Exercises "Symmetries in Particle Physics"

1. A particle is moving in an external field. Which components of the momentum $\vec{p}$ and the angular momentum $\vec{L}$ are conserved?
   (a) Field of an infinite homogeneous plane.
   (b) Field of an infinite homogeneous circular cylinder.
   (c) Field of an infinite homogeneous prism.
   (d) Field of two points.
   (e) Field of an infinite homogeneous half plane.
   (f) Field of a homogeneous cone.
   (g) Field of a homogeneous annulus.
   (h) Field of an infinite homogeneous helix.

2. The Lagrange function
$$L' = L + \frac{df\big(q(t),t\big)}{dt}$$
differs from the Lagrange function L only by a total derivative. Show that L and L' lead to the same equations of motion.

3. The action for a particle moving in the potential $U(\vec{x}) = k/|\vec{x}|^2$ is invariant under a scaling transformation $t' = \lambda^\alpha t$, $\vec{x}\,' = \lambda^\beta \vec{x}$ for certain values of α and β ($\lambda > 0$). Determine α and β. By using Noether's theorem and the conservation of energy, derive the constant of motion associated with this scaling transformation. Check the time indepence of the obtained quantity by using the equations of motion.

4. Derive the field equation following from the action of a real scalar field $\varphi(x)$:
$$S = \int d^4x \, \frac{1}{2} \left(\partial^\mu \varphi \, \partial_\mu \varphi - m^2 \varphi^2\right).$$

5. Derive the field equation following from the action of a complex scalar field $\phi(x)$:
$$S = \int d^4x \, \left(\partial^\mu \phi^* \, \partial_\mu \phi - m^2 \phi^* \phi\right).$$

6. Derive the field equation following from the action of a massive (real) vector field $A_\mu(x)$:
$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu\right), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

7. Derive the inhomogeneous Maxwell's equations following from the action of a massless (real) vector field $A_\mu(x)$ coupled to an external conserved current $j^\mu(x)$ (Heaviside system with $c = 1$):

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \right), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

8. Establish the connection between the components of the electromagnetic field strength tensor $F_{\mu\nu}$ and the components of $\vec{E}$ and $\vec{B}$.

   Remember:

$$\begin{aligned} A^\mu &= (\phi, \vec{A}), \\ j^\mu &= (\rho, \vec{j}), \\ \vec{E} &= -\vec{\nabla}\phi - \dot{\vec{A}}, \\ \vec{B} &= \vec{\nabla} \times \vec{A}. \end{aligned}$$

9. The action

$$S = \int d^4x \left(\partial^\mu \phi^* \, \partial_\mu \phi - m^2 \phi^* \phi - V(\phi^* \phi) \right).$$

   of a selfinteracting complex scalar field $\phi(x)$ is invariant under a (global) U(1) gauge transformation

$$\phi(x) \to e^{-i\alpha} \phi(x).$$

   Derive the associated Nother current $j_\mu(x)$. Check the continuity equation $\partial_\mu j^\mu = 0$ by using the equations of motion.

10. The action integral of scalar electrodynamics,

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \phi)^* \, D_\mu \phi - m^2 \phi^* \phi \right),$$

   with the covariant derivative

$$D_\mu \phi = (\partial_\mu + iqA_\mu)\phi,$$

   describes a particle with mass m and electromagnetic charge q.

   Show the invariance of the Lagrangian under a *local* gauge transformation

$$\phi(x) \to e^{-iq\alpha(x)} \phi(x), \quad A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x).$$

   Hint: Determine the behaviour of $D_\mu \phi$ under the local gauge transformation.

11. Review the following group theoretic concepts:

    (a) group axioms,
    (b) Abelian group,
    (c) subgroup,
    (d) invariant subgroup.

12. Recapitulate the definitions of the following groups and verify the group axioms:

    (a) Lorentz group $\mathcal{L}$,
    (b) proper Lorentz group $\mathcal{L}_+$,
    (c) orthochronous Lorentz group $\mathcal{L}^\uparrow$,
    (d) proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$,
    (e) orthochorous Lorentz group $\mathcal{L}_0$.

13. The Poincaré group $\mathcal{P}$ consists of all elements of the form (L, a), with a translation four-vector a and $L \in \mathcal{L}$. The composition law is given by

$$(L', a')(L, a) = (L'L, L'a + a').$$

    Explain the physical origin of the composition law and verify the group properties of $\mathcal{P}$.

14. Show that the set $\mathcal{T}$ of pure translations $(\mathbb{1}, a)$ forms an Abelian invariant subgroup of $\mathcal{P}$.

15. Determine the canonical energy-momentum tensor Θ^μ_ν of φ^4 theory defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4.$$

    Verify $\partial_\mu\Theta^\mu{}_\nu = 0$ by using the equation of motion.

16. Determine the energy-momentum four-vector P^μ of φ^4 theory. What is the explicit form of the field energy $H = P^0$ and the field momentum $\vec{P}$?

17. Determine the canonical energy-momentum tensor of a (free) massive vector field. Verify $\partial_\mu\Theta^\mu{}_\nu = 0$ by using the equation of motion.

18. Determine the Belinfante-symmetrized energy-momentum tensor of a (free) massive vector field. Verify $\partial_\mu T^\mu{}_\nu = 0$ by using the equation of motion.

19. Use the result of the previous problem with $m = 0$ to derive the symmetric energy-momentum $T^{\mu\nu}$ of the electromagneic field. Express its components in terms of the components of $\vec{E}$ and $\vec{B}$. Discuss in particular the form and the physical interpretation of the energy density T^{00}, the momentum density T^{0i} and Maxwell's stress tensor $-T^{ij}$.

20. Alternative derivation of the energy density of the electromagnetic field: Consider the electromagnetic field energy contained in the spatial domain V,

$$\mathcal{E}_V^{\text{field}} = \int\limits_V d^3x \, \frac{1}{2} \left(\vec{E}^2 + \vec{B}^2 \right),$$

in the presence of charged point particles with energies

$$\mathcal{E}_a = \sqrt{m_a^2 + \vec{p}_a^{\,2}}, \qquad a = 1, \ldots .$$

Assume for simplicity that the particles are confined to the region V and do not pass its boundary ∂V. Compute

$$\frac{d}{dt} \mathcal{E}_V^{\text{field}} = \frac{d}{dt} \int\limits_V d^3x \, \frac{1}{2} \left(\vec{E}^2 + \vec{B}^2 \right)$$

by using Maxwell's equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \cdot \vec{E} = \rho, \quad \vec{\nabla} \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t}, \quad \vec{\nabla} \cdot \vec{B} = 0,$$

to show that the time rate of change of the total energy contained in the region V (field energy plus energies of the particles) can be expressed in terms of a surface integral over the boundary of V:

$$\frac{d}{dt} \left(\mathcal{E}_V^{\text{field}} + \sum_a \mathcal{E}_a \right) = - \int\limits_{\partial V} d\vec{f} \cdot \vec{S}.$$

21. Discuss the time-derivative of the total momentum contained in V,

$$\frac{d}{dt} \left(\vec{P}_V^{\text{field}} + \sum_a \vec{p}_a \right) = \ldots ,$$

in analogy to the previous problem. As your final result you should be able to derive Maxwell's stress tensor as the integrand of a surface integral.

22. Determine the electric field between the plates of an infinitely extended capacitor with homogeneous surface charge densities $\pm\sigma$ (see Fig. 1). Use Maxwell's stress tensor to compute the forces acting on the spatial domains V and V'.

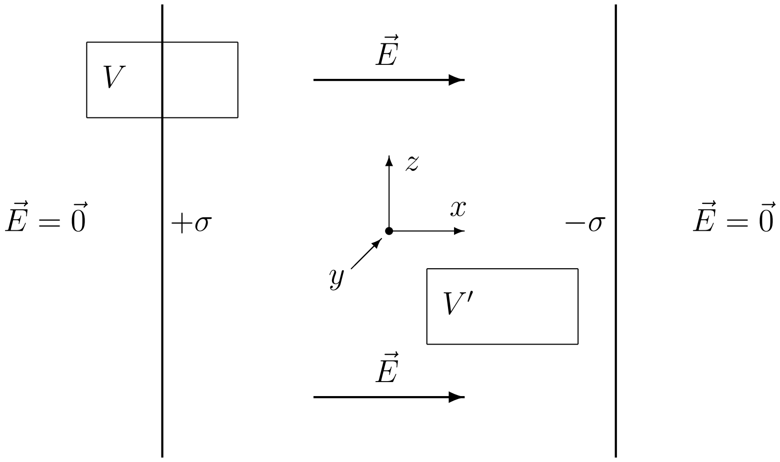

Figure 1: Electric field in a capacitor.

23. Two equal charges q are sitting at the points $\pm a\vec{e}_z/2$ (see Fig. 2). Sketch the electric field lines between the two charges. Use Maxwell's stress tensor to compute the force acting on a half-sphere with radius $R \to \infty$ located in the lower (upper) half-space with $z < 0$ ($z > 0$).

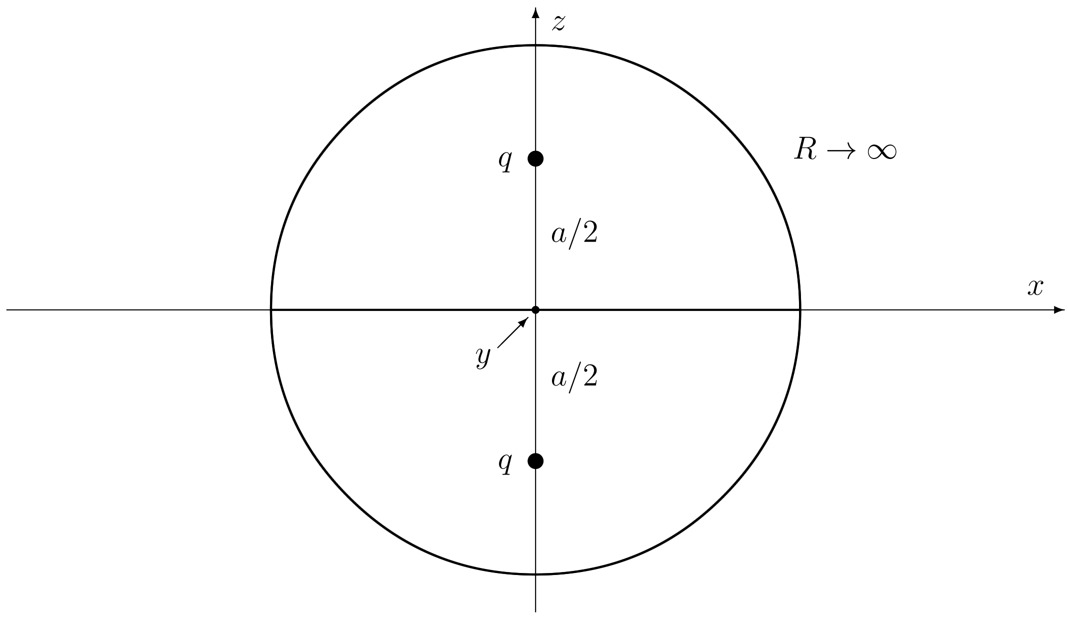

Figure 2: Forces between two charges.

24. Same as the previous problem but with two opposite charges $\pm q$ at the points $\pm a\vec{e}_z/2$.

25. The angular momentum of the electromagnetic field contained in a spatial region V is given by

$$\vec{J}_V^{\text{field}} = \int_V d^3x \; \vec{x} \times \left(\vec{E} \times \vec{B}\right).$$

Compute the time derivative

$$\frac{d}{dt}\left(\vec{J}_V^{\text{field}} + \sum_a \vec{L}_a\right) = \ldots$$

of the total angular momentum contained in V analogously to the approach employed in problems 20 and 21. In your final result, you should arrive at a surface integral, which can be interpreted as the torque acting on the system.

26. Demonstrate the group properties for $\mathrm{O}(3,\mathbb{R})$ and $\mathrm{SO}(3,\mathbb{R})$.

27. Show that the rotation of a vector $\vec{x}$ in three-dimensional space by the angle α around the rotation axis $\vec{n}$ ($|\vec{n}| = 1$, right-hand rule) is described by

$$\vec{x}\,' = R\left(\vec{\alpha}\right)\vec{x} = \cos\alpha\,\vec{x} + (1 - \cos\alpha)\,\vec{n}\,(\vec{n}\cdot\vec{x}) + \sin\alpha\,\vec{n}\times\vec{x}, \qquad \vec{\alpha} = \alpha\vec{n}.$$

Hint: Decompose the vector $\vec{x}$ in components parallel and orthogonal to the rotation axis and use the linearity of the transformation $R\left(\vec{\alpha}\right)$.

What is the matrix representation of $R\left(\vec{\alpha}\right)$ with respect to the standard orthonormal basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$?

28. Show that a real, orthogonal and unimodular 3×3 matrix O can always be written in the form $O = R\left(\vec{\alpha}\right)$ for an appropriate choice of the rotation vector $\vec{\alpha}$.

Hint: Interpret O as an element of $L(\mathbb{C}^3)$. What are the implications for the eigenvalues and eigenvectors of O from the conditions $O^* = O$, $O^T O = \mathbb{1}$ and $\det O = 1$? Use the spectral theorem for normal operators to derive your final result.

29. The generators of $\mathrm{SO}(3,\mathbb{R})$ in the defining representation were found to be $(\Lambda_k)_{ij} = -\varepsilon_{kij}$. Show that the matrix $\vec{n}\cdot\vec{\Lambda}$ ($|\vec{n}| = 1$) satisfies the relations

$$\left(\vec{n}\cdot\vec{\Lambda}\right)^2 = \vec{n}\,\vec{n}^T - \mathbb{1}, \quad \left(\vec{n}\cdot\vec{\Lambda}\right)^3 = -\vec{n}\cdot\vec{\Lambda}.$$

Use them to compute the power series expansion

$$\exp\left(\vec{\alpha}\cdot\vec{\Lambda}\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\vec{\alpha}\cdot\vec{\Lambda}\right)^k$$

and compare your result with $R\left(\vec{\alpha}\right)$.

30. Show that the vector space $\mathbb{R}^3$ with the usual cross product as the multiplication is a Lie algebra. Determine its structure constants!

31. The generators t_1, t_2, t_3 in an arbitrary representation of SO(3) satisfy the commutation relations $[t_i, t_j] = \varepsilon_{ijk} t_k$. A vector operator $\vec{v}$ can be defined by the property
$$\left[\vec{v}, \vec{\varepsilon} \cdot \vec{t}\,\right] = \vec{\varepsilon} \times \vec{v}.$$
Use these relations to show that
$$\left[\vec{v}^{\,2}, \vec{\varepsilon} \cdot \vec{t}\,\right] = 0.$$

32. Let (V, D) be a representation of the group G. A vector $v \in V$ is called cyclic if the linear span of $\{D(g)v | g \in G\}$ coincides with the whole vector space V. Show that a representation is irreducible if and only if every nonzero vector in V is cyclic.

33. Let $g \to D(g)$ be a matrix representation of a group G. Show that
$$g \to \left(D(g)^{-1}\right)^T$$
is also a representation (contragredient representation).

34. Let (V, D) and (V', D') be linear representations of some group G. Verifiy that the Kronecker product (tensor product) $(V \otimes V', D \otimes D')$ defined by
$$(D \otimes D')(g) := D(g) \otimes D'(g)\ \forall\, g \in G$$
is indeed a representation.

35. Let $g \to D(g)$ be a matrix representation of a group G in a complex vector space V. Show that $g \to D(g)^*$ is also a representation.

36. Give the definition of SU(2) and demonstrate its group properties.

37. Show that an element of SU(2) can always be written in the form
$$U(\vec{\alpha}) = \exp\left(\vec{\alpha} \cdot \vec{t}\,\right), \qquad \vec{t} = -i\vec{\sigma}/2,$$
with the Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. What can you say about the range of the real parameters $\vec{\alpha}$? Determine the commutator $[t_i, t_j]$. What do you find for the structure constants of SU(2)?

38. Write $U(\vec{\alpha})$ as a linear combination of the unit matrix $\mathbb{1}_2$ and the Pauli matrices. Perform the computation in at least two different ways!

39. Show that $\vec{\alpha} \to U(\vec{\alpha})$ and $\vec{\alpha} \to U(\vec{\alpha})^*$ are equivalent representations of SU(2).

40. We consider the space of tensors T_{ij} of rank 2 in three dimensions. The projection operator on the subspace of multiples of δ_{ij} is given by

$$(P_0)_{ij\,kl} = \frac{1}{3}\delta_{ij}\delta_{kl}.$$

Show that $P_0 P_0 = P_0$ is indeed fulfilled.

41. In the tensor space of the previous problem, the projection operator on the subspace of antisymmetric tensors is given by

$$(P_A)_{ij\,kl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right).$$

Verify $P_A P_A = P_A$.

42. Again in the same tensor space, the projection operator on the subspace of symmetric tensors is given by

$$(P_S)_{ij\,kl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right).$$

Verify $P_S P_S = P_S$ and $P_A + P_S = \mathbb{1} \otimes \mathbb{1}$.

43. Compute $P_0 P_S$ and $P_S P_0$. Use these results to show that $P_S - P_0$ is a projector (on the subspace of tracelesss symmetric tensors). Verify also:

$$\begin{aligned}
P_0 P_A = P_A P_0 &= 0, \\
P_0(P_S - P_0) = (P_S - P_0)P_0 &= 0, \\
P_A(P_S - P_0) = (P_S - P_0)P_A &= 0, \\
P_0 + P_A + (P_S - P_0) &= \mathbb{1} \otimes \mathbb{1}.
\end{aligned}$$

44. Express $\Lambda_m \otimes \Lambda_m$ in terms of P_0, P_A and P_S, where

$$(\Lambda_m)_{ik} = -\varepsilon_{mik}.$$

45. Show that

$$\frac{1}{2}\mathrm{Tr}\left(\sigma_i U(\vec{\alpha})\sigma_j U(\vec{\alpha})^\dagger\right) = \left(R(\vec{\alpha})\right)_{ij},$$

where $U(\vec{\alpha}) \in \mathrm{SU}(2)$ was defined in ex. 37 and $R(\vec{\alpha}) \in \mathrm{SO}(3)$ in ex. 27.

46. Show: $U \in \mathrm{SU}(2)$ can be expressed in terms of $R \in \mathrm{SO}(3)$ by the relation

$$U = \pm\frac{\mathbb{1}_2 + R_{ij}\sigma_i\sigma_j}{2\sqrt{1 + \mathrm{Tr}R}}.$$

47. The operator ad_X, acting on the space $L(V)$ of linear operators on a (finite-dimensional) vector space V, is defined by

$$\mathrm{ad}_X Y := [X, Y], \quad X, Y \in L(V).$$

Be $t_1, \ldots, t_n$ a basis for the generators of some Lie group G in a faithful representation with commutation relations

$$[t_a, t_b] = C_{ab}{}^c t_c.$$

Determine the commutator

$$[\mathrm{ad}_{t_a}, \mathrm{ad}_{t_b}].$$

48. Show that an SU(2) matrix U can always be written in the form

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix},$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.

49. We consider the two-dimensional representation $D^{(1/2)}$ of SU(2) with the generators

$$\vec{S} = \vec{\sigma}/2.$$

The spinors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are eigenvectors of S_3 and form a basis of the representation space. Determine $S_\pm \chi_\pm$ (in all possible combinations).

50. The totally symmetric part of $D^{(1/2)} \otimes D^{(1/2)} \otimes D^{(1/2)}$ corresponds to the irreducible representation $D^{(3/2)}$. Express the canonical basis

$$|3/2,\ 3/2\rangle\,,\ |3/2,\ 1/2\rangle\,,\ |3/2,\ -1/2\rangle\,,\ |3/2,\ -3/2\rangle$$

as linear combinations of the totally symmetric three-fold tensor products of $\chi_\pm$.

Hint: Start with

$$|3/2,\ 3/2\rangle = \chi_+ \otimes \chi_+ \otimes \chi_+$$

and apply

$$J_- = S_- \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes S_- \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes S_-.$$

51. Compute $U\chi_\pm$, using the SU(2) parametrization introduced in ex. 48.

52. Compute

$$D^{(3/2)}(\vec{\alpha})\,|3/2,\ m\rangle = U(\vec{\alpha}) \otimes U(\vec{\alpha}) \otimes U(\vec{\alpha})\,|3/2,\ m\rangle\,, \quad m = 3/2, \ldots, -3/2$$

using the result of the previous problem. Determine the matrix representation of $D^{(3/2)}(\vec{\alpha})$ (expressed in terms of the parameters a, b) with respect to the canonical basis and check its unitarity.

53. An infinitesimal Lorentz transformation

$$\begin{aligned} t' &= t + \vec{u}\cdot\vec{x}, \\ \vec{x}' &= \vec{x} + \underbrace{\vec{\alpha}\times\vec{x}}_{\text{rotation}} + \underbrace{\vec{u}\,t}_{\text{boost}} \end{aligned}$$

can be written as

$$x' = L(\vec{\alpha},\vec{u})x$$

with

$$L(\vec{\alpha},\vec{u}) = \mathbb{1} + \vec{\alpha}\cdot\vec{M} + \vec{u}\cdot\vec{N}.$$

Show that the generators $\vec{M},\vec{N}$ assume the form

$$M_i = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & \Lambda_i \end{pmatrix}, \qquad N_i = \begin{pmatrix} 0 & \vec{e}_i^{\,T} \\ \vec{e}_i & 0_3 \end{pmatrix},$$

where the 3×3 matrices Λ_i were defined in ex. 29 and $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ denotes the standard orthonormal basis of $\mathbb{R}^3$.

54. Verify the commutation relations

$$[M_i, M_j] = \varepsilon_{ijk}M_k, \quad [N_i, N_j] = -\varepsilon_{ijk}M_k, \quad [N_i, M_j] = \varepsilon_{ijk}N_k.$$

55. Show that the complex linear combinations

$$\vec{M}^{\pm} = \frac{1}{2}\left(\vec{M} \pm i\vec{N}\right)$$

satisfy the commutation relations

$$\left[M_i^{\pm}, M_j^{\pm}\right] = \varepsilon_{ijk}M_k^{\pm}, \quad \left[M_i^{+}, M_j^{-}\right] = 0.$$

56. Determine the explicit form of the finite boost

$$L\left(\vec{0},\vec{v}\right) = \exp\left(\vec{u}\cdot\vec{N}\right) = \exp\left(u\,\vec{n}\cdot\vec{N}\right), \quad |\vec{n}| = 1,\, u \geq 0.$$

Find the relation of the velocity $\vec{v}$ with the rapidity vector $\vec{u}$ by comparing your result with

$$L\left(\vec{0},\vec{v}\right) = \begin{pmatrix} \gamma & \gamma\vec{v}^T \\ \gamma\vec{v} & \mathbb{1}_3 + \frac{\gamma^2}{1+\gamma}\vec{v}\vec{v}^T \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-\vec{v}^2}}$$

57. Verify the relation

$$L\left(\vec{\alpha},\vec{0}\right)L\left(\vec{0},\vec{v}\right) = L\left(\vec{0},R(\vec{\alpha})\vec{v}\right)L\left(\vec{\alpha},\vec{0}\right).$$

58. Compute $\mathrm{Tr}(\bar{\sigma}^\mu \sigma^\nu)$, where $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$, $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$.

59. Compute $\det(A \otimes \mathbb{1}_2)$, where A is a 2×2 matrix.

    Hint: Consider the matrix representation of $A \otimes \mathbb{1}_2$ with respect to the basis $\{e_1 \otimes e_1,\, e_2 \otimes e_1,\, e_1 \otimes e_2,\, e_2 \otimes e_2\}$.

60. Show:

$$\exp(-\vec{u} \cdot \vec{\sigma}/2) = \cosh\frac{u}{2} - \vec{n} \cdot \vec{\sigma} \sinh\frac{u}{2}, \qquad \vec{u} = u\,\vec{n}, |\vec{n}| = 1, u \geq 0.$$

61. The formula

$$L^\mu_{\ \nu} = \frac{1}{2} \mathrm{Tr}\left(\bar{\sigma}^\mu A \sigma_\nu A^\dagger\right)$$

    maps $A \in \mathrm{SL}(2, \mathbb{C})$ to $L = (L^\mu_{\ \nu}) \in \mathcal{L}^\uparrow_+$. Determine L for $A = \exp(-u\sigma^3/2)$.

62. Determine $L \in \mathcal{L}^\uparrow_+$ for the general boost $A = \exp(-\vec{u} \cdot \vec{\sigma}/2)$

63. Show that only $-A \in \mathrm{SL}(2, \mathbb{C})$ effects the same transformation,

$$X \to X' = AXA^\dagger,$$

    as does $A \in \mathrm{SL}(2, \mathbb{C})$.

64. The infinitesimal Lorentz transformation discussed in ex. 53 can also be written in the form

$$x'^{\,\mu} = x^\mu + \omega^\mu_{\ \nu} x^\nu.$$

    Discuss the relation between the parameters $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and the rotation angle $\vec{\alpha}$ and the boost vector $\vec{u}$.

65. Write the infinitesimal Lorentz transformation $L(\vec{\alpha}, \vec{u})$ defined in ex. 53 in the form

$$L(\vec{\alpha}, \vec{u}) = \mathbb{1} + \frac{1}{2} \omega_{\alpha\beta} \Sigma^{\alpha\beta}$$

    and express the generators $\Sigma^{\alpha\beta} = -\Sigma^{\beta\alpha}$ in terms of the elements of the previously defined matrices M_i and N_i.

66. Determine the commutation relations of the generators $\Sigma^{\alpha\beta}$ of $\mathcal{L}^\uparrow_+$,

$$\left[\Sigma^{\alpha\beta}, \Sigma^{\gamma\delta}\right] = \ldots$$

    and show the equivalence of your result with the formulae given in ex. 54.

67. Prove the following relations for the spinors φ, χ:

$$\varphi\chi = \chi\varphi, \quad \bar{\varphi}\bar{\chi} = \bar{\chi}\bar{\varphi}, \quad \varphi\sigma^\mu\bar{\chi} = -\bar{\chi}\bar{\sigma}^\mu\varphi.$$

68. Show:

$$\int d^4x\, \bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi = \int d^4x\, \chi\sigma^\mu\partial_\mu\bar{\chi}.$$

69. Verify the following relations:

$$\begin{aligned}
\left(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu\right)_\alpha^{\ \beta} &= 2g^{\mu\nu}\delta_\alpha^{\ \beta}, \\
\left(\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}_{\ \dot{\beta}} &= 2g^{\mu\nu}\delta^{\dot{\alpha}}_{\ \dot{\beta}}.
\end{aligned}$$

70. Compute $\sigma^\mu p_\mu\, \bar{\sigma}^\nu p_\nu$ and $\bar{\sigma}^\mu p_\mu\, \sigma^\nu p_\nu$ using the relations displayed in the previous problem.

71. The γ-matrices in the Weyl basis are defined by

$$\gamma^\mu := \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

Discuss the index structure and verify the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$$

using the formulae shown in 69.

72. Show:

$$\gamma_\mu^\dagger = \beta\gamma_\mu\beta, \quad \beta = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

73. In two-component notation, the Dirac equation is given by

$$i\sigma^\mu\partial_\mu\bar{\varphi} = m\chi, \quad i\bar{\sigma}^\mu\partial_\mu\chi = m\bar{\varphi}.$$

Show that χ and $\bar{\varphi}$ fulfil the Klein Gordon equation,

$$\left(\Box + m^2\right)\chi = 0, \quad \left(\Box + m^2\right)\bar{\varphi} = 0.$$

74. Show:

$$S = \begin{pmatrix} e^{-i(\vec{\alpha}-i\vec{u})\cdot\frac{\vec{\sigma}}{2}} & 0 \\ 0 & e^{-i(\vec{\alpha}+i\vec{u})\cdot\frac{\vec{\sigma}}{2}} \end{pmatrix} = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}$$

with

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right].$$

75. Verify the spinor relations $\bar{u}(\vec{p},r)u(\vec{p},s) = 2m\delta_{rs}$, $\bar{v}(\vec{p},r)v(\vec{p},s) = -2m\delta_{rs}$.

76. Compute: $\gamma_\mu\gamma^\mu, \quad \gamma_\mu\gamma^\nu\gamma^\mu, \quad \gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu.$

77. Show that $V_{\alpha\dot{\alpha}} := \sigma^\mu_{\alpha\dot{\alpha}} V_\mu \Rightarrow V^\mu = \frac{1}{2}\left(\bar{\sigma}^\mu\right)^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}}$.

Instructions for problems 78-81: Let $(L,a) \in \mathcal{P}^\uparrow_+ \to U(L,a)$ be a faithful representation of $\mathcal{P}^\uparrow_+$ in a space $\mathcal{H}$. An infinitesimal transformation $(L^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu)$ is written in the form

$$U(L,a) \simeq \mathbb{1}_\mathcal{H} - \frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta} + ia_\mu P^\mu,$$

where the factor i is pulled out to obtain hermitean generators $J^{\alpha\beta}, P^\mu$ in the case of a unitary representation. Verify the relations given in 78-81.

Hint: Start with

$$U(L,0)^{-1}U(L',a')U(L,0) = \ldots, \quad U(\mathbb{1},a)^{-1}U(L',a')U(\mathbb{1},a) = \ldots$$

for finite (L,a) but infinitesimal (L',a').

78. $U(L,0)^{-1}J^{\alpha\beta}U(L,0) = L^\alpha{}_\mu L^\beta{}_\nu J^{\mu\nu}$

79. $U(L,0)^{-1}P^\mu U(L,0) = L^\mu{}_\nu P^\nu$

80. $U(\mathbb{1},a)^{-1}J^{\alpha\beta}U(\mathbb{1},a) = J^{\alpha\beta} + a^\alpha P^\beta - a^\beta P^\alpha$

81. $U(\mathbb{1},a)^{-1}P^\mu U(\mathbb{1},a) = P^\mu$

82. Verify the Poincaré algebra,

$$\begin{aligned}
i\left[J^{\alpha\beta}, J^{\gamma\delta}\right] &= g^{\alpha\gamma}J^{\beta\delta} - g^{\beta\gamma}J^{\alpha\delta} + g^{\alpha\delta}J^{\gamma\beta} - g^{\beta\delta}J^{\gamma\alpha}, \\
i\left[J^{\alpha\beta}, P^\gamma\right] &= g^{\gamma\alpha}P^\beta - g^{\gamma\beta}P^\alpha, \\
i\left[P^\alpha, P^\beta\right] &= 0,
\end{aligned}$$

using the results obtained in 78-81 for infinitesimal (L,a).

Compare the first formula with your result obtained for ex. 66!

83. Show that

$$(L,a) \to \begin{pmatrix} L & a \\ 0 & 1 \end{pmatrix}$$

is a 5×5 matrix representation of the Poincaré group. Note that this representation is reducible, but not decomposable ($\mathcal{P}^\uparrow_+$ is not semisimple).

84. Derive the commutation relations of the Poincaré algebra using the representation shown in the previous problem.

Free scalar quantum field theory: The commutation relations for the creation and annihilation operators of a quantized hermitean scalar field $\phi(x)$ with mass m are given by

$$\left[a(p), a(p')^\dagger\right] = \underbrace{(2\pi)^3 2p^0 \delta^{(3)}(\vec{p} - \vec{p}')}_{\delta(p,p')} \ , \quad [a(p), a(p')] = 0,$$

where $p^0 = \sqrt{m^2 + \vec{p}^2}$.

The ground state (vacuum state) $|0\rangle$ is characterized by $a(p)|0\rangle = 0 \ \forall \ \vec{p}$ and the normalization condition $\langle 0|0\rangle = 1$. The one-particle momentum eigenstate $|p\rangle$ is defined by $|p\rangle = a(p)^\dagger|0\rangle$. The general form of a normalizable one-particle state $|\psi^{(1)}\rangle$ is given by

$$|\psi^{(1)}\rangle = \int \underbrace{\frac{d^3p}{(2\pi)^3 2p^0}}_{d\mu(p)} \ |p\rangle \, \psi^{(1)}(p).$$

In the case of n particles, one defines $|p_1, \ldots p_n\rangle = a(p_1)^\dagger \ldots a(p_n)^\dagger|0\rangle$, being an eigenstate of the four-momentum operator P^μ with eigenvalue $p_1^\mu + \ldots p_n^\mu$. This n particle state obeys the normalization condition

$$\langle p_1, \ldots p_n | k_1, \ldots k_n\rangle = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} \delta(p_i, k_{\sigma(i)}).$$

The Fourier decomposition of the real scalar field is given by

$$\phi(x) = \int d\mu(p) \ \left[a(p)e^{-ipx} + a(p)^\dagger e^{ipx}\right].$$

85. A Poincaré transformation (L, a) is represented by a unitary operator $U(L, a)$ acting on the Fock space of the scalar field theory. Make an educated guess how $U(L, a)$ acts on the basis vectors

$$|0\rangle, \quad |p\rangle, \quad |p_1, p_2\rangle, \ldots.$$

86. Using your result for $U(L, a)$, verify the composition rule $U(L', a')U(L, a) = U(L'L, L'a + a')$ of the Poincaré group.

87. Determine $U(L, a)a^\dagger(p)U(L, a)^{-1}$ and $U(L, a)a(p)U(L, a)^{-1}$.

88. Determine $U(L, a)\phi(x)U(L, a)^{-1}$.

89. Determine the action of a Poincaré transformation on a general one-particle state

$$U(L, a)|\psi^{(1)}\rangle = \ldots.$$

Discuss in particular the behaviour of the momentum space wave function $\psi^{(1)}(p)$ under this transformation.

90. The Pauli-Lubanski vector is defined by

$$W_\mu := \frac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}J^{\alpha\beta}P^\gamma.$$

Prove the following properties of W_μ and $W^2 := W_\mu W^\mu$:

$$\begin{aligned} W_\mu P^\mu &= 0, \\ [P_\mu, W_\nu] &= 0, \\ \left[U(L,a), W^2\right] &= 0, \\ [W_\mu, W_\nu] &= -i\varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma. \end{aligned}$$

91. Determine the Pauli-Lubanski vector for a Dirac field and compute W^2.

92. Determine the Pauli-Lubanski vector for a Weyl field and compute W^2.

93. Determine the Pauli-Lubanski vector for a vector field V^μ and compute W^2. What is the result for W^2 if the vector field fulfills $\partial_\mu V^\mu = 0$?

94. Determine the Pauli-Lubanski vector for an antisymmetric tensor field $T^{\mu\nu}$ and compute W^2.

    Hint: An algebraic computer programme like FORM might facilitate the task.

95. Derive the commutation relations of the generators T_x, T_y, M of the euclidean group $E(2)$ of translations and rotations in the two-dimensional plane,

$$\vec{x}' = D\vec{x} + \vec{b},$$

by employing the matrix representation

$$\left(D, \vec{b}\right) \to \begin{pmatrix} D & \vec{b} \\ 0 & 1 \end{pmatrix}, \quad D \in \mathrm{SO}(2),\, \vec{b} \in \mathbb{R}^2.$$