21. Minimal subtraction

back to φ^4 theory!

form of the relations

\[ \lambda = \lambda (m_{ph}, \lambda_{ph}), \quad m = m (m_{ph}, \lambda_{ph}) \]

requires a computation of the 2- and 4-point function to the relevant order in perturbation theory and depends also on the convention used for the definition of \( \lambda_{ph} \) → rather cumbersome!

different approach: minimal subtraction method (\'t Hooft)

MS minimal subtraction scheme

MS modified minimal subtraction scheme

basic observation: in dimensional regularization, divergences appear as singularities in the variable \( d \rightarrow \) finite part of any divergent loop integral may be defined in a uniform manner by removing these singularities
example:

\[ B(0, m^2) = \frac{i}{d} \int \frac{d^d K}{(2\pi)^d} \frac{1}{(K^2 - m^2 + i\varepsilon)^2} = \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \frac{m^{d-4}}{(d-4)} \]

\[ \uparrow \]

defined on p. 5/14

\[ \times \Gamma(x) = \Gamma(x + 1) \Rightarrow B(0, m^2) = -\frac{2}{(4\pi)^2} \frac{\Gamma(3 - \frac{d}{2}) m^{d-4}}{(d-4) (4\pi)^{d/2}} \]

\[ \rightarrow \text{pole at } d = 4 \text{ with residue } -\frac{2}{(4\pi)^2} \]

\[ \rightarrow \text{define } B(0, m^2)_{\text{ren}} := B(0, m^2) + \frac{2}{(4\pi)^2} \frac{\mu^{d-4}}{d-4} \text{ finite!} \]

\[ \text{MS scheme} \]

\[ \text{remark: factor } \mu^{d-4} (\mu \text{ has dimension of a mass, but otherwise arbitrary}) \text{ was introduced as } [B] = d-4 \]

\[ \text{renormalization scale or running (mass-) scale } \mu \]

\[ \text{explicit form of } B(0, m^2)_{\text{ren}} \bigg|_{\text{MS}} = -\frac{2}{(4\pi)^2} \frac{\Gamma(3 - \frac{d}{2}) m^{d-4}}{(d-4) (4\pi)^{d/2}} + \frac{2}{(4\pi)^2} \frac{\mu^{d-4}}{(d-4)} \]

\[ = -\frac{2}{(4\pi)^2 (d-4)} \left[ \frac{\Gamma(3 - \frac{d}{2}) m^{d-4}}{(4\pi)^{d/2}} - \mu^{d-4} \right] \text{ expand around } \]

\[ d = 4 \]
\[-\frac{2}{(4\pi)^2} \left\{ \left[ \Gamma(1) - \frac{1}{2} \Gamma'(1)(d-4) + \ldots \right] \right. \]
\[\times \left[ 1 + (d-4) \ln m - \frac{1}{2} (d-4) \ln 4\pi + \ldots \right] \]
\[- \left[ 1 + (d-4) \ln \mu + \ldots \right] \}
\[-\frac{2}{(4\pi)^2} \left[ - \frac{1}{2} \Gamma'(1) - \frac{1}{2} \ln (4\pi) + \frac{\ln m}{\mu} \right] \]

Ugly terms \( \Gamma'(1) = -\gamma_E \) and \( \ln (4\pi) \) can be avoided in modified minimal subtraction scheme (\text{MS}) :

\[ B_{\overline{\text{MS}}}^{\text{ren}}(\sigma) = B(\sigma) + \frac{2\mu^{d-4}}{(4\pi)^2} \frac{\Gamma(3-\frac{d}{2})}{(4\pi)^{d-4}} (d-4) \frac{\ln \frac{m}{\mu}}{\Lambda_d} \]

\[ \Lambda_d := \frac{\Gamma(3-\frac{d}{2})}{(d-4)(4\pi)^{d-4}} = - \frac{1}{2} \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d-4}} \]

\[ = \frac{1}{d-4} - \frac{1}{2} \left[ \Gamma'(1) + \ln (4\pi) \right] + O(d-4) \]

Now: \[ B_{\overline{\text{MS}}}^{\text{ren}}(\sigma) = -\frac{2}{(4\pi)^2} \frac{\ln \frac{m}{\mu}}{\mu} + O(d-4) \]
this procedure may be applied to any function $\mathcal{X}$ that contains a simple pole at $d = 4$:

→ determine the residue $R = \lim_{d \to 4} (d - 4) \mathcal{X}$

→ choose the number $p$ such that $R \mu^p$ has the same dimension as $\mathcal{X}$

→ in $\overline{\text{MS}}$, the finite part of $\mathcal{X}$ is given by

\[
\mathcal{X}_{\text{ren}} = \mathcal{X} - R \mu^p \Lambda_d
\]

**Example:**

\[
\Delta(0) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{m^2 - k^2 - i\epsilon} = \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{d/2}} m^{d-2}
\]

→ residue

\[
\frac{2i m^2}{(4\pi)^2}
\]

→ dimension of $\Delta(0)$: $d - 2$

→ $\Delta(0)^{\overline{\text{MS}}}_\text{ren} = \Delta(0) - \frac{2i m^2}{(4\pi)^2} \mu^d \Lambda_d$

→ $\Delta(0)^{\overline{\text{MS}}}_\text{ren} = i \frac{m^2}{(4\pi)^2} \left( \ln \frac{m^2}{\mu^2} - 1 \right)$
extension to graphs with $L$ loops:

$\to$ contains in general divergences $\sim \frac{1}{(d-4)^k}$, $1 \leq k \leq L$

$\to$ corresponding divergent part may be defined by

$$\left[R_0 \Lambda_d^L + R_1 \Lambda_d^{L-1} + \ldots + R_{L-1} \Lambda_d \right] \mu^0$$

$R_0$ is the residue of the pole of highest order,

$R_1$ is the residue of the pole of order $L-1$ that remains when the term $R_0 \Lambda_d^L \mu^0$ is removed,

etc.