25. Background field method

we had: $Z[J] = \int [dQ]\, e^{i[S[Q] + J\cdot Q]}$

we define: $\tilde{Z}[J,\phi] := \int [dQ]\, e^{i[S[Q+\underset{\substack{\uparrow \\ \text{background field}}}{\phi}] + J. Q]}$

$$W[J] = -i \ln Z[J] \;\longrightarrow\; \tilde{W}[J,\phi] := -i \ln \tilde{Z}[J,\phi]$$

$$\bar{Q} = \frac{\delta W}{\delta J} \;\longrightarrow\; \tilde{Q} := \frac{\delta \tilde{W}}{\delta J}$$

$$\Gamma[\bar{Q}] = W[J] - J\cdot \bar{Q} \;\longrightarrow\; \tilde{\Gamma}[\tilde{Q},\phi] = \tilde{W}[J,\phi] - J\cdot\tilde{Q}$$

relation between $Z[J]$ and $\tilde{Z}[J,\phi]$:

$$\tilde{Z}[J,\phi] = \int [dQ]\, e^{i[S[\underbrace{Q+\phi}_{Q'}] + J. Q]} \qquad \text{shift of variable}$$

$$= \int [dQ']\, e^{i[S[Q'] + J. Q']}\; e^{-iJ\cdot\phi}$$

$$= Z[J]\, e^{-J.\phi}$$

$$\Rightarrow \tilde{W}[J,\phi] = -i \ln \tilde{Z}[J,\phi] = -i \ln Z[J] \; J\cdot\phi$$

$$= W[J] - J\cdot\phi$$

$$\Rightarrow \tilde{Q} = \frac{\delta\tilde{W}}{\delta J} = \underbrace{\frac{\delta W}{\delta J}}_{\bar{Q}} - \phi = \bar{Q} - \phi$$

$$\begin{aligned}\tilde{\Gamma}[\tilde{Q},\phi] &= \tilde{W}[J,\phi] - J.\tilde{Q} \\ &= W[J] - J.\phi - J.(\bar{Q}-\phi) \\ &= W[J] - J.\bar{Q} \\ &= \Gamma[\bar{Q}] \\ &= \Gamma[\tilde{Q}+\phi] \qquad \Rightarrow \quad \tilde{\Gamma}[\tilde{Q},\phi] = \Gamma[\tilde{Q}+\phi]\end{aligned}$$

$\tilde{Q}=0$

$\Rightarrow \quad \tilde{\Gamma}[0,\phi] = \Gamma[\phi] \quad \rightarrow$ effective action $\Gamma[\phi]$

can be determined by computing $\tilde{\Gamma}[0,\phi]$

<u>interpretation</u>: $\tilde{\Gamma}[\tilde{Q},\phi]$ is a conventional effective action in the presence of the background field ϕ $\rightarrow$ consists of all 1-P-I graphs contributing to Green functions $\rightarrow$ 1-P-I Green functions are generated by taking derivatives of the effective action $\rightarrow$ derivatives of $\tilde{\Gamma}[\tilde{Q},\phi]$ with respect to $\tilde{Q}$ would generate 1-P-I Green functions in the presence of the background field ϕ $\rightarrow$ $\tilde{\Gamma}[0,\phi]$ has <u>no</u> dependence on $\tilde{Q}$ $\rightarrow$ generates <u>no</u> graphs with external lines $\rightarrow$ $\tilde{\Gamma}[0,\phi]$ is the sum of all 1-P-I <u>vacuum</u> graphs in the presence of the field ϕ

advantage of the background field approach: effective action $\Gamma[\phi] = \tilde{\Gamma}[0,\phi]$ can be computed by summing only <u>vacuum graphs</u> (= graphs with no external lines)

gauge theories and the background field gauge

reminder (ch. 16):

$$Z[j] = \int [dQ]\, \Delta_f[Q]\, e^{i\{\underbrace{S[Q]}_{\text{gauge inv.}} - \frac{1}{2} f\cdot f + j\cdot Q\}}$$

gauge field Q^a_μ

gauge invariant action $S[Q] = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu}(x) F_a^{\mu\nu}(x)$

$$F^a_{\mu\nu} = \partial_\mu Q^a_\nu - \partial_\nu Q^a_\mu - g f_{abc} Q^b_\mu Q^c_\nu$$

$$j\cdot Q = \int d^4x\, j^\mu_a(x)\, Q^a_\mu(x)$$

$$f\cdot f = \int d^4x\, f_a[Q]\, f_a[Q] \qquad \text{gauge fixing term}$$

infinitesimal gauge transformation:

$$\delta Q^a_\mu = \frac{1}{g} \underbrace{(\delta_{ab}\partial_\mu + g\, Q^c_\mu f_{cab})}_{D_{\mu,ab}} \underbrace{\delta\alpha_b}_{\text{infinitesimal gauge parameter}}$$

$$\delta f_a[Q] = \frac{\delta f_a[Q]}{\delta Q^c_\mu}\,\delta Q^c_\mu = \frac{1}{g}\frac{\delta f_a[Q]}{\delta Q^c_\mu} D_{\mu,cb}\,\delta\alpha_b$$

$$= \frac{1}{g} M^f_{ab}\,\delta\alpha_b$$

corresponding integral kernel: $M^f_{ab}(x,y) = \int d^4z \frac{\delta f_a[Q(x)]}{\delta Q^c_\mu(z)} D_{\mu,cb}(z,y)$

$$D_{\mu,cb}(z,y) = \left[\delta_{cb}\frac{\partial}{\partial z^\mu} + g\, f_{cab}\, Q^c_\mu(z)\right]\delta^{(4)}(z-y)$$

$$\Delta_f[Q] = \det\left(\frac{1}{g}M^f\right) \sim \int [dc\, d\bar{c}]\, e^{iS_{FP}}$$

$$S_{FP} = -\int d^4x\, d^4y\, \bar{c}_a(x)\, M^f_{ab}(x,y)\, c_b(y)$$

introduce <u>background</u> <u>field</u> A^a_μ : $Q^a_\mu \to Q^a_\mu + A^a_\mu$

$$\tilde{Z}[j,A] = \int [dQ]\, \tilde{\Delta}_f[Q,A]\, e^{i\left\{S[Q+A] - \frac{1}{2}\tilde{f}[Q,A]^2 + j\cdot Q\right\}}$$

$$\tilde{f}_a[Q,A] = f_a[Q+A] \quad \Rightarrow \quad f_a[Q] = \tilde{f}_a[Q-A,A]$$

$$\tilde{\Delta}_f[Q,A] = \det\left(\frac{1}{g}\tilde{M}_f\right)$$

$$\tilde{M}^f_{ab} = \frac{\delta \tilde{f}_a[Q,A]}{\delta Q^c_\mu}\tilde{D}_{\mu,cb} \;,\; \tilde{D}_{\mu,cb} = \delta_{cb}\partial_\mu + g f_{cab}(Q^c_\mu + A^c_\mu)$$

$$\underbrace{\tilde{Z}[j,A]}_{e^{i\tilde{W}[j,A]}} = \underbrace{Z[j]}_{e^{iW[j]}}\, e^{-ij\cdot A} \quad \Rightarrow \quad \tilde{W}[j,A] = W[j] - j\cdot A$$

Legendre transformation :

$$\Gamma[\bar{Q}] = W[j] - j\cdot\bar{Q} \;,\quad \frac{\delta W}{\delta j} = \bar{Q} \;,\quad \frac{\delta \Gamma}{\delta \bar{Q}} = -j$$

$$\tilde{\Gamma}[\tilde{Q},A] = \tilde{W}[j,A] - j\cdot\tilde{Q} \;,\quad \frac{\delta \tilde{W}}{\delta j} = \tilde{Q} \;,\quad \frac{\delta\tilde{\Gamma}}{\delta\tilde{Q}} = -j$$

$$\tilde{W} = W - j\cdot A \quad \Rightarrow \quad \tilde{Q} = \frac{\delta\tilde{W}}{\delta j} = \underbrace{\frac{\delta W}{\delta j}}_{\bar{Q}} - A \;\Rightarrow\; \tilde{Q} = \bar{Q} - A$$

$$\tilde{\Gamma}[\tilde{Q},A] = \underbrace{\tilde{W}[j,A]}_{W[j]-j\cdot A} - j\cdot\tilde{Q}$$

$$= W[j] - j\cdot(\tilde{Q}+A)$$

$$= \Gamma[\tilde{Q}+A]$$

$$\Rightarrow \quad \tilde{\Gamma}[0,A] = \Gamma[A]$$

essential point: $\exists\ \tilde{f}_a[Q,A]$ such that $\tilde{\Gamma}[0,A] = \Gamma[A]$

invariant under the gauge transformation

$$\delta A^a_\mu = \frac{1}{g}\underbrace{\left(\delta_{ab}\,\partial_\mu + g\,A^c_\mu f_{cab}\right)}_{D^A_{\mu,ab}}\delta\alpha_b$$

$\tilde{f}_a[Q,A] = \sqrt{\xi}\,D^A_{\mu,ab}\,Q^\mu_b$ background field gauge

assertion: $\tilde{Z}[j,A]$ is invariant under the transformation

$$\delta A^a_\mu = \frac{1}{g}D^A_{\mu,ab}\,\delta\alpha_b\,, \quad \delta j^a_\mu = f_{abc}\,\delta\alpha_b\,j^c_\mu$$

change of integration variables $Q^a_\mu \to Q^a_\mu + f_{abc}\,\delta\alpha_b\,Q^c_\mu$

in $\tilde{Z}[j,A] = \int[dQ]\,\tilde{\Delta}_f[Q,A]\,e^{i\left\{S[Q+A]-\frac{1}{2}\tilde{f}\cdot\tilde{f}+j\cdot Q\right\}}$

orthogonal matrix $R_{ab} = \delta_{ab} + f_{abc}\,\delta\alpha_c$

$$\left.\begin{aligned} Q_\mu &\to R^T Q_\mu \\ j_\mu &\to R^T j_\mu \end{aligned}\right\} \Rightarrow \quad j\cdot Q \text{ invariant}$$

$$D_\mu^A \to R^T D_\mu^A R$$

$$\tilde{f} \to R^T \tilde{f} \quad \Rightarrow \quad \tilde{f}\cdot\tilde{f} \text{ invariant}$$

$$\delta(Q_\mu + A_\mu) = \frac{1}{g}\tilde{D}_\mu\,\delta\alpha \quad \Rightarrow S[Q+A] \text{ invariant}$$

$[dQ]$ invariant

$$\tilde{\Delta}_f[Q,A] = \det\left(\frac{1}{g}\tilde{M}_f\right) \quad \text{invariant} \qquad (\tilde{M}_f \to R^T \tilde{M}_f R)$$

$$\Rightarrow \quad \tilde{Z}[j,A] \text{ invariant} \Rightarrow \tilde{W}[j,A] \text{ invariant}$$

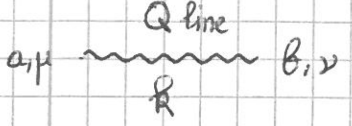
$$\tilde{\Gamma}[\tilde{Q},A] = \tilde{W}[j,A] - j\cdot\tilde{Q}$$

$$\Rightarrow \quad \tilde{\Gamma}[\tilde{Q},A] \text{ is invariant under } \quad \delta A_\mu^a = \frac{1}{g} D^A_{\mu,ab}\,\delta\alpha_b$$

$$\delta\tilde{Q}_\mu^a = f_{abc}\,\delta\alpha_b\,\tilde{Q}_\mu^c$$

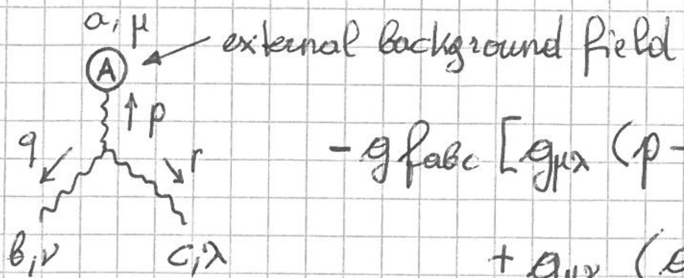
$$\Rightarrow \quad \tilde{\Gamma}[0,A] \quad (A\text{-}) \text{ gauge invariant}$$

Feynman rules

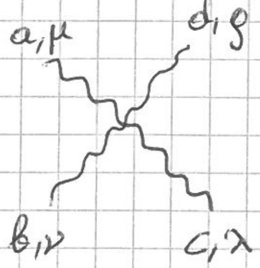


$$\frac{-i\delta_{ab}}{k^2+i\varepsilon}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\left(1-\frac{1}{\xi}\right)\right]$$

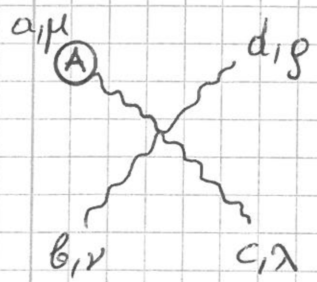
$$\frac{i\delta_{ab}}{k^2+i\varepsilon}$$



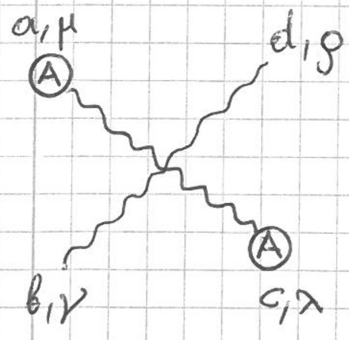
$$-g f_{abc}\left[g_{\mu\lambda}(p-r-\xi q)_\nu + g_{\nu\lambda}(r-q)_\mu + g_{\mu\nu}(q-p+\xi r)_\lambda\right]$$



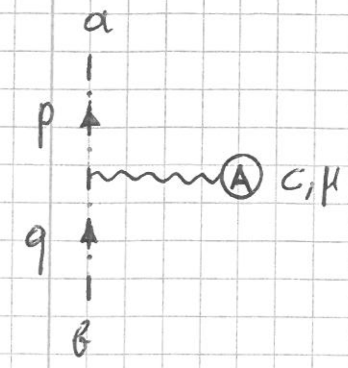
and



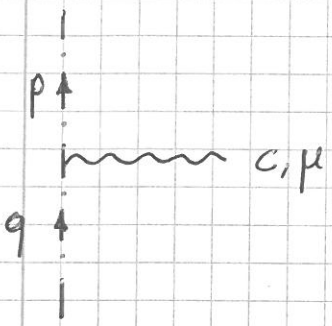
$$-ig^2\left[f_{abx}f_{xcd}(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}) + f_{adx}f_{xbc}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}) + f_{acx}f_{xbd}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\rho}g_{\nu\lambda})\right]$$



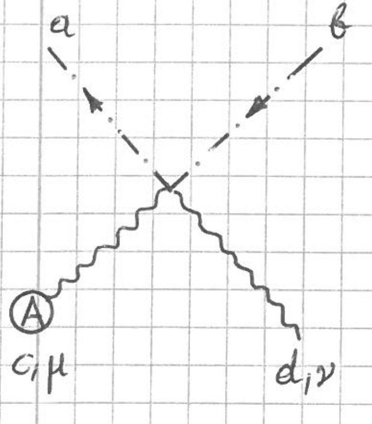
$$-ig^2\left[f_{abx}f_{xcd}(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}+\xi g_{\mu\nu}g_{\lambda\rho}) + f_{adx}f_{xbc}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-\xi g_{\mu\rho}g_{\nu\lambda}) + f_{acx}f_{xbd}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\rho}g_{\nu\lambda})\right]$$



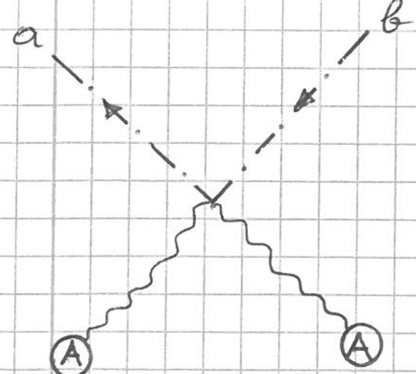
$$g f_{acb} (p+q)_\mu$$



$$g f_{acb}\, p_\mu$$



$$-ig^2 f_{acx} f_{xdb}\, g_{\mu\nu}$$



$$-ig^2 g_{\mu\nu} \left(f_{acx} f_{xdb} + f_{adx} f_{xcb} \right)$$

Lit.: L. F. Abbott, Nucl. Phys. B 185 (1981) 189; Acta Physica Polonica B 13 (1982) 33

renormalization

divergences occuring in $\tilde{\Gamma}[0,A]$ must be renormalized $\rightarrow$

bare quantities A, g, ξ related to renormalized

quantities A_r, g_r, ξ_r by

$$A^\mu = Z_A^{1/2} A_r^\mu$$

$$g = Z_g \, g_r$$

$$\xi = Z_\xi \, \xi_r$$

remark: Q, c, $\bar{c}$ appear only inside loops $\rightarrow$ <u>no</u> renormalization required

$\tilde{\Gamma}[0,A]$ gauge invariant $\Rightarrow$ Z_g and Z_A are related!

infinites in $\tilde{\Gamma}[0,A]$ must be $\sim F^a_{\mu\nu} F_a^{\mu\nu} \times$ const

$$F^a_{\mu\nu} = Z_A^{1/2} \left[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - Z_g Z_A^{1/2} f_{abc} A^b_\mu A^c_\nu \right]$$

$$\Rightarrow \quad Z_g = Z_A^{-1/2}$$

$\rightarrow$ <u>simplifies</u> calculation of Yang-Mills β-function

Yang-Mills β function

dimensional regularization $d = 4 - 2\varepsilon$

minimal subtraction : $Z_A = 1 + \sum_{n=1}^{\infty} \frac{Z_A^{(n)}}{\varepsilon^n}$

L loops $\rightarrow$ contributions $\frac{1}{\varepsilon}, \dots, \frac{1}{\varepsilon^L}$

as in φ^4 theory : bare coupling g not dimensionless in dim. reg.

$$\int d^{4-2\varepsilon}x \, (\partial_\mu A_\nu)^2 \quad \Rightarrow \quad 2[A] + 2 - (4-2\varepsilon) = 0$$

$$\Rightarrow \quad [A] = 1 - \varepsilon$$

$$\int d^{4-2\varepsilon}x \, g^2 A^4 \quad \Rightarrow \quad 4[A] + 2[g] - (4-2\varepsilon) = 0$$

$$\Rightarrow \quad [g] = 2 - \varepsilon - 2(1-\varepsilon) = \varepsilon$$

if we wish to use a dimensionless renormalized coupling g_r

$\rightarrow \quad g = Z_g \mu^\varepsilon g_r$ with arbitrary mass parameter μ

g independent of μ $\Rightarrow$ $\mu \frac{dg}{d\mu} = 0 = \mu \frac{\partial Z_g}{\partial \mu} \mu^\varepsilon g_r +$

$$+ \varepsilon Z_g \mu^\varepsilon g_r + Z_g \mu^\varepsilon \mu \frac{\partial g_r}{\partial \mu}$$

$$\Rightarrow \quad 0 = Z_g \mu^\varepsilon \left[\varepsilon g_r + \underbrace{\mu \frac{\partial g_r}{\partial \mu}}_{\beta} + g_r \mu \frac{\partial \ln Z_g}{\partial \mu} \right]$$

$$\Rightarrow \quad \beta = -\varepsilon g_r - g_r \mu \frac{\partial \ln Z_g}{\partial \mu}$$

$Z_g = Z_A^{-1/2}$

$$\Rightarrow \quad \beta = -\varepsilon g_r + \frac{1}{2} g_r \mu \frac{\partial \ln Z_A}{\partial \mu}$$

chain rule $\quad \mu \frac{\partial}{\partial \mu} = \underbrace{\mu \frac{\partial g_r}{\partial \mu}}_{\beta} \frac{\partial}{\partial g_r} = \beta \frac{\partial}{\partial g_r}$

$$\Rightarrow \quad \beta = -\varepsilon g_r + \frac{1}{2} g_r \beta \frac{\partial \ln Z_A}{\partial g_r}$$

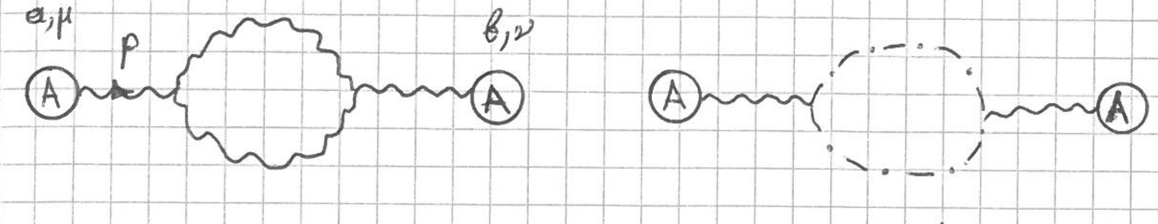
$$\beta = -\varepsilon g_r + \frac{1}{2} g_r \beta \frac{\partial}{\partial g_r} \left(\frac{Z_A^{(1)}}{\varepsilon} + O\left(\frac{1}{\varepsilon^2}\right) \right)$$

β finite $\Rightarrow$ divergences must cancel

limit $\varepsilon \to 0$: $\quad \beta = -\frac{1}{2} g_r^2 \frac{\partial Z_A^{(1)}}{\partial g_r}$

$\to$ β function determined by $Z_A^{(1)}$ (= coefficient of the $\frac{1}{\varepsilon}$ term in Z_A)

calculation of the one-loop β function: two graphs



(a) $\qquad\qquad$ (b)

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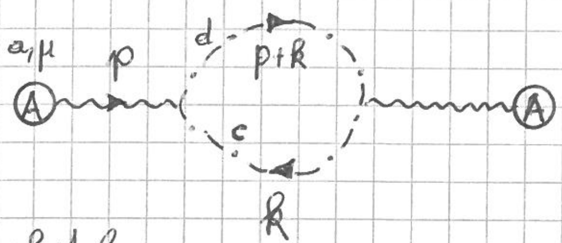
$$\text{diagram (a)}\Big|_{\frac{1}{\varepsilon}} = \frac{ig^2 C_A \delta_{ab}}{(4\pi)^2} \frac{10}{3\varepsilon} \left[g_{\mu\nu} p^2 - p_\mu p_\nu \right]$$

$$\text{diagram (b)}\Big|_{\frac{1}{\varepsilon}} = \frac{ig^2 C_A \delta_{ab}}{(4\pi)^2} \frac{1}{3\varepsilon} \left[g_{\mu\nu} p^2 - p_\mu p_\nu \right]$$

$$f_{acd} f_{bcd} = C_A \delta_{ab} \qquad (C_A(SU(N)) = N)$$

diagram (a) $\rightarrow$ home work exercise

diagram (b):



ghost loop

$$= (-) \int \frac{d^d k}{(2\pi)^d} \, g f_{dac} (p+2k)_\mu \frac{i}{k^2 + i\varepsilon} \frac{i}{(k+p)^2 + i\varepsilon} g f_{cbd} (p+2k)_\nu$$

$$= g^2 f_{dac} f_{cbd} \int \frac{d^d k}{(2\pi)^d} \int_0^1 d\alpha \frac{(p+2k)_\mu (p+2k)_\nu}{\left[\alpha (k+p)^2 + (1-\alpha) k^2 + i\varepsilon \right]^2}$$

$$= -g^2 \underbrace{f_{acd} f_{bcd}}_{C_A \delta_{ab}} \int \frac{d^d k}{(2\pi)^d} \int_0^1 d\alpha \frac{(p+2k)_\mu (p+2k)_\nu}{\left[\underbrace{k^2 + 2\alpha p.k}_{(k+\alpha p)^2 - \alpha^2 p^2} + \alpha p^2 + i\varepsilon \right]^2}$$

$$= -g^2 C_A \delta_{ab} \int \frac{d^dk}{(2\pi)^d} \int_0^1 d\alpha \frac{[p+2(k-\alpha p)]_\mu \, [p+2(k-\alpha p)]_\nu}{[k^2+\alpha(1-\alpha)p^2+i\varepsilon]^2}$$

$$= -g^2 C_A \delta_{ab} \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \frac{p_\mu p_\nu (1-2\alpha)^2 + 4 \overbrace{k_\mu k_\nu}^{\to k^2 g_{\mu\nu}/d}}{[k^2+\alpha(1-\alpha)p^2+i\varepsilon]^2}$$

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$$\overset{\downarrow}{=} -g^2 C_A \delta_{ab} \frac{i}{(4\pi)^2} \frac{1}{\varepsilon} \int_0^1 d\alpha \, [p_\mu p_\nu (1-2\alpha)^2 + 2 g_{\mu\nu} \alpha(1-\alpha)p^2]$$

+ finite terms

$$= \frac{i g^2 C_A \delta_{ab}}{(4\pi)^2} \frac{1}{3\varepsilon} (g_{\mu\nu} p^2 - p_\mu p_\nu)$$

$\Rightarrow$ divergences are cancelled by $\quad Z_A = 1 + \frac{11 C_A}{3\varepsilon} \frac{g_r^2}{(4\pi)^2}$

$$\Rightarrow \quad \beta = -\frac{11 C_A}{3} \frac{g_r^3}{(4\pi)^2} + \mathcal{O}(g_r^5)$$

<u>QCD</u>

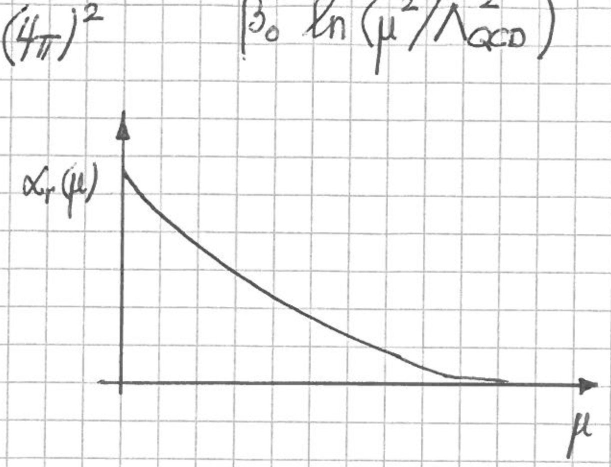
gauge group $SU(3) \;\rightarrow\; C_A(SU(3)) = 3 = N_c$

include also N_f quark flavours

$$\rightarrow \quad \beta(g) = -\beta_0 \frac{g^3}{(4\pi)^2} + \mathcal{O}(g^5), \quad \beta_0 = \frac{1}{3}(11 N_c - 2 N_f)$$

$\beta_0 > 0 \quad \Rightarrow \quad \beta$ function negative $\rightarrow$ <u>asymptotic freedom</u>

$$\frac{g_r(\mu)^2}{(4\pi)^2} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda_{QCD}^2)}$$



Λ_{QCD} characterizes the strength of the interaction, irrespective of the scale μ ($g \to \Lambda_{QCD}$: <u>dimensional transmutation</u>)

remark: mass spectrum of hadrons determined by Λ_{QCD}

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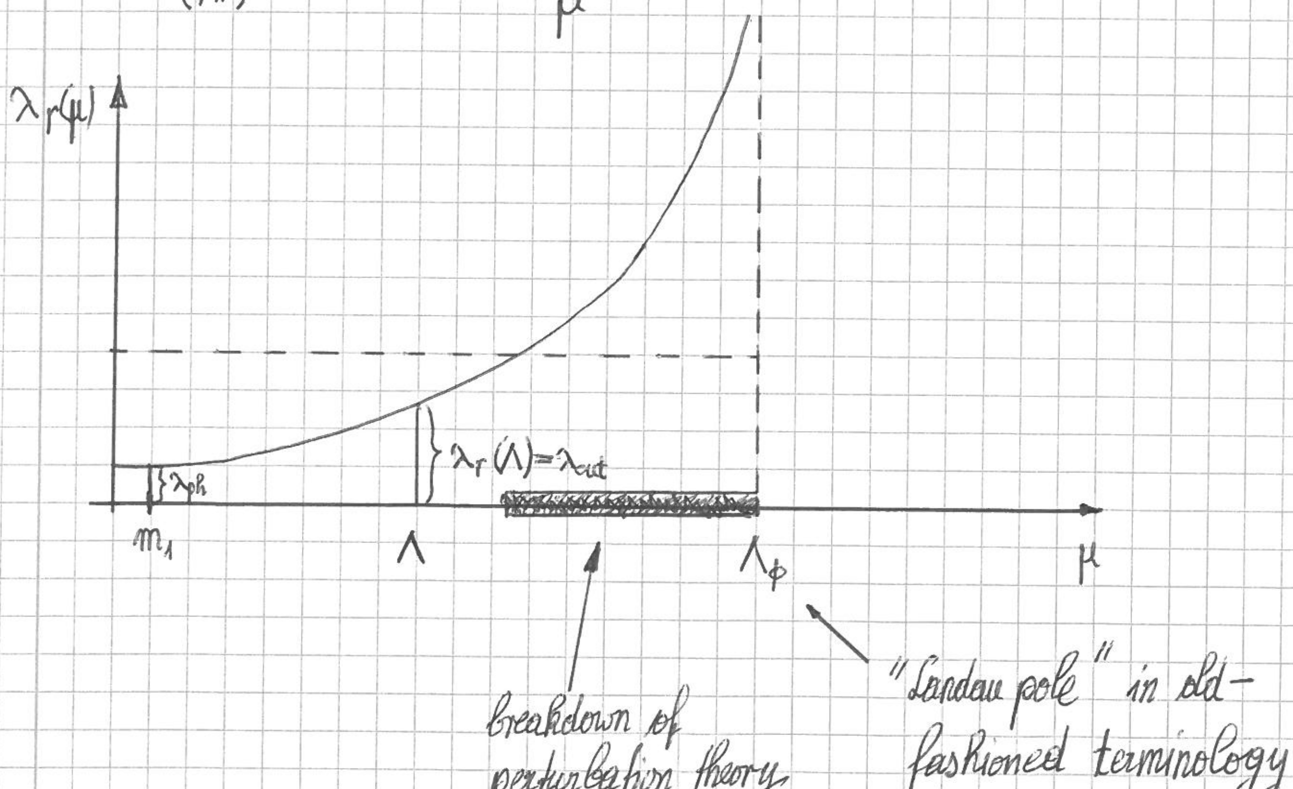
remember situation in $\lambda\phi^4$ theory:

$$\frac{\lambda_r(\mu)}{(4\pi)^2} = \frac{1}{3 \ln \frac{\Lambda_\phi}{\mu}}$$



$$\Lambda_\phi = m_1 \exp\left(\frac{(4\pi)^2}{3\lambda_{ph}}\right)$$

$\Lambda \to \infty$? (or $a = \frac{1}{\Lambda} \to 0$)

general analysis (independent of perturbation theory):

"triviality" of ϕ^4 theory: ϕ^4 Lagrangian describes a strictly local QFT in 4 dimensions if the physical coupling is set equal to zero

nevertheless ϕ^4 theory acceptable model if not taken seriously down to arbitrarily small distances

similar situation in QED (positive β function!)

$$\Lambda_{QED} \simeq m_e \exp\left(\frac{3\pi}{2\alpha_{em}}\right) \simeq 10^{277}\,GeV$$

$\to$ beyond good and evil ($p_{Planck} \simeq 10^{19}\,GeV$)

Higgs sector of SM ($\sim \phi^4$ interaction)

$\lambda = M_h^2/2v^2$ $\quad M_h \simeq 125\,GeV$, $v \simeq 246\,GeV$ $\to \lambda$ small

SM makes good sense as effective theory at present energies