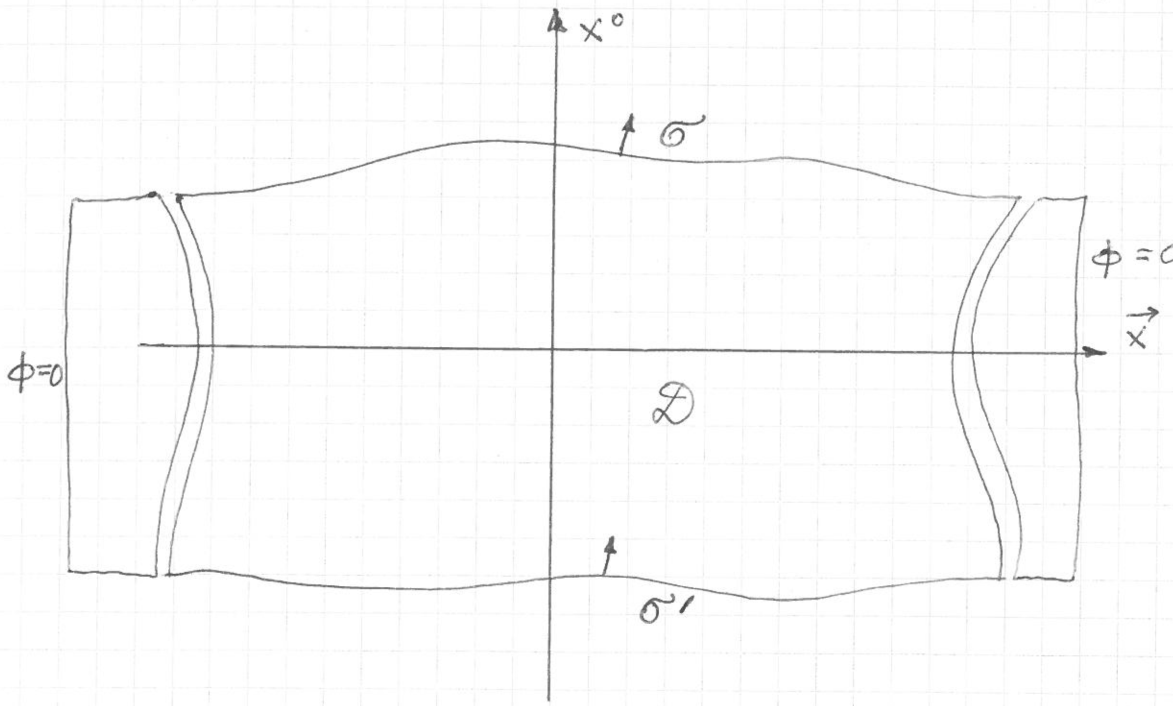
Gauß $\Rightarrow \displaystyle\int_{D} d^4x \; j^{\mu}{}_{,\mu} = 0$

D arbitrary $\Rightarrow j^{\mu}{}_{,\mu} = 0$

four-charge density j^μ satisfies an equation of continuity

total charge $Q = \int_\sigma d\sigma_\mu \, j^\mu$ independent of the choice of the spacelike hypersurface σ

assumption: $\phi(x) \to 0$ at spatial infinity



$$0 = \int_{\partial\mathcal{D}} d\sigma_\mu \, j^\mu = \int_\sigma d\sigma_\mu \, j^\mu - \int_{\sigma'} d\sigma_\mu \, j^\mu$$

special case σ = spacelike hyperplane at $x^0 = t = \text{const.}$ parametrized by x^1, x^2, x^3

$$\Rightarrow \quad d\sigma_\mu = \varepsilon_{\mu 123} \, dx^1 dx^2 dx^3 = \varepsilon_{\mu 123} \, d^3x = (d^3x, \vec{0})$$

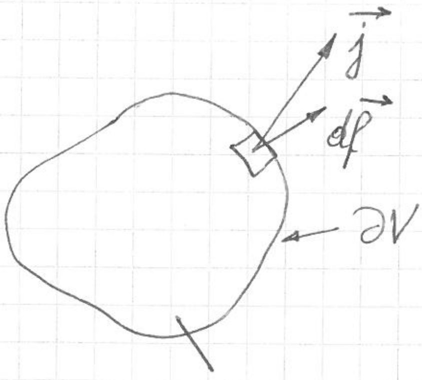
$$\Rightarrow \quad Q = \int_{\sigma = \mathbb{R}^3} d^3x \; \underbrace{j^0(t, \vec{x})}_{\text{charge density}}$$

charge Q_V contained in a finite volume $V \subset \mathbb{R}^3$

$$Q_V(t) = \int_V d^3x \; j^0(t, \vec{x})$$

$$\frac{d}{dt} Q_V(t) = \int_V d^3x \; \frac{\partial j^0}{\partial x^0} = - \int_V d^3x \; \vec{\nabla} \vec{j} = - \int_{\partial V} d\vec{f} \cdot \vec{j}$$

$\vec{j}(x)$ = density of current



summary: Noether theorem in field theory

if action S invariant under one-parameter group of transformations $x^\mu \to x'^\mu = X^\mu(x, \phi; \alpha)$, $\phi_a \to \Phi_a(x, \phi; \alpha) \Rightarrow \exists$ conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \frac{\partial \Phi_a}{\partial \alpha}\bigg|_{\alpha=0} - \Theta^\mu{}_\nu \frac{\partial X^\nu}{\partial \alpha}\bigg|_{\alpha=0}$$

symmetry transformations usually not needed in full generality:

(i) (global) internal symmetries

$$x' = x, \qquad \phi_a' = \phi_a'(\phi;\alpha)$$

(ii) spacetime symmetries

$$x'^\mu = x'^\mu(x;\alpha), \qquad \phi_a' = \phi_a'(x,\phi;\alpha)$$

generalizations of the Noether theorem:

(a) $\mathcal{L}' = \mathcal{L} + \mathcal{F}.\alpha, \qquad \mathcal{F} = \dfrac{\partial f^\mu}{\partial x^\mu} + \dfrac{\partial f^\mu}{\partial \phi_a}\phi_{a,\mu}$ total divergence

$$f^\mu = f^\mu(x,\phi)$$

$$\Rightarrow \quad j^\mu = \frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \frac{\partial \Phi_a}{\partial \alpha}\bigg|_{\alpha=0} - \Theta^\mu{}_\nu \frac{\partial X^\nu}{\partial \alpha}\bigg|_{\alpha=0} + f^\mu$$

(b) local gauge transformations $\alpha \to \alpha(x)$

Applications to Poincaré-covariant field theories

Poincaré transformation $x' = \underset{\substack{\uparrow\\ 4\times 4 \text{ matrix}}}{L}x + \underset{\substack{\uparrow\\ \in \mathbb{R}^4}}{a}$, $L^T g L = g$;

$(g_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1) = (g^{\mu\nu})$

leaves the line element $ds^2 = (dx^0)^2 - d\vec{x}^2 = dx^\mu dx^\nu g_{\mu\nu}$ invariant

Poincaré group $\mathcal{P}$ consist of elements $g = (L, a)$

composition law:

$$x'' = L'x' + a' = L'(Lx + a) + a'$$
$$= L'Lx + L'a + a'$$

$$g'g = (L', a')(L, a) = (L'L, L'a + a')$$

Lorentz group $\mathcal{L}$ = subgroup of $\mathcal{P}$ consisting of the elements $(L, 0)$

$x' = Lx$ (no shift of the origin)

- full Lorentz group $\mathcal{L}$ contains also space and time reversals $\longrightarrow$ $\mathcal{L}$ is not connected

$T = \begin{pmatrix} -1 & 0_3^T \\ 0_3 & \mathbb{1}_3 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0_3^T \\ 0_3 & -\mathbb{1}_3 \end{pmatrix}$

$$\underbrace{\det(L^T g L)}_{\underbrace{\det L^T}_{\det L}\,\det g\,\det L} = \det g$$

proper Lorentz group $\mathcal{L}_+$

$$\Rightarrow \quad (\det L)^2 = 1 \quad \Rightarrow \quad \det L = \pm 1$$

remark : $\det P = \det T = -1$

$$(L^T g L)_{00} = g_{00} = 1$$

$$(L^T)_0{}^\mu \, g_{\mu\nu} \, L^\nu{}_0 = 1$$

$$L^\mu{}_0 \, g_{\mu\nu} \, L^\nu{}_0 = 1$$

$$(L^0{}_0)^2 - \sum_{i=1}^{3} (L^i{}_0)^2 = 1$$

$$(L^0{}_0)^2 = 1 + \sum_{i=1}^{3} (L^i{}_0)^2 \Rightarrow (L^0{}_0)^2 \geq 1$$

$\Rightarrow \quad L^0{}_0 \geq 1$ (sense of time unchanged → <u>orthochronous</u> transformation)

orthochronous Lorentz group $\mathcal{L}^\uparrow$

<u>or</u>

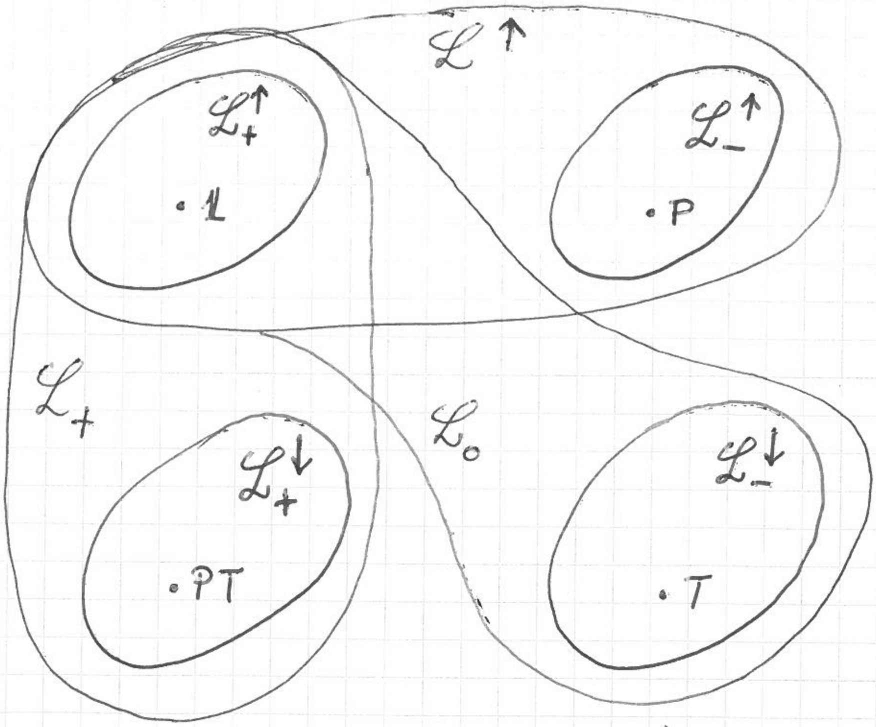
$L^0{}_0 \leq -1$ (time reversal → <u>antichronous</u> transf.)

decomposition of $\mathcal{L}$ into four components (each connected)

	$\det L$	$\operatorname{sgn} L^0{}_0$	
$\mathcal{L}_+^\uparrow$	1	1	$\mathbb{1}$
$\mathcal{L}_-^\uparrow$	−1	1	P
$\mathcal{L}_+^\downarrow$	+1	−1	PT
$\mathcal{L}_-^\downarrow$	−1	−1	T

$$\mathcal{L}_-^\uparrow = P\mathcal{L}_+^\uparrow \,, \quad \mathcal{L}_+^\downarrow = PT\mathcal{L}_+^\uparrow \,, \quad \mathcal{L}_-^\downarrow = T\mathcal{L}_+^\uparrow$$

$\mathcal{L}_+^\uparrow = \mathcal{L}_+ \cap \mathcal{L}^\uparrow$



$\mathcal{L}_+^\uparrow$ = proper orthochronous Lorentz group

$\mathcal{L}_+ = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow = \mathcal{L}_+^\uparrow \cup PT\mathcal{L}_+^\uparrow$ = proper Lorentz group

$\mathcal{L}^\uparrow = \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\uparrow = \mathcal{L}_+^\uparrow \cup P\mathcal{L}_+^\uparrow$ = orthochronous Lorentz group

$\mathcal{L}_0 = \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\downarrow = \mathcal{L}_+^\uparrow \cup T\mathcal{L}_+^\uparrow$ = orthochorous –〃– –〃–

$\mathcal{L}_-^\uparrow, \mathcal{L}_+^\downarrow, \mathcal{L}_-^\downarrow$ do not form groups!

- fundamental interactions (strong, weak, electromagnetic, gravity) respect $\mathcal{P}_+^\uparrow$; weak interaction violates P, T

remark: $\mathcal{P}_+^\uparrow$ describes transformations from one inertial frame to another one (elements can be continously connected with $\mathbb{1}$)

$(L, a) \in \mathcal{P}_+^\uparrow$ can be written in the form

$$\Big(\underbrace{L_{rot}(\vec{\alpha})}_{\text{rotation}} \; \underbrace{L_{Boost}(\vec{u})}_{\substack{\text{Boost} \\ \text{(velocity transf.)}}} \,,\; a\Big)$$

$\mathcal{L}_+^\uparrow$ has 6 parameters $(\vec{\alpha}, \vec{u})$

$\mathcal{P}_+^\uparrow$ -//- 10 -//- $(\vec{\alpha}, \vec{u}, a)$

Poincaré invariance $\longrightarrow$ 10 constants of motion

$$S = \int \underbrace{d^4x}_{inv.} \; \mathcal{L}(\phi, \phi_{,\mu})$$

S $\mathcal{P}_+^\uparrow$-invariant if $\mathcal{L}$ transforms as a scalar (no explicit dependence on x)

a) <u>translational invariance</u>

$$x'^\mu = x^\mu + \underbrace{n^\mu \alpha}_{\Delta x^\mu}$$

$$\Delta\varphi = \varphi'(x') - \varphi(x) = 0$$

$\Rightarrow$ conserved current $\quad j^\mu = -\,\Theta^\mu{}_\nu \, n^\nu$

n^ν arbitrary $\Rightarrow \quad \partial_\mu \, \Theta^\mu{}_\nu = 0$

$$P_\nu = \int_\sigma d\sigma_\mu \, \Theta^\mu{}_\nu = \int_{t=\text{const.}} d^3x \, \Theta^0{}_\nu = \text{const.}$$

$\Theta^\mu{}_\nu$ = <u>canonical energy-momentum tensor</u>

P_ν = <u>energy-momentum four-vector</u>

$$\frac{d}{dt} \int_V d^3x \, \Theta^{0\nu} = \int_V d^3x \, \underbrace{\frac{\partial}{\partial x^0} \Theta^{0\nu}}_{-\frac{\partial}{\partial x^i} \Theta^i{}_\nu} = -\int_{\partial V} df_i \, \Theta^{i\nu}$$

V : finite domain $\subset \mathbb{R}^3$

→ interpretation ?

Θ^{00} = energy density

Θ^{0i} = momentum density

Θ^{i0} = energy current density

Θ^{ij} = stress tensor density

objection: conclusion integral → integrand

arbitrariness in the localization of energy, momentum, etc.

take arbitrary tensor field $f^{\alpha\mu}{}_{\nu}$, antisymmetric in α, μ and $f^{\alpha\mu}{}_{\nu}(x) \xrightarrow[|\vec{x}| \to \infty]{} 0$

$$\Rightarrow \quad \partial_\mu \left(\partial_\alpha f^{\alpha\mu}{}_{\nu} \right) = 0$$

and

$$\underset{\substack{\sigma \\ \uparrow \\ \text{spacelike hypersurface}}}{\int} d\sigma_\mu \, \partial_\alpha f^{\alpha\mu}{}_{\nu} = \int_{\mathbb{R}^3} d^3x \, \partial_\alpha f^{\alpha 0}{}_{\nu} = \int_{\mathbb{R}^3} d^3x \, \partial_i f^{i0}{}_{\nu}$$

$$= \int_{\partial\mathbb{R}^3} df_i \, f^{i0}{}_{\nu} = 0$$

$\Rightarrow \quad T^\mu{}_\nu := \Theta^\mu{}_\nu + \partial_\alpha f^{\alpha\mu}{}_\nu$ also divergence-free,

same value for total energy-momentum P_ν

but: energy-momentum contained in a finite spatial volume depends on the choice of $f^{\alpha\mu}{}_\nu$ (further arguments necessary to fix it)

infinitesimal Lorentz transformation

we know: $\quad L^T g L = g \qquad \overbrace{(L^T)_\mu{}^\alpha}^{L^\alpha{}_\mu} g_{\alpha\beta} L^\beta{}_\nu = g_{\mu\nu}$

infinitesimal transformation $\quad \underset{\in \mathcal{L}_+^\uparrow}{L} = \mathbb{1} + \omega \qquad L^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$

$$(\mathbb{1} + \omega^T)\, g\, (\mathbb{1} + \omega) = g + \omega^T g + g\omega = g$$

$$\Rightarrow \quad \omega^T g + g\omega = 0$$

$$\omega^\alpha{}_\mu g_{\alpha\nu} + g_{\mu\alpha}\omega^\alpha{}_\nu = 0$$

$$\omega_{\nu\mu} + \omega_{\mu\nu} = 0 \qquad (\omega_{\mu\nu} \text{ is antisymmetric})$$

$\rightarrow$ 6 real parameters (rotations and boosts)

$$x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu \quad \Rightarrow \quad \Delta x^\mu = \omega^\mu{}_\nu x^\nu$$

field transformation $\quad \Phi'_a(x') = D_{ab}(L)\,\Phi_b(x)$

$D(L)$ = (finite-dimensional) linear representation of $L \in \mathcal{L}_+^\uparrow$
(more precisely: $SL(2,\mathbb{C})$) $\qquad D(L_1 L_2) = D(L_1) D(L_2)$

<u>examples</u>:

(i) scalar field $\quad \phi'(x') = \phi(x)$, $D(L) = 1$
(trivial repr.)

(ii) vector field $\quad A'^\mu(x') = L^\mu{}_\nu A^\nu(x)$, $D(L) = L$
(defining repr.)

---

infinitesimal transformation $\quad L = \mathbb{1} + \omega$

$$\Rightarrow \quad D(L) = \mathbb{1} + \frac{1}{2}\,\omega_{\alpha\beta}\Sigma^{\alpha\beta} + \mathcal{O}(\omega^2) \, , \quad \Sigma^{\alpha\beta} = -\Sigma^{\beta\alpha}$$

$$D_{ab}(L) = \delta_{ab} + \frac{1}{2}\,\omega_{\alpha\beta}\Sigma^{\alpha\beta}_{ab} + \mathcal{O}(\omega^2)$$

$$\Phi'_a(x') = \Phi_a(x) + \underbrace{\frac{1}{2}\,\omega_{\alpha\beta}\Sigma^{\alpha\beta}_{ab}\,\Phi_b(x)}_{\text{skew variation } \Delta\Phi_a}$$

---

for the previous examples:

(i) $\quad \Sigma = 0$

(ii) $\quad A'^\mu(x') = (\delta^\mu{}_\nu + \omega^\mu{}_\nu)\,A^\nu(x) =$

$$= A^\mu(x) + \frac{1}{2}\,\omega_{\alpha\beta}\,(g^{\alpha\mu}g^\beta{}_\nu - g^{\beta\mu}g^\alpha{}_\nu)\,A^\nu(x)$$

$$A'_\mu(x') = A_\mu(x) + \frac{1}{2}\,\omega_{\alpha\beta}\,\underbrace{\left(\delta^\alpha_\mu\,\delta^\beta_\nu - \delta^\beta_\mu\,\delta^\alpha_\nu\right)}_{\Sigma^{\alpha\beta}_{\mu\nu}}A^\nu(x)$$

<u>divergence-free current</u>:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\phi_{a,\mu}}\,\frac{1}{2}\,\omega_{\alpha\beta}\,\Sigma^{\alpha\beta}_{ab}\,\phi_b - \Theta^{\mu\alpha}\,\omega_{\alpha\beta}\,x^\beta$$

$$= \frac{1}{2}\left[\frac{\partial\mathcal{L}}{\partial\phi_{a,\mu}}\,\Sigma^{\alpha\beta}_{ab}\,\phi_b - \left(\Theta^{\mu\alpha}x^\beta - \Theta^{\mu\beta}x^\alpha\right)\right]\omega_{\alpha\beta}$$

$\omega_{\alpha\beta}$ arbitrary

$$\Rightarrow \quad \partial_\mu\underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi_{a,\mu}}\,\Sigma^{\alpha\beta}_{ab}\,\phi_b - \left(\Theta^{\mu\alpha}x^\beta - \Theta^{\mu\beta}x^\alpha\right)\right]}_{\mathcal{M}^{\mu\alpha\beta}} = 0$$

$$M^{\alpha\beta} = \int_\sigma d\sigma_\mu\;\mathcal{M}^{\mu\alpha\beta} = \int_{\mathbb{R}^3} d^3x\;\mathcal{M}^{0\alpha\beta} = \text{const.}$$

$$M^{\alpha\beta} = \int d^3x\;\Big(x^\alpha\Theta^{0\beta} - x^\beta\Theta^{0\alpha} + \underbrace{\underbrace{\frac{\partial\mathcal{L}}{\partial\dot\phi_a}}_{\pi_a}\,\Sigma^{\alpha\beta}_{ab}\,\phi_b}_{\text{spin contribution}}\Big)$$

energy-momentum density

<u>scalar field</u>: $\Sigma^{\alpha\beta} = 0$

$$\partial_\mu \left(x^\alpha \Theta^{\mu\beta} - x^\beta \Theta^{\mu\alpha} \right) = 0$$

$$\Rightarrow \quad \Theta^{\alpha\beta} - \Theta^{\beta\alpha} = 0$$

$\Rightarrow$ the <u>canonical</u> energy-momentum tensor of a <u>scalar</u> field is <u>symmetric</u>

<u>general case</u>:

$$\partial_\mu \mathcal{M}^{\mu\alpha\beta} = 0 \Rightarrow \quad \Theta^{\alpha\beta} - \Theta^{\beta\alpha} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \Sigma^{\alpha\beta}_{ab} \phi_b \right) = 0$$

we want to construct a <u>symmetric</u> energy-momentum tensor $T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\alpha f^{\alpha\mu\nu}$ by an appropriate choice of $f^{\alpha\mu\nu} = -f^{\mu\alpha\nu}$

$$0 = T^{\mu\nu} - T^{\nu\mu} = \Theta^{\mu\nu} + \partial_\alpha f^{\alpha\mu\nu} - \Theta^{\nu\mu} - \partial_\alpha f^{\alpha\nu\mu}$$

$$= \partial_\alpha \left(f^{\alpha\mu\nu} - f^{\alpha\nu\mu} \right) - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial \phi_{a,\alpha}} \Sigma^{\mu\nu}_{ab} \phi_b \right)$$

$$\Rightarrow \quad \partial_\alpha \left(f^{\alpha\mu\nu} - f^{\alpha\nu\mu} \right) = \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial \phi_{a,\alpha}} \Sigma^{\mu\nu}_{ab} \phi_b \right)$$

$$\Rightarrow \quad f^{\alpha\mu\nu} - f^{\alpha\nu\mu} = \underbrace{\frac{\partial \mathcal{L}}{\partial \phi_{a,\alpha}} \Sigma^{\mu\nu}_{ab} \phi_b + \partial_\sigma h^{\sigma\alpha\mu\nu}}_{= g^{\alpha\mu\nu}}$$

with $h^{\sigma\alpha\mu\nu} = - h^{\alpha\sigma\mu\nu} = - h^{\sigma\alpha\nu\mu}$

unique solution for $f^{\alpha\mu\nu}$:

$$f^{\alpha\mu\nu} - f^{\alpha\nu\mu} = g^{\alpha\mu\nu}$$

$$\underline{f^{\mu\nu\alpha}} \underbrace{- f^{\mu\alpha\nu}}_{f^{\alpha\mu\nu}} = g^{\mu\nu\alpha}$$

$$- \left(\underbrace{f^{\nu\alpha\mu}}_{\underline{-f^{\alpha\nu\mu}}} - \underbrace{f^{\nu\mu\alpha}}_{\underline{f^{\mu\nu\alpha}}} \right) = - g^{\nu\alpha\mu}$$

$$\Rightarrow \quad 2 f^{\alpha\mu\nu} = g^{\alpha\mu\nu} + g^{\mu\nu\alpha} - g^{\nu\alpha\mu}$$

$$f^{\alpha\mu\nu} = \frac{1}{2} \left(g^{\alpha\mu\nu} + g^{\mu\nu\alpha} - g^{\nu\alpha\mu} \right)$$

$$f^{\alpha\mu\nu} = \frac{1}{2}\left\{\frac{\partial \mathcal{L}}{\partial \phi_{a,\alpha}} \Sigma^{\mu\nu}_{ab}\phi_b + \frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \Sigma^{\nu\alpha}_{ab}\phi_b - \right.$$

$$\left. - \frac{\partial \mathcal{L}}{\partial \phi_{a,\nu}} \Sigma^{\alpha\mu}_{ab}\phi_b + \partial_\sigma\left(h^{\sigma\alpha\mu\nu} + h^{\sigma\mu\nu\alpha} - h^{\sigma\nu\alpha\mu}\right)\right\}$$

possible choice: $h^{\sigma\alpha\mu\nu} = 0$

$$\Rightarrow T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\alpha f^{\alpha\mu\nu} =$$

$$= \Theta^{\mu\nu} + \frac{1}{2}\partial_\alpha\left\{\frac{\partial \mathcal{L}}{\partial \phi_{a,\alpha}} \Sigma^{\mu\nu}_{ab}\phi_b + \frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \Sigma^{\nu\alpha}_{ab}\phi_b \right.$$

$$\left. - \frac{\partial \mathcal{L}}{\partial \phi_{a,\nu}} \Sigma^{\alpha\mu}_{ab}\phi_b\right\} \quad (\text{with } h = 0)$$

$$J^{\mu\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} =$$

$$= x^\alpha \Theta^{\mu\beta} - x^\beta \Theta^{\mu\alpha} + x^\alpha \partial_\sigma f^{\sigma\mu\beta} - x^\beta \partial_\sigma f^{\sigma\mu\alpha}$$

$$= x^\alpha \Theta^{\mu\beta} - x^\beta \Theta^{\mu\alpha} + \partial_\sigma\left(x^\alpha f^{\sigma\mu\beta} - x^\beta f^{\sigma\mu\alpha}\right)$$

$$\underbrace{- f^{\alpha\mu\beta} + f^{\beta\mu\alpha}}$$

$$f^{\mu\alpha\beta} - f^{\mu\beta\alpha} = \frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \Sigma^{\alpha\beta}_{ab}\phi_b + \partial_\sigma h^{\sigma\mu\alpha\beta}$$

$$\mathcal{J}^{\mu\alpha\beta} = \overbrace{\frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \Sigma_{ab}^{\alpha\beta} + x^{\alpha}\Theta^{\mu\beta} - x^{\beta}\Theta^{\mu\alpha}}^{\mathcal{M}^{\mu\alpha\beta}}$$

$$+ \partial_{\sigma}\left(x^{\alpha} f^{\sigma\mu\beta} - x^{\beta} f^{\sigma\mu\alpha} + h^{\sigma\mu\alpha\beta}\right)$$

$$\Rightarrow \int_{\sigma} d\sigma_{\mu}\, \mathcal{J}^{\mu\alpha\beta} = \int d\sigma_{\mu}\, \mathcal{M}^{\mu\alpha\beta}$$

choice of $h^{\sigma\mu\alpha\beta}$ $\rightarrow$ affects only <u>localization</u> of angular momentum

"correct" choice of $h^{\sigma\mu\alpha\beta}$?

where does a localization of field energy and field momentum play a rôle? $\rightarrow$ <u>general relativity</u>: energy-momentum tensor acts as the source of the gravitational field $\rightarrow$ <u>symmetric</u> $T^{\mu\nu}$ with $h^{\sigma\mu\alpha\beta} = 0$ (Belinfante)

What the graviton listens to

$$\delta S_{matter} = -\frac{1}{2}\int d^4x \sqrt{-g}\, T^{\mu\nu}\, \delta g_{\mu\nu} =$$

$$= \frac{1}{2}\int d^4x \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \;,\quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

$$-g = 1 + \eta^{\mu\nu} h_{\mu\nu} \;,\quad \sqrt{-g} = 1 + \frac{1}{2}\eta^{\mu\nu} h_{\mu\nu}$$

$$S_{matter} = \int d^4x \sqrt{-g}\, \underbrace{\left(A + g^{\mu\nu} B_{\mu\nu} + g^{\mu\nu} g^{\lambda\rho} C_{\mu\nu\lambda\rho} + \dots\right)}_{\mathcal{L}}$$

$C_{\mu\nu\lambda\rho} = C_{\lambda\rho\mu\nu}$

$$\Rightarrow \quad T_{\mu\nu} = 2\left(B_{\mu\nu} + 2\, C_{\mu\nu\lambda\rho}\eta^{\lambda\rho} + \dots\right) - \eta_{\mu\nu}\mathcal{L}$$

remark : $T^{\mu}{}_{\mu} = -\left(4A + 2\eta^{\mu\nu} B_{\mu\nu}\right)$ <u>no</u> contribution from $C_{\mu\nu\lambda\rho}$

<u>example</u> : massive spin 1 field

$$\mathcal{L} = -\frac{1}{4} g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} + \frac{m^2}{2} g^{\mu\nu} A_\mu A_\nu$$

$$\Rightarrow \quad T_{\mu\nu} = -F_{\mu\lambda} F_{\nu}{}^{\lambda} + m^2 A_\mu A_\nu - \eta_{\mu\nu}\mathcal{L}$$

symmetric energy-momentum tensor $T^{\mu\nu} = T^{\nu\mu}$, $\partial_\mu T^{\mu\nu} = 0$

correctly localizing energy and momentum

$$J^{\mu\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} \quad , \quad \partial_\mu J^{\mu\alpha\beta} = 0$$

$$\Rightarrow \quad J^{\alpha\beta} = \int_\sigma d\sigma_\mu \, J^{\mu\alpha\beta} = \int_\sigma d\sigma_\mu \, (x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha})$$

$$= \int_{\mathbb{R}^3} d^3x \, (x^\alpha T^{0\beta} - x^\beta T^{0\alpha}) = \text{const.}$$

$\Rightarrow$ density of angular momentum:

$$\mathcal{J}_k := \frac{1}{2} \varepsilon_{kij} \, J^{0ij} = \varepsilon_{kij} \, x^i \underbrace{T^{0j}}_{\text{momentum density}}$$

$\mathcal{E} := T^{00}$ energy density

$\wp^i := T^{0i}$ momentum density

$\mathcal{J}_k = \varepsilon_{kij} \, x^i \wp^j$ angular momentum density

total angular momentum $\vec{J} = \int_{\mathbb{R}^3} d^3x \; \vec{x} \times \vec{\wp}$

$$J^{i0} = \int_{\mathbb{R}^3} d^3x \, (x^i \mathcal{E} - x^0 \wp^i) = \text{const.}$$

dividing by the total energy $E := P^0 = \int_{\mathbb{R}^3} d^3x \, \mathcal{E} \Rightarrow$

$$\frac{\int d^3x \, \vec{x} \, \mathcal{E}}{\int d^3x \, \mathcal{E}} = \frac{\vec{P}}{E} t + \underbrace{\vec{a}_0}_{\text{const.}}$$

center of mass-energy (centroid) moves uniformly and rectilinearly with with velocity $\vec{P}/E$ with respect to the chosen reference frame

remark: position of the world line of centroid depends on the inertial frame used for its definition; only its direction is uniquely given by the 4-momentum P^μ

if $P^\mu P_\mu > 0$ (P^μ timelike) $\rightarrow$ restframe with $\vec{P} = 0$ exists $\rightarrow$ centroid with respect to rest frame $\equiv$ relativistic center of mass

angular momentum with respect to rest frame $\equiv$ spin

<u>assume</u>: P^μ timelike $(P^2 > 0)$

behaviour of P^μ, $J^{\mu\nu}$ under translations: P^μ unchanged (P^μ is a genuine 4-vector), but

$$J^{\mu\nu} \to \bar{J}^{\mu\nu} = \int d^3\bar{x}\, \left(\bar{x}^\mu T^{0\nu} - \bar{x}^\nu T^{0\mu}\right)$$

$$= J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu$$

under $x \to \bar{x} = x - a$

u^μ = 4-velocity of the inertial frame used to define the centroid (space-time split)

world line of centroid = set of points a with

$$\bar{J}^{\mu\nu} u_\nu = 0$$

$\to$ rest frame of u^μ: $u = (1, \vec{0})$

$$\Rightarrow \quad \bar{J}^{i0} = 0 = J^{i0} - a^i P^0 + a^0 P^i$$

$$\Rightarrow \quad a^i = \underbrace{\frac{P^i}{P^0}\, a^0}_{\frac{P^i}{E} t} + \underbrace{\frac{J^{i0}}{P^0}}_{a^i_0} \quad \checkmark$$

four-dimensional version:

$$0 = \bar{J}^{\mu\nu} u_\nu = J^{\mu\nu} u_\nu - a^\mu P.u + P^\mu a.u$$

$$\Rightarrow \quad a^\mu = \frac{P^\mu}{P.u} \underbrace{a.u}_{\lambda} + \frac{J^{\mu\nu} u_\nu}{P.u}$$

relativistic center of mass: choose reference frame with $u = P/\sqrt{P^2}$

$\Rightarrow$ world line of relativistic center of mass:

$$a^\mu = \frac{P^\mu}{\sqrt{P^2}} \lambda + \frac{J^{\mu\nu} P_\nu}{P^2}$$

(λ = proper time of relativistic center of mass)

<u>spin tensor</u> $S^{\mu\nu}$ = tensor of angular momentum with respect to the center of mass world line

$S^{\mu\nu}$ satisfies

$$S^{\mu\nu} P_\nu = 0 \qquad (u^\mu \sim P^\mu)$$

$S^{\mu\nu}$ contains the <u>same</u> information as the <u>relativistic spin vector</u>

$$S_\mu := \frac{1}{2} \varepsilon_{\mu\alpha\beta\gamma} S^{\alpha\beta} P^\gamma = \frac{1}{2} \varepsilon_{\mu\alpha\beta\gamma} J^{\alpha\beta} P^\gamma$$

the spin tensor $S_{\alpha\beta}$ can be reconstructed from the spin 4-vector as

$$S_{\alpha\beta} = - \varepsilon_{\alpha\beta\mu\nu} S^\mu P^\nu / P^2 =$$

$$= J_{\alpha\beta} + (P_\alpha J_{\beta\gamma} - P_\beta J_{\alpha\gamma}) P^\gamma / P^2$$

<u>remark</u>: this formula can also be obtained from

$$S^{\mu\nu} = J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu$$

with $a^\mu = \frac{P^\mu}{\sqrt{P^2}} \lambda + \frac{J^{\mu\nu} P_\nu}{P^2}$

S_μ is orthogonal to P^μ:

$$S_\mu P^\mu = 0$$

S_μ is <u>spacelike</u>: $S_\mu S^\mu = -\frac{1}{2} S_{\mu\nu} S^{\mu\nu} \underbrace{P^2}_{>0} < 0$