26. Determinants of differential operators

finite-dimensional Gaussian integral:

$$I = \int \underbrace{d\varphi_1 \dots d\varphi_N}_{d^N\varphi} \, e^{-\frac{1}{2}\varphi^T A \varphi} \qquad A^T = A$$

dependence of I on parameters of A?

$$\varphi = R\varphi' \qquad R^T A R = \hat{A} \text{ diagonal}, \quad d^N\varphi = d^N\varphi'$$
$$= \mathrm{diag}(a_1, \dots, a_N)$$

$\Rightarrow$ I product of one-dimensional Gaussian integrals

$$I = \prod_{n=1}^{N} \int d\varphi_n \, e^{-\frac{1}{2} a_n \varphi_n^2} = \prod_{n=1}^{N} \left(\frac{2\pi}{a_n}\right)^{1/2} = K (\det A)^{-1/2}$$

$$K = (2\pi)^{N/2}$$

finite-dimensional Grassmann integral:

$$I = \int [d\chi] \, e^{-\frac{1}{2} D^{kl} \chi_k \chi_l} \qquad D^T = -D$$

dependence of I on D?

consider $D \to D + \delta D$

$$\Rightarrow \quad \delta I = \int [d\chi] \, e^{-\frac{1}{2} D^{kl} \chi_k \chi_l} \left(-\frac{1}{2} \delta D^{mn} \chi_m \chi_n\right)$$

p. 10/4: $$\underbrace{\langle\langle \chi_m \chi_n \rangle\rangle}_{D^{-1}_{mn}} = \frac{\int [d\chi] \, e^{-\frac{1}{2} D^{kl} \chi_k \chi_l} \, \chi_m \chi_n}{\int [d\chi] \, e^{-\frac{1}{2} D^{kl} \chi_k \chi_l}}$$

$$\Rightarrow \quad \delta I = -\frac{1}{2}\,\delta D^{mn} D^{-1}_{mn}\, I = \frac{1}{2}\,\delta D^{nm} D^{-1}_{mn}\, I$$

$$= \frac{1}{2}\,\mathrm{Tr}(\delta D\, D^{-1})\, I$$

compare with

$$\det(D+\delta D) = \det\left[D\,(\mathbb{1} + D^{-1}\delta D)\right] = \det D \det(\mathbb{1} + D^{-1}\delta D)$$

$$= \det D\,\left[1 + \mathrm{Tr}\,(D^{-1}\delta D) + \dots\right]$$

$\uparrow$

$$\det(\mathbb{1}+\varepsilon) = 1 + \mathrm{Tr}\,\varepsilon + \mathcal{O}(\varepsilon^2)$$

$$\Rightarrow \quad \delta \det D = \det D\ \mathrm{Tr}\,(D^{-1}\delta D)$$

$$\Rightarrow \quad \delta \ln\det D = \mathrm{Tr}\,(D^{-1}\delta D)$$

$$\Rightarrow \quad \delta \ln I = \frac{1}{2}\,\delta \ln\det D$$

$$\Rightarrow \quad \ln I = \frac{1}{2}\ln\det D + \ln K$$

$$\Rightarrow \quad I = K(\det D)^{1/2}$$

remarks:

a) complex bosonic variables $\int\left(\prod_n d\mathrm{Re}\,\varphi_n\, d\mathrm{Im}\,\varphi_n\right) e^{-\varphi_m^* A_{mn}\varphi_n}$

$$= K(\det A)^{-1}$$

b) $\int [d\psi\, d\bar\psi]\, e^{-\bar\psi^T D \psi} = K \det D$

determinant of a <u>differential operator</u>

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu \varphi(x)\, \partial^\mu \varphi(x) - (m^2 - i\varepsilon)\varphi(x)^2\right] + f(x)\,\varphi(x)^2$$

<u>external field</u> $f(x)$ coupled to $\varphi(x)^2$

$$S[\varphi, f] = \frac{1}{2}\int d^dx\; \varphi(x)\left[-\Box - m^2 + i\varepsilon + 2f(x)\right]\varphi(x)$$

$$I[f] = \int [d\varphi]\, e^{i S[\varphi, f]}$$

$$= \int [d\varphi]\, e^{-\frac{1}{2}\int d^dx\, \varphi(x)\, \underbrace{i\left[\Box + m^2 - i\varepsilon - 2f(x)\right]}_{A}\varphi(x)}$$

$$A = i\left[\Box + m^2 - i\varepsilon - 2f(x)\right]$$

particle with spacetime dependent "mass" $m^2 - 2f(x)$

formally $\quad I[f] = K(\det A)^{-1/2}$

remarks: 1) in principle, one could solve the eigenvalue equation $A u(x) = a\, u(x)$ and diagonalize the quadratic form, such that $\det A$ stands for the product of all the eigenvalues of A; however, a product of ∞ many factors only makes sense if these are countable (purely discrete spectrum) and tend to one; neither of these two conditions are met here!

2) formulas nicer in euclidean field theory $\rightarrow$ Gauss integral instead of Fresnel integral $\rightarrow$ $A \longrightarrow -\Delta_d + m^2 - 2f(x)$

→ quantum mechanical problem of particle in d space dimensions in potential $m^2 - 2f(x)$; BUT: problems addressed in 1) NOT resolved

consider expansion of $I[f]$ in powers of f:

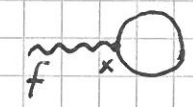
$$I[f] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^d x_1 \, f(x_1) \ldots \int d^d x_n \, f(x_n)$$

$$\times \int [d\varphi] \, e^{-\frac{i}{2}\int d^d x \, \varphi(x) (\Box + m^2 - i\varepsilon) \varphi(x)} \, \varphi(x_1)^2 \ldots \varphi(x_n)^2$$

$$= I[0] \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^d x_1 \, f(x_1) \ldots \int d^d x_n \, f(x_n) \langle\!\langle \varphi(x_1)^2 \ldots \varphi(x_n)^2 \rangle\!\rangle$$

$$\langle\!\langle \varphi(x) \varphi(y) \rangle\!\rangle = \frac{1}{i} \Delta(x-y)$$

$$I[f]/I[0] = 1 + i \int d^d x \, f(x) \overbrace{\langle\!\langle \varphi(x)^2 \rangle\!\rangle}^{\frac{1}{i}\Delta(0)}$$ <u>divergent</u> for $d \to 4$

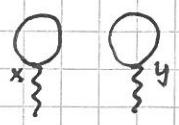


$$+ \frac{i^2}{2!} \int d^d x \, d^d y \, f(x) f(y) \langle\!\langle \varphi(x)^2 \varphi(y)^2 \rangle\!\rangle$$

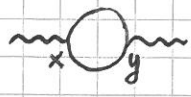
$$+ \frac{i^3}{3!} \int d^d x_1 \, d^d x_2 \, d^d x_3 \, f(x_1) f(x_2) f(x_3) \langle\!\langle \varphi(x_1)^2 \varphi(x_2)^2 \varphi(x_3)^2 \rangle\!\rangle$$

$$+ \ldots$$

$$\langle\!\langle \varphi(x)^2 \varphi(y)^2 \rangle\!\rangle = \langle\!\langle \varphi(x)^2 \rangle\!\rangle \langle\!\langle \varphi(y)^2 \rangle\!\rangle$$



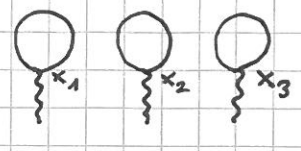
$$+ 2 \langle\!\langle \varphi(x) \varphi(y) \rangle\!\rangle \langle\!\langle \varphi(x) \varphi(y) \rangle\!\rangle$$



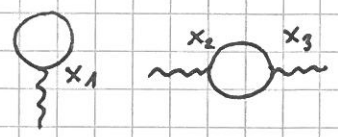
<u>divergent</u> for $d \to 4$

$$\langle\!\langle \varphi(x_1)^2\, \varphi(x_2)^2\, \varphi(x_3)^2 \rangle\!\rangle =$$

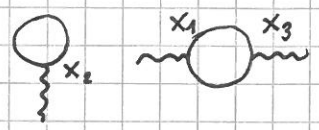
$$= \langle\!\langle \varphi(x_1)^2 \rangle\!\rangle \langle\!\langle \varphi(x_2)^2 \rangle\!\rangle \langle\!\langle \varphi(x_3)^2 \rangle\!\rangle$$



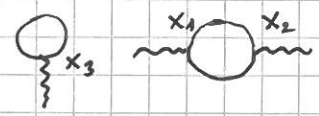
$$+\ 2 \langle\!\langle \varphi(x_1)^2 \rangle\!\rangle \langle\!\langle \varphi(x_2)\varphi(x_3) \rangle\!\rangle^2$$



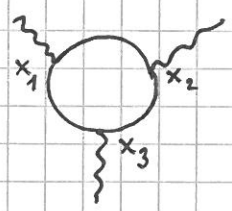
$$+\ 2 \langle\!\langle \varphi(x_2)^2 \rangle\!\rangle \langle\!\langle \varphi(x_1)\varphi(x_3) \rangle\!\rangle^2$$



$$+\ 2 \langle\!\langle \varphi(x_3)^2 \rangle\!\rangle \langle\!\langle \varphi(x_1)\varphi(x_2) \rangle\!\rangle^2$$

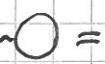


$$+\ 4 \langle\!\langle \varphi(x_1)\varphi(x_2) \rangle\!\rangle \langle\!\langle \varphi(x_2)\varphi(x_3) \rangle\!\rangle \langle\!\langle \varphi(x_3)\varphi(x_1) \rangle\!\rangle$$



– <u>finite!</u>

divergences reflect the fact that we are dealing with a product of infinitelly many eigenvalues → <u>regularization</u> required

dimensional regularization:  $= \Delta(0) \int d^d x\, f(x)$

$$\Delta(0) = i\, \frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{d/2}}\, m^{d-2} \to \frac{i}{d-4}\, \frac{m^2}{8\pi^2} + \dots$$

$= \int d^d x\, d^d y\, f(x)\, f(y)\, \Delta(x-y)^2$

$$\int d^dx\, e^{ipx}\, \Delta(x)^2 = \int d^dx\, e^{ipx} \int \frac{d^dk}{(2\pi)^d} \frac{e^{-ikx}}{m^2-i\varepsilon-k^2} \int \frac{d^dl}{(2\pi)^d} \frac{e^{-ilx}}{m^2-i\varepsilon-l^2}$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{1}{m^2-i\varepsilon-k^2} \frac{1}{m^2-i\varepsilon-l^2} (2\pi)^d \delta^{(d)}(p-k-l)$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{1}{m^2-i\varepsilon-k^2} \frac{1}{m^2-i\varepsilon-(k-p)^2} \overset{\text{p. 5/14}}{=} i\,B(p^2,m^2)$$

$$\xrightarrow[d\to 4]{} -\frac{i}{8\pi^2(d-4)} + \text{finite terms}$$

$$\Delta(x)^2 = i \int \frac{d^dp}{(2\pi)^d} e^{-ipx}\, B(p^2,m^2)$$

$$\xrightarrow[d\to 4]{} -\frac{i}{8\pi^2(d-4)} \delta^{(d)}(x) + \dots$$

$\Rightarrow$ $= \int d^dx\, d^dy\, f(x) f(y)\, \Delta(x-y)^2$

$$\to -\frac{i}{8\pi^2(d-4)} \int d^dx\, f(x)^2 + \dots$$

$\to$ counterterms $\Delta\mathcal{L} = c_1 f(x) + c_2 f(x)^2$ must be added:

$\mathcal{L} \to \mathcal{L} + \Delta\mathcal{L}$; c_1, c_2 must be tuned in such a way that generating functional

$$Z[f] = e^{iW[f]} = \frac{I[f]}{I[0]}\, e^{i\int d^dx\, \overbrace{[c_1 f(x) + c_2 f(x)^2]}^{\Delta\mathcal{L}}}$$

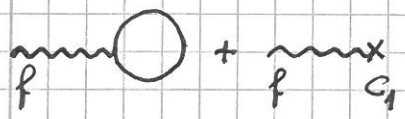
finite in the limit $d \to 4$

generating functional for Green functions of the <u>composite field</u> $\varphi(x)^2$:

$$Z[f] = \exp(iW[f]) =$$

$$= \frac{\int [d\varphi]\, e^{i\int d^dx \left(\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2-i\varepsilon}{2}\varphi^2 + f\varphi^2 + c_1 f + c_2 f^2\right)}}{\int [d\varphi]\, e^{i\int d^dx \left(\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2-i\varepsilon}{2}\varphi^2\right)}}$$

<u>renormalization</u>:

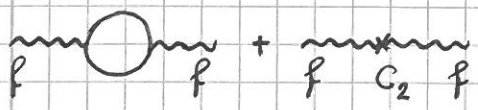
1) 

$= [\Delta(0) + ic_1] \int d^dx\, f(x)$

$\Rightarrow$ $\Delta(0) + ic_1$ must tend to a <u>finite limit</u> for $d \to 4$

(finite part of c_1 remains <u>undetermined</u>)

e.g. $\overline{MS}$: $c_1 = -\Lambda_d \frac{m^2}{8\pi^2} + c_1^{ren}$

2) 

finite for $d \to 4$

$\overline{MS}$: $c_2 = \Lambda_d \frac{1}{8\pi^2} + c_2^{ren}$

remarks: a) divergences could not be absorbed in the parameters m^2, $f(x)$ → divergences represent contributions to the <u>vacuum energy</u>.

b) generating functional is unique only up to the choice of the finite parts of the constants c_1, c_2 (related to the freedom in the normalization of the vacuum energy); conversly, any two regularization and renormalization procedures yield the same result when the cutoff is removed, except for a finite difference of the form

$$\overline{W}[f] = W[f] + \int d^dx \, [\bar{c}_1 f(x) + \bar{c}_2 f(x)^2]$$

result may be used to <u>define</u> the determinant of the differential operator A:

$$\det A / \det A_0 = Z[f]^{-2} \Rightarrow \det A = \det A_0 \, e^{-2iW[f]}$$

$f=0$

$$A = i\,[\Box + m^2 - i\varepsilon - 2f(x)], \quad A_0 = i\,(\Box + m^2 - i\varepsilon)$$

det A unique up to three constants reflecting the inherent ambiguities in the determinant of a differential operator (normalization factor $\det A_0$ and finite parts of the two counterterms)

remarks: composite <u>operator</u> $\phi^2(x)$ (discussion in operator language)

start with $\langle 0|T\,\phi(x_1)\,\phi(x_2)\,|0\rangle = \frac{1}{i}\Delta(x_1-x_2)$

consider limit $x_2 \to x_1$ (<u>point splitting technique</u>)

coincidence limit is singular ($x = x_1 - x_2$ may be viewed as a regularization parameter):

$$\Delta(x) = \frac{i}{4\pi^2(-x^2+i\varepsilon)} + \frac{im^2}{16\pi^2}\ln\left[(-x^2+i\varepsilon)m^2 e^{\gamma-1}\right] + O(x^2\ln x^2)$$

(expansion for small x^2)

$\to$ definition of $\langle 0|\phi^2(x)|0\rangle$ requires regularization <u>and renormalization</u> $\to$ counterterm $c_1 f(x)$

---

now consider $\Pi(x_1-x_2) = \langle 0|T\,\phi(x_1)^2\phi(x_2)^2|0\rangle_c$

$\Pi(x) \sim \frac{1}{(x^2)^2}$ for $d=4$ $\to$ integral $\int d^dx\,\Pi(x)\,t(x)$ only makes sense for test function $t(x)$ with $t(0)=0$!

however, $\tilde{\Pi}(x) := \Pi(x) - \delta^{(d)}(x)\int d^dx\,\Pi(x)$ does approach a well-defined limit for $d\to 4$; moreover, any distribution that is defined on <u>all</u> test functions and agrees with $\Pi(x)$ on those with $t(0)=0$ differs from $\tilde{\Pi}(x)$ only by a multiple of the δ-function; $\to$ counterterm $c_2 f(x)^2$