27. Chiral anomalies

consider free massless Dirac field $\psi(x)$

$$-i\gamma^\mu \partial_\mu \psi(x) = 0$$

p. 9/10

$\Rightarrow$ Green function $S_\circ(x) = i\not\partial \, \Delta_\circ(x) = i\not\partial \, \dfrac{i}{4\pi^2(-x^2+i\varepsilon)}$

$m=0$

$$= \frac{-x_\mu \gamma^\mu}{2\pi^2(x^2-i\varepsilon)^2}$$

p. 9/9 : $\langle 0|T\psi(x)\,\bar\psi(y)|0\rangle = \dfrac{1}{i}\, S(x-y)$

vector current: $V^\mu(x) = :\bar\psi(x)\gamma^\mu\psi(x):$

$$= \bar\psi(x)\gamma^\mu\psi(x) - \bar\psi(x)\gamma^\mu\psi(x)$$

$$= \bar\psi(x)\gamma^\mu\psi(x) - i\,\mathrm{Tr}(\gamma^\mu S(0))$$

axial current: $A^\mu(x) = :\bar\psi(x)\gamma^\mu\gamma_5\psi(x):$

$$= \bar\psi(x)\gamma^\mu\gamma_5\psi(x) - i\,\mathrm{Tr}(\gamma^\mu\gamma_5 S(0))$$

left-handed current: $L^\mu(x) = :\bar\psi(x)\dfrac{1-\gamma_5}{2}\psi(x): = \dfrac{1}{2}(V^\mu - A^\mu)$

current conservation $\partial_\mu V^\mu = \partial_\mu A^\mu = \partial_\mu L^\mu = 0$

two-point function of vector current:

$$G^{\mu\nu}(x-y) = \langle 0|T\,V^\mu(x)\,V^\nu(y)|0\rangle = \mathrm{Tr}[\gamma^\mu S_\circ(x-y)\gamma^\nu S_\circ(y-x)]$$

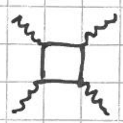
distribution $G^{\mu\nu}(x)$ is well defined only on test functions vanishing at $x=0$ together with their first and second derivatives $\rightarrow$ on these, it obeys the Ward identity $\partial_\mu G^{\mu\nu}(x) = 0$

$\rightarrow$ extension of $G^{\mu\nu}(x)$ to arbitrary test functions is not unique ; $\exists$ extensions which obey $\partial_\mu G^{\mu\nu} = 0$ and are Lorentz covariant $\rightarrow$ these two requirements determine $G^{\mu\nu}(x)$ up to one free parameter ; two extensions of this type differ by

$$\overline{G}^{\mu\nu}(x) = G^{\mu\nu}(x) + ic\,(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)\,\delta^{(4)}(x)$$

three-point function $\langle 0|T\,V^\lambda(x_1)\,V^\mu(x_2)\,V^\nu(x_3)|0\rangle$ vanishes because of charge conjugation invariance

p. 9/16 : $V^\mu \xrightarrow{C} -V^\mu$

four-point function $\langle 0|T\,V^{\mu_1}(x_1)\,V^{\mu_2}(x_2)\,V^{\mu_3}(x_3)\,V^{\mu_4}(x_4)|0\rangle_c$ contains a logarithmic divergence (box graph )

$\sim c\,(g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3})\,\delta^{(4)}(x_1-x_2)\,\delta^{(4)}(x_1-x_3)$
$\times\ \delta^{(4)}(x_1-x_4)$, but current conservation $\Rightarrow c=0$

Green functions with more than four currents are unambigious and conserved

$\Rightarrow$ 1) Green functions of the free vector current can be chosen such that

$$\partial_{\mu_1} \langle 0 | T\, V^{\mu_1} \dots V^{\mu_n} | 0 \rangle = 0$$ <u>Ward identities</u>

2) only the two-point function has a renormalization ambiguity

<u>generating functional</u>

$$e^{iW_V[f]} = \langle 0 | T\, e^{i \int d^4x\, f_\mu(x) V^\mu(x)} | 0 \rangle$$

$$W_V[f] = \frac{i}{2!} \int d^4x_1\, d^4x_2\, f_{\mu_1}(x_1)\, f_{\mu_2}(x_2) \langle 0 | T\, V^{\mu_1}(x_1)\, V^{\mu_2}(x_2) | 0 \rangle$$

$$+ \frac{i^3}{4!} \int d^4x_1 \dots d^4x_4\, f_{\mu_1}(x_1) \dots f_{\mu_4}(x_4) \langle 0 | T V^{\mu_1}(x_1) \dots V^{\mu_4}(x_4) | 0 \rangle_c$$

$$+ \dots$$

compact form of Ward identities: $W_V[f]$ <u>invariant</u> under the <u>gauge transformation</u> $f_\mu(x) \to f_\mu(x) + \partial_\mu \alpha(x)$:

$$W_V[f + \partial\alpha] = W_V[f]$$

renormalization ambiguity of two-point function $\Rightarrow$

$\Rightarrow$ generating functional unique up to local polynomial in the external field:

$$\overline{W}_V[f] = W_V[f] + \frac{\bar{c}}{4}\int d^4x \, \overbrace{(\partial_\mu f_\nu - \partial_\nu f_\mu)^2}^{\text{gauge invariant}}$$

Green functions of the free left-handed currents

$$e^{iW_L[f]} = \langle 0 | T \, e^{i\int d^4x \, f_\mu(x) L^\mu(x)} | 0 \rangle$$

as before, the short distance singularity of the free propagator generates ambiguities in the Green functions with up to four currents:

- 2- and 4-point function can be renormalized in such a manner that current conservation holds
- impossible for 3-point function: independently of how one extends the distribution $\langle 0 | T \, L^\lambda(x) L^\mu(y) L^\nu(z) | 0 \rangle$ to the space of all test functions, at least one of the currents fails to be conserved at $x = y = z$; if the 3-point function is renormalized in such a way that it is symmetric with respect to the interchange of any two of the three currents, one finds

$$\partial_\lambda \langle 0|T L^\lambda(x) L^\mu(y) L^\nu(z)|0\rangle = -\frac{1}{12\pi^2}\varepsilon^{\mu\nu\alpha\beta}\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial z^\beta}\left[\delta^{(4)}(x-y)\,\delta^{(4)}(x-z)\right]$$

$\Rightarrow$ generating functional is <u>not</u> gauge invariant:

$$W_L[f+\partial\alpha] = W_L[f] + \frac{1}{24\pi^2}\int d^4x\,\alpha(x)\,\varepsilon^{\mu\nu\rho\sigma}\partial_\mu f_\nu(x)\,\partial_\rho f_\sigma(x)$$

<u>Anomaly in two-dimensional space-time:</u>

$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$, $\mu = 0,1$ $\quad g^{00}=1,\ g^{11}=-1,\ g^{01}=0$

possible representation by 2×2 matrices:

$$\gamma^0 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \gamma^1 = i\sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\gamma_5 := \gamma_0\gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma_z$$

$$\frac{1-\gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad \frac{1+\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^0 = \bar\psi\gamma^0\gamma_5\psi = \bar\psi\underbrace{\gamma^0\gamma_0}_{\mathbb{1}}\underbrace{\gamma_1}_{-\gamma^1}\psi = -\bar\psi\gamma^1\psi = -V^1$$

$$A^1 = \bar\psi\gamma^1\gamma_5\psi = \bar\psi\gamma^1\gamma_0\gamma_1\psi = -\bar\psi\underbrace{\gamma^1\gamma_1}_{+\mathbb{1}}\underbrace{\gamma_0}_{\gamma^0}\psi = -\bar\psi\gamma^0\psi = -V^0$$

$$L^- = L^0 - L^1 = (V^0 - A^0) - (V^1 - A^1) = (V^0 + V^1) - (V^1 + V^0)$$
$$\equiv 0$$

$x^+ := x^0 + x^1 \,, \quad x^- = x^0 - x^1 \,, \quad J^\pm = J^0 \pm J^1$

$$\partial_+ := \frac{\partial}{\partial x^+} \,, \quad \partial_- := \frac{\partial}{\partial x^-}$$

$$\partial_\mu J^\mu = \partial_0 J^0 + \partial_1 J^1 = \frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} = \partial_+ J^+ + \partial_- J^-$$

$$0 = \partial_\mu L^\mu = \partial_+ L^+ + \partial_- \underbrace{L^-}_{0} = \partial_+ L^+$$

$\Rightarrow \quad L^+ = L^+(x^-) \quad \Rightarrow \quad L^+$ is a free field

as $\quad \Box = 4 \partial_- \partial_+$

free propagator in $d=2$:

$$S_0(x) = \frac{x^\mu \gamma_\mu}{2\pi (x^2 - i\varepsilon)} = i \not{\partial} \Delta_0(x)$$

$$\Delta_0(x) = \frac{1}{4\pi i} \ln(-x^2 + i\varepsilon) \qquad \text{scalar propagator in } d=2$$

$$\Box \Delta_0(x) = \delta^{(2)}(x)$$

remark: scalar propagator in $d=2$ can be obtained from scalar propagator in $d=2$ by method of Hadamard:

$$\Delta_0^{(d=2)} = \int dx^2 dx^3 \, \Delta_0^{(d=4)}(x) + C$$

$$= \int dx^2 dx^3 \frac{i}{4\pi^2(-x^2 + i\varepsilon)} + C = \frac{i}{4\pi^2} \int_0^\infty dr \, r \int_0^{2\pi} d\varphi \, \frac{1}{-(x^0)^2 + (x^1)^2 + r^2 + i\varepsilon} + C$$

polar coordinates $x^2 = r\cos\varphi$, $x^3 = r\sin\varphi$

variable transformation $u = r^2 \Rightarrow du = 2r\,dr$

$$\Rightarrow \Delta_0^{(d=2)}(x^0, x^1) = \frac{i}{4\pi} \int_0^\infty du \, \frac{1}{-x^2 + i\varepsilon + u} + C$$

$$= \frac{i}{4\pi} \ln(-x^2 + i\varepsilon + u) \Big|_0^\infty + C = \frac{1}{4\pi i} \ln(-x^2 + i\varepsilon)$$

with appropriate choice of C

two-point function of L^+:

$$\langle 0 | T\, L^+(x)\, L^+(y) | 0 \rangle = -\operatorname{Tr}(\gamma^+ \not{z}\, \gamma^+ \not{z}) \frac{1}{4\pi^2 (z^2 - i\varepsilon)^2}$$

with $z = x - y$
$$= \frac{i}{\pi} \partial_- \partial_- \Delta_0(z)$$

$$\Rightarrow \partial_+ \langle 0 | T\, L^+(x)\, L^+(y) | 0 \rangle = \frac{i}{\pi} \partial_- \underbrace{\partial_+ \partial_-}_{\frac{1}{4}\Box} \Delta_0(z)$$

$$= \frac{i}{4\pi} \partial_- \delta^{(2)}(z)$$

<u>d=2</u>: • <u>Ward identity</u> of <u>two-point function</u> contains an <u>anomaly</u>

• all Green functions containing <u>more</u> than two currents vanish (reason $L^+(x)$ is a free field)

$\Rightarrow$ generating functional can be given in closed form:

$$W_L[f] = \frac{1}{8\pi} \int d^2x\, d^2y\; \partial_- f_+(x)\, \Delta_0(x-y)\, \partial_- f_+(y)$$

<u>not</u> gauge invariant!

general case:

d odd $\rightarrow$ no analogue of γ_5: $\gamma^0 \gamma^1 \dots \gamma^{d-1} \sim \mathbb{1}$

(example in $d=3$: $\gamma^0 = \sigma_x$, $\gamma^1 = i\sigma_x$, $\gamma^2 = i\sigma_z$

$\Rightarrow \gamma^0 \gamma^1 \gamma^2 = i \mathbb{1}_2$) $\longrightarrow$ no left-handed or axial currents $\longrightarrow$ no anomalies

d even $\rightarrow$ anomaly in the Ward identities for Green functions with $\frac{d}{2}+1$ left-handed currents (vacuum polarization diagram in $d=2$, triangle graph in $d=4$, ...)

fermions in an external field

"physical" interpretation of generating functional: vacuum-to-vacuum transition amplitude in the presence of an external field:

$$\langle 0 \text{ out} | 0 \text{ in} \rangle_f = Z[f] = e^{iW[f]}$$

decomposition of a Dirac spinor into two Weyl spinors:

$$\psi = \begin{bmatrix} \chi_\alpha \\ \bar{\varphi}^{\dot\alpha} \end{bmatrix}, \quad \psi_L = \begin{bmatrix} \chi_\alpha \\ 0 \end{bmatrix}, \quad \psi_R = \begin{bmatrix} 0 \\ \bar{\varphi}^{\dot\alpha} \end{bmatrix}$$

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu_{\alpha\dot\beta} \\ \bar{\sigma}^{\mu\dot\alpha\beta} & 0 \end{bmatrix}, \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\bar{\sigma}^{\mu\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}, \quad \varepsilon^{12} = +1, \quad \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha{}^\gamma$$

$$\varepsilon_{12} = -1$$

$$\Rightarrow \bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$$

$$\chi^\alpha = \varepsilon^{\alpha\beta}\chi_\beta, \quad \chi_\alpha = \varepsilon_{\alpha\beta}\chi^\beta$$

$$\bar{\chi}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar{\chi}_{\dot\beta}, \quad \bar{\chi}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar{\chi}^{\dot\beta}$$

$\varphi^\alpha\chi_\alpha =: \varphi\chi = \chi\varphi$, $\quad \bar{\varphi}_{\dot\alpha}\bar{\chi}^{\dot\alpha} =: \bar{\varphi}\bar{\chi} = \bar{\chi}\bar{\varphi}$ Lorentz scalars

$\varphi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar{\chi}^{\dot\alpha} =: \varphi\sigma^\mu\bar{\chi}$ 4-vector

$$\varphi\sigma^\mu\bar{\chi} = -\bar{\chi}\bar{\sigma}^\mu\varphi$$

$$(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha{}^\beta$$

$$(\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu)^{\dot\alpha}{}_{\dot\beta} = 2g^{\mu\nu}\delta^{\dot\alpha}{}_{\dot\beta}$$

$$\Rightarrow \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -\delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{bmatrix}$$

$$\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi = \varphi i\sigma^\mu\partial_\mu\bar{\varphi} + \bar{\chi}i\bar{\sigma}^\mu\partial_\mu\chi - m(\varphi\chi + \bar{\chi}\bar{\varphi})$$

consider Lagrangian of Weyl field $\chi(x)$

(describing left-handed fermions $\psi_L = \begin{bmatrix} \chi \\ 0 \end{bmatrix}$) in

the presence of an external field $f_\mu(x)$:

$$\mathcal{L} = \overline{\psi}_L \, i \gamma^\mu (\partial_\mu - i f_\mu) \psi_L = \overline{\chi} \, i \overline{\sigma}^\mu (\partial_\mu - i f_\mu) \chi$$

f_μ couples to $L^\mu = \overline{\psi}_L \gamma^\mu \psi_L = \overline{\chi} \overline{\sigma}^\mu \chi$

$$\Rightarrow \quad Z_L[f] = e^{iW_L[f]} = \langle 0 \text{ out} | 0 \text{ in} \rangle_f$$

anomaly in the left-handed current implies that vacuum-to-vacuum transition amplitude <u>not</u> gauge invariant:

$$\langle 0 \text{ out} | 0 \text{ in} \rangle_{f + \partial\alpha} = \langle 0 \text{ out} | 0 \text{ in} \rangle_f \, e^{\frac{i}{24\pi^2} \int d^4x \, \alpha \, \varepsilon^{\mu\nu\rho\sigma} \partial_\mu f_\nu \partial_\rho f_\sigma}$$

remark: anomaly affects only the <u>phase</u> of the vacuum-to-vacuum transition amplitude $\rightarrow$ transition <u>probability</u> is gauge invariant

path integral representation:

$$Z_L[f] = e^{iW_L[f]} = \langle 0 \text{ out} | 0 \text{ in} \rangle_f = \int [d\chi] \, e^{-i \int d^4x \, \overline{\chi} D_L \chi}$$

Weyl operator $D_L^f = -i \overline{\sigma}^\mu (\partial_\mu - i f_\mu)$

$$\langle 0\,\text{out}\,|\,0\,\text{in}\rangle_f = \det D_L^f \quad (\text{normalization } \det D_L^0 = 1)$$

$$D_L^f = D_L^0 - f\,, \qquad f = \bar\sigma^\mu f_\mu$$

$$iW_L[f] = \ln\det D_L^f = \ln\det D_L^0\,(\mathbb{1} - \overbrace{(D_L^0)^{-1}}^{S_0} f)$$

$$= \underbrace{\ln\det D_L^0}_{0} - \mathrm{Tr}(f S_0) - \frac{1}{2}\mathrm{Tr}(f S_0 f S_0) - \dots$$

$$\mathrm{Tr}\dots = \int d^4x\ \mathrm{tr}\dots$$

$f \to f + \delta f$:

$$\delta \ln\det D_L = \mathrm{Tr}(\delta D_L\, D_L^{-1})$$

$$= -\int d^4x\ \delta f_\mu(x)\ \mathrm{tr}[\bar\sigma^\mu S_f(x,x)]$$

$S_f(x,y)$ = propagator in external field :

$$-i\bar\sigma^\mu(\partial_\mu - i f_\mu(x))\, S_f(x,y) = \delta^{(4)}(x-y)$$

propagator singular at $y = x$ $\Rightarrow$ $S_f(x,x)$ does <u>not</u> make sense as it stands $\rightarrow$ split off singular part :

$$S_f(x,y) = \underbrace{P_f(x,y)}_{\substack{\text{singular}\\\text{part}}} + \underbrace{\hat S_f(x,y)}_{\text{well defined distribution for } x\to y}$$

essential point: $P_f(x,y)$ is a polynomial in f_μ

and its derivatives

e.g. for $d=2$: $P_f(x,y) = \frac{(x-y)^\mu \sigma_\mu}{2\pi\,[(x-y)^2 - i\varepsilon]}\,[1 + i\,(x-y)^\mu f_\nu(x)]$

(in this case linear in f)

singular part of $P_f(x,y)$ unique, but one can always add

a local polynomial of f_μ:

$d=2$: $\overline{P_f}(x,y) = P_f(x,y) + c\,\sigma^\mu f_\mu$ also possible

→ change of the renormalized determinant defined by

$$\delta \ln \det D_L = -\int d^4x\; \delta f_\mu(x)\; \mathrm{tr}[\bar\sigma^\mu \hat S_f(x,x)]$$

$t \to f^t_\mu(x) := t\, f_\mu(x)$, $t \in [0,1]$ interpolates between

$f_\mu = 0$ and $f_\mu = f_\mu(x)$

$$\Rightarrow \ln\det D_L = -\int_0^1 dt \int d^4x\; f_\mu(x)\; \mathrm{tr}[\bar\sigma^\mu \hat S_{f^t}(x,x)]$$

explicit expression for the generating functional in terms

of the truncated external field propagator (unique only

up to local polynomial)

behaviour of generating functional under gauge transformation $f'_\mu = f_\mu + \partial_\mu \alpha$:

$$\rightarrow \quad D_L' = e^{i\alpha} D_L e^{-i\alpha} \Rightarrow S_{f'}(x,y) = e^{i\alpha(x)} S_f(x,y) e^{-i\alpha(x)}$$

polynomial $P_f(x,y)$ can be chosen such that the finite part $\hat{S}_f(x,x)$ is gauge invariant

$$\rightarrow \ln\det D_L' = \ln\det D_L \underbrace{- \int_0^1 dt \int d^4x \, \partial_\mu\alpha \; \mathrm{tr}[\bar{\sigma}^\mu \hat{S}_{f_t}(x,x)]}_{+ \int_0^1 dt \int d^4x \, \alpha \, \partial_\mu \mathrm{tr}[\bar{\sigma}^\mu \hat{S}_{f_t}(x,x)]}$$

$\hat{S}_f(x,y)$ satisfies the differential equation

$$D_L \hat{S}_f(x,y) = - D_L P_f(x,y) \quad \underline{\text{for } x \neq y}$$

$$\Rightarrow \quad i\,\partial_\mu \mathrm{tr}[\bar{\sigma}^\mu \hat{S}_f(x,x)] = \mathrm{tr}[D_L^x P_f(x,y) + P_f(x,y) \overleftarrow{D}_L^y]_{x=y}$$

→ calculation of anomaly $\hat{=}$ analysis of the <u>short-distance singularities</u> of the fermion propagator in an external field right-handed currents:

generating functional of the right-handed current =

= determinant of the right-handed Weyl operator

$$Z_R[f] = e^{iW_R[f]} = \det D_R$$

$$D_R = -i\sigma^\mu(\partial_\mu - if_\mu)$$

vector currents:

generating functional of the vector current =

= determinant of the Dirac operator

$$Z_V[f] = e^{iW_V[f]} = \det \not{D}$$

$$\not{D} = -i\gamma^\mu(\partial_\mu - if_\mu) = \begin{bmatrix} 0 & D_R \\ D_L & 0 \end{bmatrix}$$

it is possible to renormalize the determinants in such a manner that the product rule

$$\det \not{D} = \det D_R \, \det D_L$$

holds, i.e. $W_V[f] = W_R[f] + W_L[f]$

$$W_A[f] = W_R[f] - W_L[f]$$

we know already: generating functional of the vector current can be chosen to be gauge invariant

→ with this choice, anomalies in W_R and W_L are of opposite sign

fermions with internal quantum numbers

consider N Weyl fields $\chi_a(x)$ $(1 \le a \le N)$, where the index a labels the various different flavours and colours

remark: all fermions can always be brought to left-handed form (start with $\psi_R = \begin{bmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{bmatrix} \rightarrow \psi_R{}^c = \begin{bmatrix} \chi^{\alpha} \\ 0 \end{bmatrix}$ is left-handed)

corresponding left-handed currents: $L^{\mu}_{ab} = :\bar{\chi}_a \bar{\sigma}^{\mu} \chi_b:$

→ we need an $N \times N$ matrix $f^{\mu}_{ab}(x)$ of external fields

$$Z_L[f] = e^{iW_L[f]} = \langle 0 | T\, e^{i\int d^4x\, \mathrm{tr}(f^{\mu} L_{\mu})} | 0 \rangle = \det D_L$$

$$D_L = -i\bar{\sigma}^{\mu}(\partial_{\mu} - i f_{\mu})$$ (matrix in spin space as well as in the space of internal quantum numbers)

gauge transformation $f_\mu' = U f_\mu U^\dagger + i\, U \partial_\mu U^\dagger$

$$D_L' = U D_L U^\dagger$$

infinitesimal transformation $\quad U = \mathbb{1} + i\alpha + \dots$

$$\delta f_\mu = \nabla_\mu \alpha = \partial_\mu \alpha - i [f_\mu, \alpha]$$

Bardeen (1969)

$$\to \quad \delta \ln \det D_L = \frac{i}{24\pi^2} \int d^4x\, \varepsilon^{\mu\nu\rho\sigma} \operatorname{tr}\left[\alpha \left\{ \partial_\mu f_\nu \partial_\rho f_\sigma - \frac{i}{2} \partial_\mu (f_\nu f_\rho f_\sigma) \right\}\right]$$

→ anomalies in 3- and 4-point function (in $d=4$)

→ for $N=1$ vanishing of the second term → recover expression given on p. 27/5 (no anomalies in the 4-point function if there is only a single fermion)

→ expression $\{\dots\}$ is not gauge covariant (→ Wess-Zumino consistency condition (1971))

fermion mass terms

so far : $m = 0$

$m \neq 0$ → free currents in general not conserved

example :

$$\partial_\mu \left\{ \bar{u} \gamma^\mu \frac{1-\gamma_5}{2} d \right\} = \frac{i}{2} \bar{u} \left\{ (m_u - m_d) + \gamma_5 (m_u + m_d) \right\} d \neq 0$$

→ introduce external scalar and pseudoscalar fields s_{ab}, p_{ab} to keep track of the mass terms in the generating functional [Gasser, Leutwyler (1984)]

→ investigate the transformation properties of the generating functional in the presence of s and p

→ leading short-distance singularities of the fermion propagator are <u>not</u> affected by s or p

→ anomalies <u>not</u> modified

→ anomalies can be analyzed on the basis of the corresponding massless theory

<u>interacting gauge fields</u>

SM gauge group $SU(3) \times SU(2) \times U(1)$

denote $8+3+1$ gauge fields by $G^i_\mu(x)$

T_i = generators of the gauge group acting on the fermions collected in a <u>left-handed</u> field

$$\psi_L = \begin{bmatrix} L \\ e_R^c \\ q_L \\ u_R^c \\ d_R^c \end{bmatrix} \quad \hat{=} \text{ Weyl field } \chi$$

gauge fields interact with the fermions through the matrix $G_\mu(x) = \sum_i G^i_\mu(x)\, T_i$

→ structure of the SM Lagrangian :

$$\mathcal{L} = \mathcal{L}_G + i\,\bar{\chi}\,\bar{\sigma}^\mu(\partial_\mu - i\,G_\mu)\chi + \mathcal{L}_H$$

↑
coupling constants
absorbed in G_μ

ignoring $\mathcal{L}_H$ one has the path integral

$$\int [dG][d\chi]\; e^{\,i\int d^4x\,\mathcal{L}_G - i\int d^4x\,\bar{\chi} D_L \chi}$$

$$D_L = -i\,\bar{\sigma}^\mu(\partial_\mu - i\,G_\mu)$$

fermions can be integrated out

$$\rightarrow \quad \int [dG]\; e^{\,i\int d^4x\,\mathcal{L}_G}\,\det D_L$$

consistency of perturb. theory requires gauge invariant term $\det D_L$ → no anomalies in the Ward identities for the currents $\bar{\chi}\,\bar{\sigma}^\mu T_i\,\chi$

condition for <u>cancellation of anomalies</u> :

$$\xrightarrow[\alpha = \alpha^i T_i]{\text{p. } 27/16} \qquad \mathrm{tr}\left(T_i\{T_k, T_\ell\}\right) = 0$$

simplest example for an inconsistent model : QED with only left-handed electrons (and right-handed positrons) :

$$\mathcal{L} = -\frac{1}{4e^2} G_{\mu\nu} G^{\mu\nu} + i \bar{\chi} \bar{\sigma}^\mu (\partial_\mu - i G_\mu) \chi$$

generator $\quad Q = -1 \quad \rightarrow \quad Q^3 = -1 \neq 0$

for ordinary QED there is no problem :

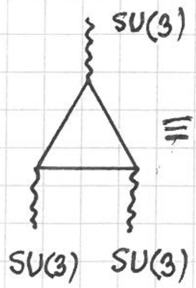
$Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \rightarrow \quad \mathrm{tr}\, Q^3 = 0 \quad \checkmark$

<u>anomaly</u> <u>cancellation in the SM:</u>

SU(3) generators $\quad T_a \qquad a = 1, .., 8$

SU(2) generators $\quad t_i \qquad i = 1, 2, 3$

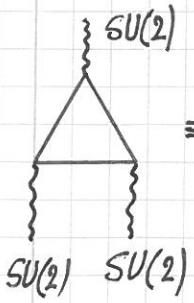
weak hypercharge $\quad Y$



$\equiv 0$

Gell-Mann matrices appear always in pairs $\frac{\lambda_a}{2}$, $-\frac{\lambda_a^T}{2}$

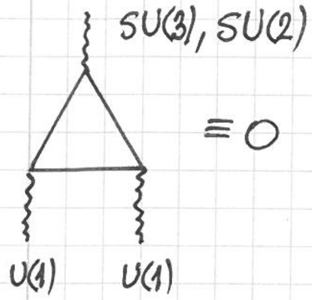
$\rightarrow \quad \mathrm{tr}\left(T_a \{T_b, T_c\}\right) \equiv 0 \; \checkmark$ (gluons couple only to <u>vector currents</u>)



$\equiv 0$

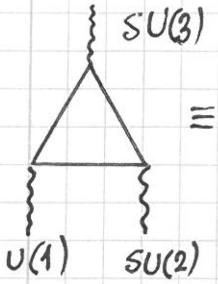
$$\{\tau_i, \tau_j\} = 2\,\delta_{ij}\,\mathbb{1}_2 \quad , \quad \mathrm{tr}\,\tau_i = 0$$

$$\Rightarrow \quad \mathrm{tr}\left(t_i\,\{t_j, t_k\}\right) \equiv 0 \quad \checkmark$$



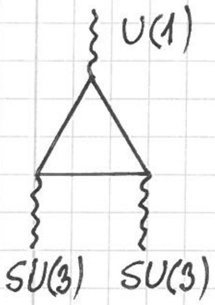
$\equiv 0$

$$\mathrm{tr}\,\lambda_a = 0 \quad , \quad \mathrm{tr}\,x_i = 0$$



$\equiv 0$

—//—



$$\mathrm{tr}\left\{\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right\} = \delta_{ab} \quad \Rightarrow 2\,Y_{q_L} + Y_{u_R^c} + Y_{d_R^c} = 0$$

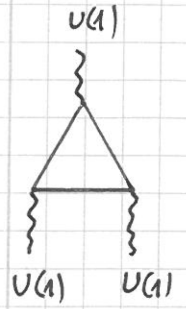
$$\Rightarrow \quad \boxed{2\,Y_{q_L} - Y_{u_R} - Y_{d_R} = 0}$$

p. 19/7 : $Y_{q_L} = \frac{1}{3}$, $Y_{u_R} = \frac{4}{3}$, $Y_{d_R} = -\frac{2}{3}$

$$3 \times 2\,Y_{q_L} + 2\,Y_L = 0$$

$$\Rightarrow \quad \boxed{3Y_{q_L} + Y_L = 0}$$

p. 19/7 : $Y_L = -1$



$$\mathrm{tr}\, Y^3 = 0$$

$$\Rightarrow \underset{\uparrow \atop \text{colour}}{3}\left(2Y_{q_L}^3 + Y_{u_R^c}^3 + Y_{d_R^c}^3\right) + 2Y_L^3 + Y_{e_R^c}^3 = 0$$

$$\Rightarrow \boxed{3\left(2Y_{q_L}^3 - Y_{u_R}^3 - Y_{d_R}^3\right) + 2Y_L^3 - Y_{e_R}^3 = 0}$$

$$\underbrace{3\left[2\left(\tfrac{1}{3}\right)^3 - \left(\tfrac{4}{3}\right)^3 - \left(-\tfrac{2}{3}\right)^3\right]}_{-\frac{6}{3}} + \underbrace{2(-1)^3 - (-2)^3}_{6} = 0$$

contributions from quarks and leptons cancel ($N_c = 3$ essential)

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extensions of the SM: constraints from anomaly cancellation!

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anomalies of global symmetries

anomalies associated to generators of the gauge group must vanish

BUT: global symmetries can be anomalous!

introduce external fields coupled to the associated currents and consider generating functional

$$e^{iW_L[f]} = \int [dG]\, e^{i\int d^4x\, \mathcal{L}_G} \det D_L$$

$$D_L = -i\,\bar{\sigma}^\mu (\partial_\mu - iG_\mu - if_\mu)$$

$[T_i, f_\mu] = 0$ (global currents must be gauge invariant)

(anomalous) Ward identities obtained from formula on p. 27/16 with the replacement $f_\mu \to G_\mu + f_\mu$ (involves terms ~ <u>winding-number density</u> $\varepsilon^{\mu\nu\rho\sigma} G_{\mu\nu} G_{\rho\sigma}$)

Lit.: H. Leutwyler: Anomalies, Helvetica Physica Acta, Vol. 59 (1986) 201–219;
Chiral Fermion Determinants and their Anomalies, Physics Letters, Vol. 152 (1985) 78–82