21. Minimal subtraction

back to φ^4 theory!

form of the relations

$$\lambda = \lambda(m_{ph}, \lambda_{ph}) \ , \ m = m(m_{ph}, \lambda_{ph})$$

requires a computation of the 2- and 4-point function to the relevant order in perturbation theory and depends also on the convention used for the definition of λ_{ph} $\longrightarrow$ rather cumbersome!

different approach: <u>minimal subtraction</u> method ('t Hooft)

MS minimal subtraction scheme

$\overline{\text{MS}}$ modified minimal subtraction scheme

<u>basic observation</u>: in dimensional regularization, divergences appear as singularities in the variable d

$\rightarrow$ finite part of any divergent loop integral may be defined in a uniform manner by removing these singularities example:

$$B(0,m^2) = \frac{1}{i}\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-m^2+i\varepsilon)^2} = \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\, m^{d-4}$$

defined on p. 5/14 see p. 5/18

$$x\Gamma(x) = \Gamma(x+1) \Rightarrow B(0,m^2) = -\frac{2\Gamma(3-\frac{d}{2})\, m^{d-4}}{(d-4)\,(4\pi)^{d/2}}$$

$\rightarrow$ pole at $d=4$ with residue $-\frac{2}{(4\pi)^2}$

$\rightarrow$ <u>define</u> $B(0,m^2)_{ren}\Big|_{MS} := B(0,m^2) + \frac{2\mu^{d-4}}{(4\pi)^2}\frac{1}{d-4}$ - <u>finite!</u>

MS scheme

remark: factor μ^{d-4} (μ has dimension of a mass, but otherwise arbitrary) was introduced as $[B] = d-4$

<u>renormalization scale</u> or <u>running</u> (mass-) <u>scale</u> μ

explicit form of $B(0,m^2)_{ren}\Big|_{MS}$:

$$B(0,m^2)_{ren}\Big|_{MS} = -2\,\frac{\Gamma(3-\frac{d}{2})\, m^{d-4}}{(d-4)\,(4\pi)^{d/2}} + \frac{2\,\mu^{d-4}}{(4\pi)^2(d-4)}$$

$$= -\frac{2}{(4\pi)^2(d-4)}\underbrace{\left[\frac{\Gamma(3-\frac{d}{2})\; m^{d-4}}{(4\pi)^{\frac{d-4}{2}}} - \mu^{d-4}\right]}_{\text{expand around } d=4}$$

$$= -\frac{2}{(4\pi)^2(d-4)}\Big\{\left[\Gamma(1) - \tfrac{1}{2}\Gamma'(1)(d-4) + \ldots\right]$$

$$\times\left[1 + (d-4)\ln m - \tfrac{1}{2}(d-4)\ln 4\pi + \ldots\right]$$

$$- \left[1 + (d-4)\ln\mu + \ldots\right]\Big\}$$

$$= -\frac{2}{(4\pi)^2}\left[-\frac{1}{2}\Gamma'(1) - \frac{1}{2}\ln(4\pi) + \ln\frac{m}{\mu}\right]$$

ugly terms $\Gamma'(1) = -\gamma_E$ and $\ln(4\pi)$ can be avoided in modified minimal subtraction scheme ($\overline{MS}$):

$$B_{ren}^{\overline{MS}}(0) = B(0) + \frac{2\mu^{d-4}}{(4\pi)^2}\underbrace{\frac{\Gamma(3-\frac{d}{2})}{(4\pi)^{\frac{d-4}{2}}(d-4)}}_{\Lambda_d}$$

$$\Lambda_d := \frac{\Gamma(3-\frac{d}{2})}{(d-4)(4\pi)^{\frac{d-4}{2}}} = -\frac{1}{2}\,\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d-4}{2}}}$$

$$= \frac{1}{d-4} - \frac{1}{2}\left[\Gamma'(1) + \ln(4\pi)\right] + \mathcal{O}(d-4)$$

now: $B_{ren}^{\overline{MS}}(0) = -\dfrac{2}{(4\pi)^2}\ln\dfrac{m}{\mu} + \mathcal{O}(d-4)$

this procedure may be applied to any function X that contains a simple pole at $d = 4$:

→ determine the residue $R = \lim_{d \to 4} [(d-4) X]$

→ choose the number p such that $R\mu^p$ has the same dimension as X

→ in $\overline{MS}$, the finite part of X is given by

$$X_{ren} = X - R\mu^p \Lambda_d$$

example: $\Delta(0) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{m^2 - k^2 - i\varepsilon} = i \frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{d/2}} m^{d-2}$

→ residue $\frac{2im^2}{(4\pi)^2}$

→ dimension of $\Delta(0)$: $d-2$

→ $\Delta(0)_{ren} \overset{\overline{MS}}{=} \Delta(0) - \frac{2im^2}{(4\pi)^2} \mu^{d-4} \Lambda_d$

→ $\Delta(0)_{ren}^{\overline{MS}} = i \frac{m^2}{(4\pi)^2} \left(\ln \frac{m^2}{\mu^2} - 1 \right)$

<u>extension</u> to graphs with L loops:

→ contains in general divergences $\sim \frac{1}{(d-4)^k}$ $1 \leq k \leq L$

→ corresponding divergent part may be defined by

$$[R_0 \Lambda_d^L + R_1 \Lambda_d^{L-1} + \ldots + R_{L-1} \Lambda_d] \mu^p$$

R_0 is the residue of the pole of highest order,

R_1 is the residue of the pole of order $L-1$ that remains when the term $R_0 \Lambda_d^L \mu^p$ is removed,

etc.