25. Background field method

We had: \[ Z[J] = \int [dQ] e^{i[S[Q] + J \cdot Q]} \]

We define: \[ \tilde{Z}[J, \phi] := \int [dQ] e^{i[S[Q + \phi] + J \cdot Q]} \]

\[ W[J] = -i \ln Z[J] \quad \rightarrow \quad \tilde{W}[J, \phi] := -i \ln \tilde{Z}[J, \phi] \]

\[ \bar{Q} = \frac{\delta W}{\delta J} \quad \rightarrow \quad \bar{Q} := \frac{\delta \tilde{W}}{\delta J} \]

\[ \Gamma[\bar{Q}] = W[J] - J \cdot \bar{Q} \quad \rightarrow \quad \tilde{\Gamma}[\bar{Q}, \phi] = \tilde{W}[J, \phi] - J \cdot \bar{Q} \]

Relation between \( Z[J] \) and \( \tilde{Z}[J, \phi] \):\[
\tilde{Z}[J, \phi] = \int [dQ] e^{i[S[Q + \phi] + J \cdot Q]} \\
= \int [dQ'] e^{i[S[Q'] + J \cdot Q']} e^{-ij \cdot \phi} \\
= Z[J] e^{-J \cdot \phi} \\
\Rightarrow \quad \tilde{W}[J, \phi] = -i \ln \tilde{Z}[J, \phi] = -i \ln Z[J] \quad \text{with} \quad J \cdot \phi \\
= W[J] - J \cdot \phi \\
\Rightarrow \quad Q = \frac{\delta \tilde{W}}{\delta J} = \frac{\delta W}{\delta J} - \phi = \bar{Q} - \phi \]
\[ \tilde{\Gamma}[\tilde{Q}, \phi] = \tilde{W}[J, \phi] - j \cdot \tilde{Q} \]
\[ = W[J] - j \cdot \phi - j \cdot (\tilde{Q} - \phi) \]
\[ = W[J] - j \cdot \tilde{Q} \]
\[ = \Gamma[\tilde{Q}] \]
\[ = \Gamma[\tilde{Q} + \phi] \Rightarrow \tilde{\Gamma}[\tilde{Q}, \phi] = \Gamma[\tilde{Q} + \phi] \]

\[ \tilde{Q} = 0 \Rightarrow \tilde{\Gamma}[0, \phi] = \Gamma[\phi] \rightarrow \text{effective action} \Gamma[\phi] \]

can be determined by computing \( \tilde{\Gamma}[0, \phi] \)

interpretation: \( \tilde{\Gamma}[\tilde{Q}, \phi] \) is a conventional effective action in the presence of the background field \( \phi \rightarrow \) consists of all 1-P-I graphs contributing to Green functions

\[ \rightarrow \text{1-P-I Green functions are generated by taking derivatives of the effective action} \rightarrow \text{derivatives of } \tilde{\Gamma}[\tilde{Q}, \phi] \text{ with respect to } \tilde{Q} \text{ would generate 1P-I Green functions in the presence of the background field} \]

\[ \rightarrow \tilde{\Gamma}[0, \phi] \text{ has no dependence on } \tilde{Q} \rightarrow \text{generates no graphs with external lines} \rightarrow \tilde{\Gamma}[0, \phi] \text{ is the sum of all 1-P-I vacuum graphs in the presence of the field } \phi \]

advantage of the background field approach: effective action \( \Gamma[\phi] = \tilde{\Gamma}[0, \phi] \) can be computed by summing only vacuum graphs \((= \text{graphs with no external lines})\)
gauge theories and the background field gauge

reminder (ch. 16):

\[ Z[j] = \int [dQ] \Delta_p [Q] e^{i \{ S[Q] - \frac{i}{2} j \cdot A + j \cdot Q \}} \]

gauge field \( Q^a_\mu \)

gauge invariant action \( S[Q] = -\frac{1}{4} \int d^4 x \; F^a_{\mu\nu}(x) F_a^{\mu\nu}(x) \)

\[ F^a_{\mu\nu} = \partial_\mu Q^a_\nu - \partial_\nu Q^a_\mu - g f^{abc} Q^b_\mu Q^c_\nu \]

\[ j \cdot Q = \int d^4 x \; j_a^\mu(x) Q^a_\mu(x) \]

\[ j \cdot \bar{\rho} = \int d^4 x \; \bar{\rho}_a [Q] \rho_a [Q] \quad \text{gauge fixing term} \]

infinitesimal gauge transformation:

\[ S Q^a_\mu = \frac{1}{g} \left( \delta_{ab} \partial_\mu + g Q^c_\mu f_{cab} \right) S x_b \]

\[ D_{\mu, ab} \quad \text{infinitesimal gauge parameter} \]

\[ S \rho_a [Q] = \frac{S \rho_a [Q]}{S Q^c_\mu} \quad S Q^c_\mu = \frac{1}{g} \frac{S \rho_a [Q]}{S Q^c_\mu} \quad D_{\mu, cb} \quad S x_b \]

\[ = \frac{1}{g} \quad M_{ab} \quad S x_b \]

\[ \text{corresponding integral kernel: } M_{ab}(x,y) = \int d^4 z \frac{S \rho_a [Q]}{S Q^c_\mu} D_{\mu, cb}(z,y) \]

\[ D_{\mu, cb}(z,y) = [ S_{cb} \frac{d}{dz} + g f_{cab} Q^c_\mu(z) ] S^c_\mu(z-y) \]
\[ \Delta_f [Q] = \det \left( \frac{1}{g} M_f \right) \sim \int [dc \, dc] \, e^{iS_{FP}} \]
\[ S_{FP} = -\int d^4x \, d^4y \, \tilde{c}_a(x) \, M_{ab}^f (x, y) \, c_b(y) \]

Introduce background field \( A_\mu^a : Q_\mu^a \to Q_\mu^a + A_\mu^a \)

\[ \tilde{\mathbf{Z}} [j, A] = \int [dQ] \tilde{\Delta}_f [Q, A] \, e^{i \{ J_{Q+A}, A \} - \frac{1}{2} \tilde{F}[Q, A]^2 + j \cdot Q} \]

\[ \tilde{\mathfrak{F}}_a [Q, A] = \mathfrak{F}_a [Q + A] \Rightarrow \mathfrak{F}_a [Q] = \tilde{\mathfrak{F}}_a [Q - A, A] \]

\[ \tilde{\Delta}_f [Q, A] = \det \left( \frac{1}{g} \tilde{M}_f \right) \]

\[ \tilde{M}_{ab}^f = \left. S_{\tilde{\mathfrak{F}}_a [Q, A]} \right|_{Q^c} D_\mu^c, c \quad \tilde{D}_\mu^c, c = S_{c, b} D_\mu^c + g f_{cab} (Q_\mu^c + A_\mu^c) \]

\[ \tilde{\mathbf{Z}} [j, A] = \mathbf{Z} [j] \, e^{-ij \cdot A} \Rightarrow \tilde{\mathbf{W}} [j, A] = \mathbf{W} [j] - j \cdot A \]

Legendre transformation:

\[ \mathbf{W} [Q] = \mathbf{W} [j] - j \cdot Q, \quad \frac{\delta \mathbf{W}}{\delta j} = \bar{Q}, \quad \frac{\delta \mathbf{W}}{\delta Q} = -j \]

\[ \tilde{\mathbf{W}} [\tilde{Q}, \tilde{A}] = \tilde{\mathbf{W}} [j, A] - j \cdot \tilde{Q}, \quad \frac{\delta \tilde{\mathbf{W}}}{\delta j} = \bar{\tilde{Q}}, \quad \frac{\delta \tilde{\mathbf{W}}}{\delta \tilde{Q}} = -j \]

\[ \tilde{\mathbf{W}} = \mathbf{W} - j \cdot A \Rightarrow \tilde{Q} = \frac{\delta \tilde{\mathbf{W}}}{\delta j} \frac{\delta \mathbf{W}}{\delta j} - A \Rightarrow \bar{\tilde{Q}} = \bar{Q} - A \]
\[ \tilde{\Gamma}[\tilde{Q}, A] = \tilde{W}[j, A] - j \cdot \tilde{Q} \]
\[ = W[j] - j \cdot (\tilde{Q} + A) \]
\[ = \Gamma[\tilde{Q} + A] \]

\[ \Rightarrow \tilde{\Gamma}[0, A] = \Gamma[A] \]

Essential point: \( \exists \tilde{\Gamma}[Q, A] \) such that \( \tilde{\Gamma}[0, A] = \Gamma[A] \)

Invariant under the gauge transformation

\[ \delta A^a_\mu = \frac{1}{g} \left( \delta_{ab} \partial_\mu + g A^c_\mu f_{cab} \right) \delta x^c \]
\[ D^A_{\mu, ab} \]

\[ \tilde{\Gamma}[Q, A] = \Gamma[Q, A] \quad \text{background field gauge} \]

Section: \( \tilde{Z}[j, A] \) is invariant under the transformation

\[ \delta A^a_\mu = \frac{1}{g} D^A_{\mu, ab} \delta x^b, \quad \delta j^a_\mu = f_{abc} \delta x^c j^b_\mu \]

Change of integration variables \( Q^a_\mu \to Q^a_\mu + f_{abc} \delta x^c Q^c_\mu \)

in \( \tilde{Z}[j, A] = \int [dQ] \tilde{\Gamma}[Q, A] e^{i \left\{ S[Q + A] - \frac{1}{2} j \cdot \tilde{Q} + j \cdot Q \right\}} \)
orthogonal matrix \( \mathbf{R}_{ab} = \delta_{ab} + f_{abc} \delta_{ac} \)

\[
\begin{align*}
Q_{\mu} & \to \mathbf{R}^T Q_{\mu} \\
\mathbf{j}_{\mu} & \to \mathbf{R}^T \mathbf{j}_{\mu} \} \Rightarrow j \cdot Q \text{ invariant} \\
D_{\mu}^A & \to \mathbf{R}^T D_{\mu}^A \mathbf{R} \\
\tilde{\mathbf{\mathcal{P}}} & \to \mathbf{R}^T \tilde{\mathbf{\mathcal{P}}} \Rightarrow \tilde{\mathbf{\mathcal{P}}} \cdot \tilde{\mathbf{\mathcal{P}}} \text{ invariant} \\
S(Q_{\mu} + A_{\mu}) = \frac{1}{g} \mathbf{\hat{D}}_{\mu} \delta x & \Rightarrow S[Q + A] \text{ invariant} \\
[dQ] \text{ invariant} \\
\tilde{\Delta}_p [Q, A] = \det \left( \frac{1}{g} \mathbf{\hat{M}}_f \right) \text{ invariant} \quad (\mathbf{\hat{M}}_f \to \mathbf{R}^T \mathbf{\hat{M}}_f \mathbf{R}) \\
\Rightarrow \tilde{\mathbf{\mathcal{Z}}} [q, A] \text{ invariant} \Rightarrow \tilde{\mathbf{\mathcal{W}}} [q, A] \text{ invariant} \\
\tilde{\Gamma} [\tilde{Q}, A] = \tilde{\mathbf{\mathcal{W}}} [q, A] - q \cdot \tilde{Q} \\
\Rightarrow \tilde{\Gamma} [\tilde{Q}, A] \text{ is invariant under } \delta A_{\mu}^a = \frac{1}{g} D_{\mu}^A, ab \delta x_b \\
\delta \tilde{Q}_{\mu}^a = f_{abc} \delta x_b \tilde{Q}_{\mu}^c \\
\Rightarrow \tilde{\Gamma} [Q, A] \text{ (A-) gauge invariant}
Feynman rules

\[
\frac{-iS_{ab}}{R^2 + i\varepsilon} \left[ g_{\mu} - \frac{R_{\mu}}{R^2} \left(1 - \frac{1}{3}\right) \right]
\]

External background field

\[
-g_{abc} \left[ g_{\mu} (p - r - s q) + g_{\mu} (r - q) \mu + g_{\nu} (q - p + s r) \nu \right]
\]

\[
-ig^2 \left[ f_{abc} f_{x cd} (g_{\mu} g_{\nu} - g_{\mu} g_{\nu} g_{\alpha}) + f_{abc} f_{x bc} (g_{\mu} g_{\nu} - g_{\mu} g_{\nu}) + f_{acx} f_{x bd} (g_{\mu} g_{\nu} - g_{\mu} g_{\nu}) \right]
\]

\[
-ig^2 \left[ f_{abc} f_{x cd} (g_{\mu} g_{\nu} - g_{\mu} g_{\nu} + s g_{\mu} g_{\nu}) + f_{abc} f_{x bc} (g_{\mu} g_{\nu} - g_{\mu} g_{\nu} - s g_{\mu} g_{\nu}) + f_{acx} f_{x bd} (g_{\mu} g_{\nu} - g_{\mu} g_{\nu}) \right]
\]
\[ A \mu \] \[ g_{\alpha \beta \sigma} (p+q)_\mu \]

\[ A \mu \] \[ g_{\mu \nu} \]

\[ -ig^2 f_{\alpha \beta \sigma} f_{\chi \lambda \tau} g_{\mu \nu} \]

\[ -ig^2 g_{\mu \nu} (f_{\alpha \beta \sigma} f_{\chi \lambda \tau} + f_{\alpha \chi \tau} f_{\beta \lambda \sigma}) \]

divergences occurring in \( \tilde{\Gamma}[0,A] \) must be renormalized → bare quantities \( A, g, \xi \) related to renormalized quantities \( A_r, g_r, \xi_r \) by

\[
A^\mu = Z_A^{\frac{\nu}{2}} A_r^\mu
\]

\[
g = Z_g \ g_r
\]

\[
\xi = Z_\xi \ \xi_r
\]

remark: \( Q, c, \overline{c} \) appear only inside loops → no renormalization required

\( \tilde{\Gamma}[0,A] \) gauge invariant ⇒ \( Z_g \) and \( Z_A \) are related!

infinities in \( \tilde{\Gamma}[0,A] \) must be \( \sim F_{\mu \nu}^a F_a^{\mu \nu} \cdot \text{const} \)

\[
F_{\mu \nu}^a = Z_A^{\frac{\nu}{2}} \left[ \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - Z_g \ Z_A^{\frac{\nu}{2}} \ p_{abc} \ A_\mu^b A_\nu^c \right]
\]

⇒ \( Z_g = Z_A^{-\frac{\nu}{2}} \)

→ simplifies calculation of Yang-Mills β-function
Yang-Mills β function

dimensional regularization \( d = 4 - 2\epsilon \)

minimal subtraction: \( Z_A = 1 + \sum_{n=1}^{\infty} \frac{Z^{(n)}_A}{\epsilon^n} \)

1 loops \( \rightarrow \) contributions \( \frac{1}{\epsilon}, \ldots, \frac{1}{\epsilon^L} \)

as in \( \varphi^4 \) theory: bare coupling \( g \) not dimensionless in dim. reg.

\[ \int \! d^{4-2\epsilon} \mathbf{x} \left( \partial_{\mu} A_{\nu} \right)^2 \quad \Rightarrow \quad 2 [A] + 2 - (4-2\epsilon) = 0 \]

\[ \Rightarrow \quad [A] = 1 - \epsilon \]

\[ \int \! d^{4-2\epsilon} \mathbf{x} \, g^2 A^4 \quad \Rightarrow \quad 4 [A] + 2 [g] - (4-2\epsilon) = 0 \]

\[ \Rightarrow \quad [g] = 2 - \epsilon - 2 (1 - \epsilon) = \epsilon \]

if we wish to use a dimensionless renormalized coupling \( g_r \)

\[ \Rightarrow \quad g = Z_g \mu^\epsilon g_r \quad \text{with arbitrary mass parameter } \mu \]

\( g \) independent of \( \mu \)

\[ \mu \frac{dg}{d\mu} = 0 = \mu \frac{dZ_g}{d\mu} \mu^\epsilon g_r + \]

\[ + \epsilon Z_g \mu^\epsilon g_r + Z_g \mu^\epsilon \mu \frac{d^2 g_r}{d\mu^2} \]

\[ \Rightarrow 0 = Z_g \mu^\epsilon \left[ \epsilon g_r + \mu \frac{d^2 g_r}{d\mu^2} + g_r \mu \frac{d\ln Z_g}{d\mu} \right] \]
\[ \Rightarrow \quad \beta = -\varepsilon g_r - g_r \mu \frac{\partial \ln Z_g}{\partial \mu} \]

\[ Z_g = Z_A^{-\frac{1}{2}} \]

\[ \Rightarrow \quad \beta = -\varepsilon g_r + \frac{1}{2} g_r \beta \frac{\partial \ln Z_A}{\partial g_r} \]

Chain rule:
\[ \mu \frac{\partial}{\partial \mu} = \mu \frac{\partial g_r}{\partial \mu} \frac{\partial}{\partial g_r} = \beta \frac{\partial}{\partial g_r} \]

\[ \Rightarrow \quad \beta = -\varepsilon g_r + \frac{1}{2} g_r \beta \frac{\partial \ln Z_A}{\partial g_r} \]

\[ \beta = -\varepsilon g_r + \frac{1}{2} g_r \beta \frac{\partial}{\partial g_r} \left( \frac{Z_A^{(1)}}{\varepsilon} + o\left(\frac{1}{\varepsilon^2}\right) \right) \]

\[ \beta \text{ finite } \Rightarrow \text{ divergences must cancel} \]

\[ \lim_{\varepsilon \to 0} : \quad \beta = -\frac{1}{2} g_r^2 \frac{\partial Z_A^{(1)}}{\partial g_r} \]

\[ \rightarrow \beta \text{ function determined by } Z_A^{(1)} \quad (\text{coefficient of } \frac{1}{\varepsilon} \text{ term in } Z_A) \]

calculation of the one-loop $\beta$ function: two graphs

(a) \hspace{2cm} (b)
\[
\text{diagram (a)} \bigg|_{\frac{1}{\varepsilon}} = \frac{i g^2 C_A S_{ab}}{(4\pi)^2} \frac{10}{3\varepsilon} \left[ g_{\mu\nu} p^2 - p_\mu p_\nu \right]
\]

\[
\text{diagram (b)} \bigg|_{\frac{1}{\varepsilon}} = \frac{i g^2 C_A S_{ab}}{(4\pi)^2} \frac{1}{3\varepsilon} \left[ g_{\mu\nu} p^2 - p_\mu p_\nu \right]
\]

\[\text{facd} P_{abcd} = C_A S_{ab} \quad (C_A(\text{SU}(N)) = N)\]

diagram (a) \rightarrow \text{Home work exercise}

diagram (b):

\[
\begin{align*}
\alpha, \mu & \quad p & \quad d & \quad p + R & \quad c \\
\text{A} & \quad \circlearrowleft & \quad \text{R} & \quad \text{A}
\end{align*}
\]

\[
\text{ghost loop} = (-) \int \frac{d^d R}{(2\pi)^d} \ g \ P_{facd} (p+2R) \mu \frac{1}{R^2+i\varepsilon} \frac{1}{(R+p)^2+i\varepsilon} \ g \ P_{cd} (p+2R)\nu
\]

\[
= - \frac{g^2}{C_A S_{ab}} \int \frac{d^d R}{(2\pi)^d} \int \frac{dR}{(2\pi)^d} \ \frac{(p+2R)_\mu (p+2R)_\nu}{(\alpha(R+p)^2 + (1-\alpha)R^2 + i\varepsilon)^2}
\]

\[
\frac{(p+2R)_\mu (p+2R)_\nu}{[R^2 + 2\alpha R \cdot p + \alpha p^2 + i\varepsilon]^2} \left( R + \alpha p \right)^2 - \alpha^2 p^2
\]
\[-g^2 C_A S_{ab} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{\partial} \left[ p + 2(\mathbf{R} - \alpha p) \right]_\mu \left[ p + 2(\mathbf{R} - \alpha p) \right]_\nu \left[ k^2 + \alpha(1-\alpha)p^2 + \mu^2 \right]^2 \]

\[-g^2 C_A S_{ab} \int_0^1 \frac{d^4 k}{(2\pi)^4} \left[ \frac{p_\mu p_\nu (1-2\alpha)^2 + 4 R_{\mu\nu}}{[k^2 + \alpha(1-\alpha)p^2 + \mu^2]^{2}} \right] \frac{-k^2}{d} \]

\[p_{5/4} \]
\[\downarrow \]
\[-g^2 C_A S_{ab} \int_0^1 \frac{1}{(4\pi)^2} \frac{1}{\varepsilon} \left[ p_\mu p_\nu (1-2\alpha)^2 + 2 g_{\mu\nu} \alpha(1-\alpha)p^2 \right] \]

\[+ \text{finite terms} \]

\[= \frac{ig^2 C_A S_{ab}}{(4\pi)^2} \frac{1}{3\varepsilon} \left( \epsilon_{\mu\nu} p^2 - p_\mu p_\nu \right) \]

\[\Rightarrow \text{divergences are cancelled by} \quad Z_A = 1 + \frac{41C_A}{3\varepsilon} \frac{g_r^2}{(4\pi)^2} \]

\[\Rightarrow \beta = -\frac{41C_A}{3} \frac{g_r^3}{(4\pi)^2} + o(g_r^5) \]

**QCD**

Gauge group $SU(3) \rightarrow C_A(SU(3)) = 3 = N_c$

Include also $N_f$ quark flavours

\[\beta(g) = -\beta_0 \frac{g^3}{(4\pi)^2} + o(g^5) \quad, \quad \beta_0 = \frac{4}{3} \left( 11N_c - 2N_f \right) \]

$\beta_0 > 0 \Rightarrow \beta$ function negative $\Rightarrow$ asymptotic freedom
\[-\frac{g_c(\mu)^2}{(4\pi)^2} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda_{\infty}^2)}\]

\(\Lambda_{\infty}\) characterizes the strength of the interaction, irrespective of the scale \(\mu\) (\(g \to \Lambda_{\infty}\); dimensional transmutation)

Remark: mass spectrum of hadrons determined by \(\Lambda_{\infty}\)

Remember situation in \(\lambda\phi^4\) theory:

\[-\frac{\lambda_c(\mu)}{(4\pi)^2} = \frac{1}{3 \ln(\Lambda/\mu)}\]

\(\Lambda_{\phi} = m_\phi \exp\left(\frac{(4\pi)^2}{3 \lambda_c}\right)\)

"Landau pole" in old-fashioned terminology

\(\Lambda_{\phi}\) is the breakdown of perturbation theory.
\(\Lambda \to \infty \) \quad (\text{or} \quad \alpha = \frac{1}{\Lambda} \to 0)

general analysis (independent of perturbation theory):
"triviality" of \(\phi^4\) theory: \(\phi^4\) Lagrangian describes a strictly local QFT in 4 dimensions if the physical coupling is set equal to zero

nevertheless \(\phi^4\) theory acceptable model if not taken seriously
down to arbitrarily small distances

similar situation in \text{QED} \quad \text{(positive } \beta \text{ function!)}

\[ \Lambda_{\text{QED}} \approx m_e \exp \left( \frac{3\pi}{2\alpha_{\text{em}}} \right) \approx 10^{27} \text{GeV} \]

\[ \to \text{ beyond good and evil} \quad (\rho_{\text{Planck}} \approx 10^{19} \text{ GeV}) \]

Higgs sector of SM \quad \text{(}\phi^4\text{ interaction)}

\[ \alpha = \frac{M_h^2}{2\nu^2} \quad M_h \approx 125 \text{ GeV} \quad \nu \approx 246 \text{ GeV} \quad \to \alpha \text{ small} \]

SM makes good sense as effective theory at present energies