24. Functional methods

generating functional $Z[J] = \int [dQ]\, e^{i[S[Q] + J\cdot Q]}$

external source (J)

quantum field (Q)

$J\cdot Q = \int d^4x \sum_r J_r(x)\, Q_r(x)$ normalization $Z[0] = 1$

$$\langle 0|T\{\hat{Q}_{r_1}(x_1)\ldots\hat{Q}_{r_n}(x_n)\}|0\rangle = \int [dQ]\, e^{iS[Q]}\, Q_{r_1}(x_1)\ldots Q_{r_n}(x_n)$$

$$= \frac{1}{i}\frac{\delta}{\delta J_{r_1}(x_1)}\ldots\frac{1}{i}\frac{\delta}{\delta J_{r_n}(x_n)}\, Z[J]\Big|_{J=0}$$

index notation $Q_r(x) \rightarrow Q_j \qquad j \equiv (r,x)$

$$\int d^4x\, J_r(x)\, Q_r(x) \longrightarrow J_j Q_j \equiv J\cdot Q = J^T Q$$

$$\int d^4x\, d^4y\, J_r(x)\, K_{rs}(x,y)\, J_s(y) \longrightarrow J_j K_{jk} J_k \equiv J^T K J$$

example: free scalar field

$$Z_0[J] = \int [dQ]\, e^{i\int d^4x\,[-\frac{1}{2}Q(x)(\Box + m^2 - i\varepsilon)Q(x) + J(x)Q(x)]}$$

$$\rightarrow \int [dQ]\, e^{i[-\frac{1}{2}Q_j A_{jk} Q_k + J_k Q_k]} \qquad A_{jk} = A_{kj}$$

$$\equiv \int [dQ]\, e^{i[-\frac{1}{2}Q^T A Q + J^T Q]} \qquad A^T = A$$

$$A_{jk} \leftrightarrow (\Box + m^2 - i\varepsilon)\,\delta^{(4)}(x-y) = A(x,y)$$

standard procedure: change of variables

$$Q = Q' + k \quad (\text{shift}) \qquad [dQ] = [dQ']$$

$$-\frac{1}{2} Q^T A Q + J^T Q =$$

$$= -\frac{1}{2} Q'^T A Q' \qquad \sim (Q')^2$$

$$-\frac{1}{2} k^T A Q' - \frac{1}{2} Q'^T A k + J^T Q' \qquad \sim Q'$$

$$-\frac{1}{2} k^T A k + J^T k \qquad \sim (Q')^0$$

$$= -\frac{1}{2} Q'^T A Q' - Q'^T (Ak - J) - \frac{1}{2} k^T A k + J^T k$$

so far, k was arbitrary

now: choose k such that $Ak - J = 0 \Rightarrow k = A^{-1} J$

$$\Rightarrow -\frac{1}{2} k^T A k + J^T k = -\frac{1}{2} J^T A^{-1} A A^{-1} J + J^T A^{-1} J$$

$$= \frac{1}{2} J^T A^{-1} J$$

$$\Rightarrow \quad Z_0[J] = e^{\frac{i}{2} J^T A^{-1} J} \underbrace{\int [dQ'] e^{-\frac{i}{2} Q'^T A Q'}}_{Z_0[0] = 1}$$

$$\Rightarrow \quad Z_0[J] = e^{\frac{i}{2} J^T \Delta J} \quad , \Delta = A^{-1}$$

translation: $A^{-1}(x,y) = \frac{1}{\Box + m^2 - i\varepsilon} \delta^{(4)}(x-y) =$

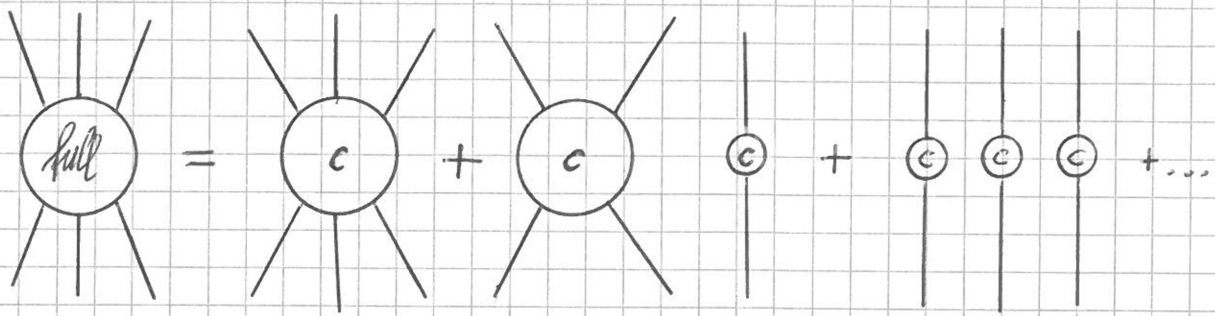
$$= \frac{1}{\Box + m^2 - i\varepsilon} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{m^2 - i\varepsilon - k^2} = \Delta(x-y)$$

$$\Rightarrow \quad Z_0[J] = e^{\frac{i}{2}\int d^4x\, d^4y\, J(x)\,\Delta(x-y)\,J(y)}$$

(c.f. p. 3/9)

back to general case:

Green function $\langle 0|T[\hat{Q}(x_1)\ldots\hat{Q}(x_n)]|0\rangle$ contains ~~also~~ disconnected pieces



only connected Green functions relevant for S matrix!

generating functional of connected Green functions:

$$W[J] = -i \ln Z[J]$$

$$Z[J] = e^{iW[J]}$$

$$\frac{\delta\, iW[J]}{i\,\delta J_1} = \frac{1}{Z[J]}\,\frac{\delta Z[J]}{i\,\delta J_1}$$

$$\Rightarrow \frac{1}{i}\,\frac{\delta\, iW[J]}{\delta J_1}\bigg|_{J=0} = \frac{1}{i}\,\frac{\delta Z[J]}{\delta J_1}\bigg|_{J=0} = \langle 0|\hat{Q}_1|0\rangle$$

VEV

(1-P fct. is connected)

$$\frac{1}{i^2}\frac{\delta^2 iW[J]}{\delta J_1\,\delta J_2} = \frac{1}{Z[J]}\frac{1}{i^2}\frac{\delta^2 Z[J]}{\delta J_1\,\delta J_2} - \frac{1}{Z[J]^2}\frac{\delta Z[J]}{i\delta J_1}\frac{\delta Z[J]}{i\delta J_2}$$

$$\Rightarrow \frac{1}{i^2}\frac{\delta^2 iW[J]}{\delta J_1\,\delta J_2}\bigg|_{J=0} = \frac{1}{i^2}\frac{\delta^2 Z[J]}{\delta J_1\,\delta J_2}\bigg|_{J=0} - \frac{1}{i}\frac{\delta Z[J]}{\delta J_1}\bigg|_{J=0}\frac{1}{i}\frac{\delta Z[J]}{\delta J_2}\bigg|_{J=0}$$

one more derivative:

$$\frac{1}{i^3}\frac{\delta^3 iW[J]}{\delta J_1\delta J_2\delta J_3} = \frac{1}{Z[J]}\frac{1}{i^3}\frac{\delta^3 Z[J]}{\delta J_1\delta J_2\delta J_3} - \frac{1}{Z[J]^2}\frac{1}{i^2}\frac{\delta^2 Z[J]}{\delta J_1\,\delta J_2}\frac{\delta Z[J]}{i\delta J_3}$$

$$- \frac{1}{Z[J]^2}\frac{1}{i^2}\frac{\delta^2 Z[J]}{\delta J_1\,\delta J_3}\frac{\delta Z[J]}{i\delta J_2}$$

$$- \frac{1}{Z[J]^2}\frac{1}{i}\frac{\delta Z[J]}{\delta J_1}\frac{1}{i^2}\frac{\delta^2 Z[J]}{\delta J_2\,\delta J_3}$$

$$+ 2\frac{1}{Z[J]^3}\frac{\delta Z[J]}{i\delta J_1}\frac{\delta Z[J]}{i\delta J_2}\frac{\delta Z[J]}{i\delta J_3}$$

express $\frac{1}{i^2}\frac{\delta^2 Z}{\delta J\,\delta J}$ in terms of $\frac{1}{i^2}\frac{\delta^2 iW}{\delta J\,\delta J}$ and $\frac{\delta iW}{i\delta J}$:

$$\frac{1}{i^3}\frac{\delta^3 iW[J]}{\delta J_1\,\delta J_2\,\delta J_3} = \frac{1}{Z[J]}\frac{1}{i^3}\frac{\delta^3 Z[J]}{\delta J_1\,\delta J_2\,\delta J_3}$$

$$-\frac{1}{Z[J]}\frac{\delta iW[J]}{i\,\delta J_3}\left(\frac{1}{i^2}\frac{\delta^2 iW[J]}{\delta J_1\,\delta J_2} + \frac{1}{Z[J]^2}\frac{\delta Z[J]}{i\,\delta J_1}\frac{\delta Z[J]}{i\,\delta J_2}\right)$$

$$-\frac{1}{Z[J]}\frac{\delta iW[J]}{i\,\delta J_2}\left(\quad \overset{-/\!/-}{1,2 \rightarrow 1,3}\quad\right)$$

$$-\frac{1}{Z[J]}\frac{\delta iW[J]}{i\,\delta J_1}\left(\quad 1,2 \rightarrow 2,3 \quad\right)$$

$$+2\frac{1}{Z[J]^3}\frac{\delta iW[J]}{i\,\delta J_1}\frac{\delta iW[J]}{\delta J_2}\frac{\delta iW[J]}{\delta J_3}$$

$$\Rightarrow \frac{1}{i^3}\frac{\delta^3 iW[J]}{\delta J_1\,\delta J_2\,\delta J_3} = \frac{1}{Z[J]}\frac{1}{i^3}\frac{\delta^3 Z[J]}{\delta J_1\,\delta J_2\,\delta J_3}$$

$$-\frac{1}{Z[J]}\left[\frac{1}{i^2}\frac{\delta^2 iW[J]}{\delta J_1\,\delta J_2}\frac{\delta iW[J]}{i\,\delta J_3} + \frac{1}{i^2}\frac{\delta^2 iW[J]}{\delta J_2\,\delta J_3}\frac{\delta iW[J]}{i\,\delta J_1} + \frac{1}{i^2}\frac{\delta^2 iW[J]}{\delta J_3\,\delta J_1}\frac{\delta iW[J]}{i\,\delta J_2}\right]$$

$$-\frac{1}{Z[J]^3}\frac{\delta iW[J]}{i\,\delta J_1}\frac{\delta iW[J]}{i\,\delta J_2}\frac{\delta iW[J]}{i\,\delta J_3}$$

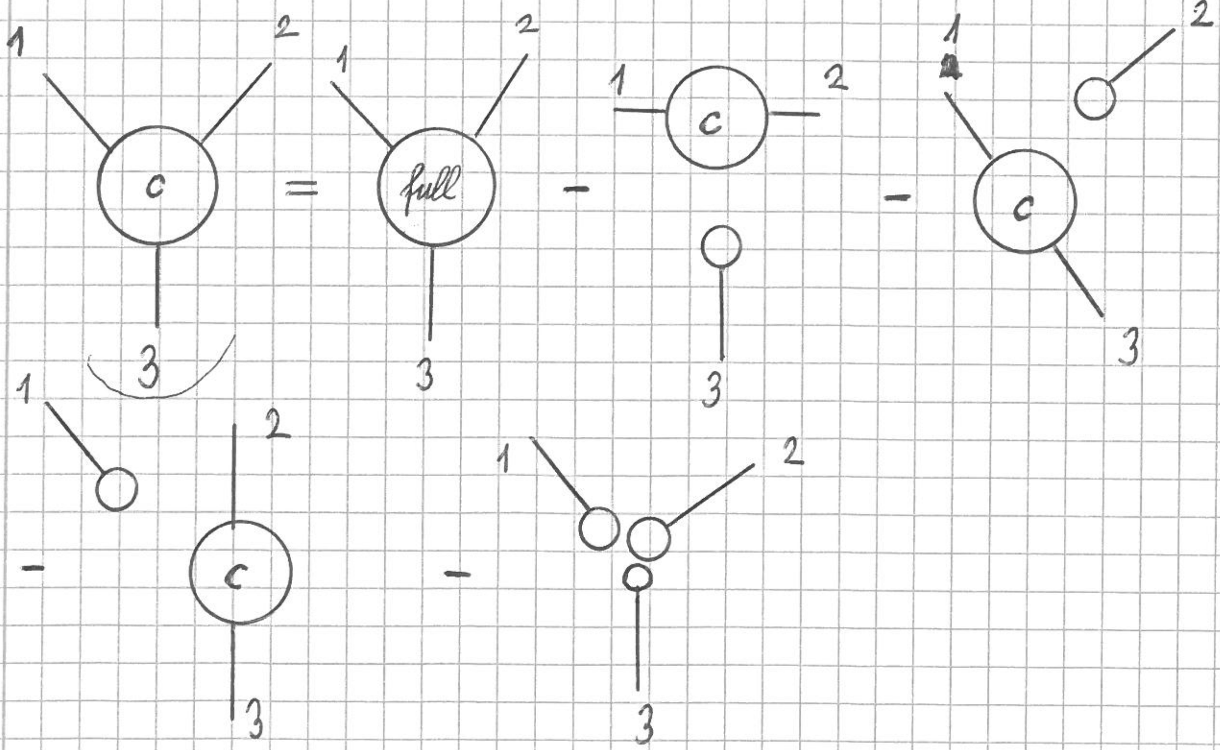
$$\Rightarrow \frac{1}{i^3}\frac{\delta^3 iW[J]}{\delta J_1 \delta J_2 \delta J_3}\bigg|_{J=0} = \frac{1}{i^3}\frac{\delta^3 Z[J]}{\delta J_1 \delta J_2 \delta J_3}\bigg|_{J=0}$$

$$-\frac{1}{i^2}\frac{\delta^2 iW[J]}{\delta J_1 \delta J_2}\bigg|_{J=0}\frac{1}{i}\frac{\delta iW[J]}{\delta J_3}\bigg|_{J=0}$$

$$-\frac{1}{i^2}\frac{\delta^2 iW[J]}{\delta J_2 \delta J_3}\bigg|_{J=0}\frac{1}{i}\frac{\delta iW[J]}{\delta J_1}\bigg|_{J=0}$$

$$-\frac{1}{i^2}\frac{\delta^2 iW[J]}{\delta J_1 \delta J_3}\bigg|_{J=0}\frac{1}{i}\frac{\delta iW[J]}{\delta J_2}\bigg|_{J=0}$$

$$-\frac{1}{i}\frac{\delta iW[J]}{\delta J_1}\bigg|_{J_1=0}\frac{1}{i}\frac{\delta iW[J]}{\delta J_2}\bigg|_{J_2=0}\frac{1}{i}\frac{\delta iW[J]}{\delta J_3}\bigg|_{J=0}$$



etc. ...

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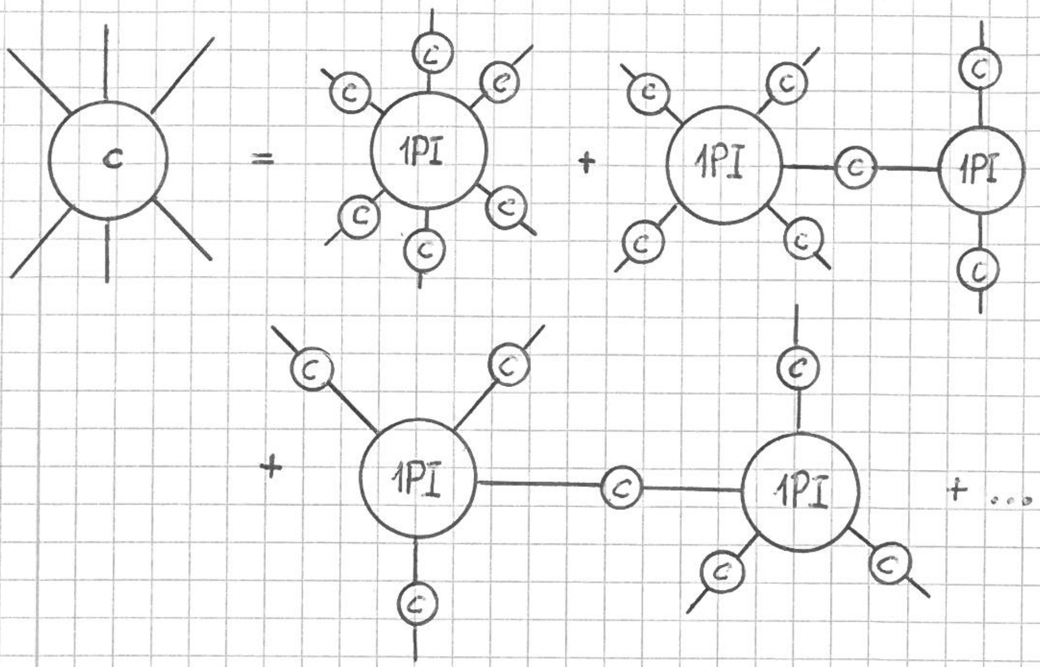
further simplification: express connected Green functions in terms of <u>one-particle-irreducible (1PI)</u> pieces



generating functional of 1PI Green functions: <u>effective action</u>

$$\Gamma[\bar{Q}] = W[J] - J\cdot\bar{Q}$$

(functional) Legendre transform

$$\bar{Q}_k := \frac{\delta W}{\delta J_k} \Rightarrow \frac{\delta\Gamma}{\delta\bar{Q}_k}\delta\bar{Q}_k = \cancel{\frac{\delta W}{\delta J_k}\delta J_k} - \cancel{\bar{Q}_k\delta J_k} - J_k\,\delta\bar{Q}_k$$

$$\Rightarrow \quad \frac{\delta\Gamma}{\delta\bar{Q}_k} = -J_k \qquad \text{"field equation"}$$

classical field equation $\dfrac{\delta S}{\delta Q_k} = -J_k$ $\qquad \begin{array}{l} S\to\Gamma \\ Q_k\to\bar{Q}_k \end{array} \to \dfrac{\delta\Gamma}{\delta\bar{Q}_k} = -J_k$

$$\frac{\delta\Gamma}{\delta\bar{Q}_k} = -J_k \quad\longrightarrow\quad \frac{\delta^2\Gamma}{\delta\bar{Q}_k\,\delta\bar{Q}_\ell} = -\frac{\delta J_k}{\delta\bar{Q}_\ell}$$

$$\overline{Q}_k = \frac{\delta W}{\delta J_k}$$

$$\delta_{kl} = \frac{\delta \overline{Q}_k}{\delta \overline{Q}_l} = \frac{\delta}{\delta \overline{Q}_l}\,\frac{\delta W}{\delta J_k} = \frac{\delta^2 W}{\delta J_s\, \delta J_k}\;\underbrace{\frac{\delta J_s}{\delta \overline{Q}_l}}_{= -\frac{\delta^2 \Gamma}{\delta \overline{Q}_s\, \delta \overline{Q}_l}}$$

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$$\Rightarrow \quad \underbrace{\frac{\delta^2 W}{\delta J_k\, \delta J_s}}_{=: D_{ks}}\;\frac{\delta^2 (-\Gamma)}{\delta \overline{Q}_s\, \delta \overline{Q}_l} = \delta_{kl}$$

($D_{ks} \to \Delta_{ks}$ for free field)

$$\Rightarrow \quad \frac{\delta^2 (-\Gamma)}{\delta \overline{Q}_s\, \delta \overline{Q}_l} = \left(D^{-1}\right)_{sl}$$

$$\underbrace{\frac{\delta^2\, i W}{i^2\, \delta J_k\, \delta J_s}}_{\frac{1}{i} D_{ks}}\;\frac{\delta^2 (-i\Gamma)}{\delta \overline{Q}_s\, \delta \overline{Q}_l} = \delta_{kl}$$

$\frac{1}{i} D_{ks}$ = connected two-point function in the presence of external source J

$$\frac{1}{i} D\;\frac{\delta^2(-i\Gamma)}{\delta \overline{Q}^2}\;\frac{1}{i} D = \frac{1}{i} D$$

$$\frac{\delta}{\delta \overline{Q}_k} = \frac{\delta J_l}{\delta \overline{Q}_k} \frac{\delta}{\delta J_l} = - \underbrace{\frac{\delta^2 \Gamma}{\delta \overline{Q}_k \delta \overline{Q}_l}}_{(D^{-1})_{kl}} \frac{\delta}{\delta J_l}$$

$$= \underbrace{i (D^{-1})_{kl}}_{\uparrow} \underbrace{\frac{1}{i} \frac{\delta}{\delta J_l}}_{\text{"creates" external line}}$$

removes (full) propagator from this line

example: start from

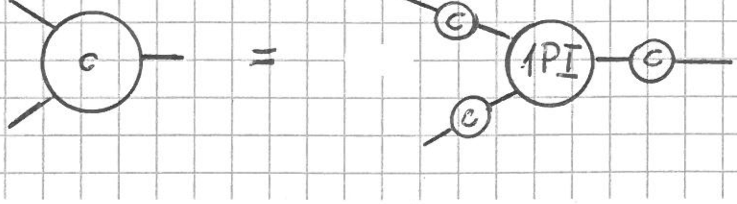
$$\frac{\delta}{\delta \overline{Q}_m} \Bigg| \frac{\delta^2 W}{\delta J_k \delta J_s} \frac{\delta^2 \Gamma}{\delta \overline{Q}_s \delta \overline{Q}_l} = - \delta_{kl}$$

$$\Rightarrow \underbrace{\frac{\delta^2 W}{\delta J_k \delta J_s}}_{D_{ks}} \frac{\delta^3 \Gamma}{\delta \overline{Q}_m \delta \overline{Q}_s \delta \overline{Q}_l} + \left(\frac{\delta}{\delta \overline{Q}_m} \underbrace{\frac{\delta^2 W}{\delta J_k \delta J_s}}_{D_{ks}} \right) \underbrace{\frac{\delta^2 \Gamma}{\delta \overline{Q}_s \delta \overline{Q}_l}}_{-D^{-1}_{sl}} = 0$$

$$\Rightarrow D_{ks} \frac{\delta^3 \Gamma}{\delta \overline{Q}_m \delta \overline{Q}_s \delta \overline{Q}_l} = (D^{-1})_{mr} \frac{\delta^3 W}{\delta J_r \delta J_k \delta J_s} (D^{-1})_{sl}$$

$$\Rightarrow \frac{\delta^3 W}{\delta J_k \delta J_r \delta J_s} = D_{kk'} D_{rr'} D_{ss'} \frac{\delta^3 \Gamma}{\delta \overline{Q}_{k'} \delta \overline{Q}_{r'} \delta \overline{Q}_{s'}}$$

$$\Rightarrow \frac{\delta^3 i W}{i^3 \delta J^3} = \left(\frac{1}{i} D \right)^3 \frac{\delta^3 i \Gamma}{\delta \overline{Q}^3}$$



next example: start from

$$\frac{\delta}{\delta \bar{Q}_{j'}} \left| \frac{\delta^3 W}{\delta J_k \, \delta J_r \, \delta J_s} = D_{kk'} \, D_{rr'} \, D_{ss'} \, \frac{\delta^3 \Gamma}{\delta \bar{Q}_{k'} \, \delta \bar{Q}_{r'} \, \delta \bar{Q}_{s'}} \right.$$

use $\displaystyle \frac{\delta}{\delta \bar{Q}_{j'}} = \left(D^{-1}\right)_{j'j} \frac{\delta}{\delta J_j}$ and

$$\frac{\delta D_{kk'}}{\delta \bar{Q}_{j'}} = D_{kp'} \, D_{q'k'} \, \frac{\delta^3 \Gamma}{\delta \bar{Q}_{j'} \, \delta \bar{Q}_{p'} \, \delta \bar{Q}_{q'}} \quad \text{etc.}$$

$$\Rightarrow \frac{\delta^4 W}{\delta J_j \, \delta J_k \, \delta J_r \, \delta J_s} = D_{jj'} \, D_{kk'} \, D_{rr'} \, D_{ss'} \, \frac{\delta^4 \Gamma}{\delta \bar{Q}_{j'} \, \delta \bar{Q}_{k'} \, \delta \bar{Q}_{r'} \, \delta \bar{Q}_{s'}}$$

$$+ \; D_{jj'} \, D_{kk'} \frac{\delta^3 \Gamma}{\delta \bar{Q}_{j'} \, \delta \bar{Q}_{k'} \, \delta \bar{Q}_{m}} \, D_{mn} \, \frac{\delta^3 \Gamma}{\delta \bar{Q}_{n} \, \delta \bar{Q}_{r'} \, \delta \bar{Q}_{s'}} \, D_{r'r} \, D_{s's}$$

$$+ \; D_{jj'} \, D_{rr'} \frac{\delta^3 \Gamma}{\delta \bar{Q}_{j'} \, \delta \bar{Q}_{r'} \, \delta \bar{Q}_{m}} \, D_{mn} \, \frac{\delta^3 \Gamma}{\delta \bar{Q}_{n} \, \delta \bar{Q}_{s'} \, \delta \bar{Q}_{k'}} \, D_{s's} \, D_{k'k}$$

$$+ \; D_{jj'} \, D_{ss'} \frac{\delta^3 \Gamma}{\delta \bar{Q}_{j'} \, \delta \bar{Q}_{s'} \, \delta \bar{Q}_{m}} \, D_{mn} \, \frac{\delta^3 \Gamma}{\delta \bar{Q}_{n} \, \delta \bar{Q}_{k'} \, \delta \bar{Q}_{r'}} \, D_{k'k} \, D_{r'r}$$