18. Spontaneous breaking of local gauge symmetries

1) Abelian gauge theory: scalar QED + scalar self interaction

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - V(\phi)$$

$D_\mu = \partial_\mu + ieq A_\mu$, ϕ complex scalar field

$$V(\phi) = r\, \phi^* \phi + \lambda (\phi^* \phi)^2$$

$\lambda > 0$ (stability of the theory)

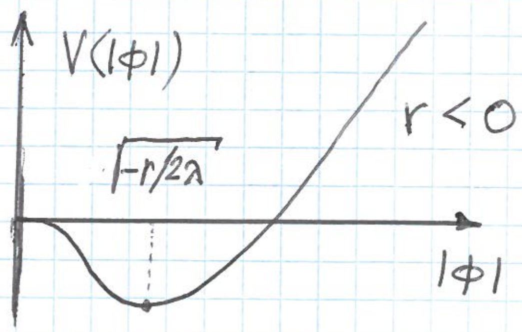
$r > 0$ no SSB $M_\phi^2 = r$ (scalar QED)

$r < 0$ SSB

gauge transformation: $\phi(x) \to e^{-iq\alpha(x)} \phi(x)$

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x)$$

$$V(|\phi|) = r\,|\phi|^2 + \lambda |\phi|^4$$



minimum of the potential determined by

$$2r\,|\phi| + 4\lambda\,|\phi|^3 = 0$$

$$\Rightarrow \quad |\phi|\left(r + 2\lambda|\phi|^2\right) = 0$$

$$\Rightarrow \quad |\phi|^2_{min} = -\frac{r}{2\lambda} = \frac{v^2}{2} \quad , \qquad v = \sqrt{-\frac{r}{\lambda}} > 0$$

$$\phi(x) = e^{i\beta(x)}\,\frac{1}{\sqrt{2}}\left(v + h(x)\right)$$

gauge transformation $\alpha(x) = \beta(x)/q$:

$$\phi(x) \rightarrow \frac{1}{\sqrt{2}}\left(v + h(x)\right) \qquad \text{\underline{"unitary gauge"}}$$

$$(D_\mu\phi)^* D^\mu\phi \rightarrow \frac{1}{2}\left[\partial_\mu h - ieq\,A_\mu(v+h)\right]$$

$$\times\left[\partial^\mu h + ieq\,A^\mu(v+h)\right] =$$

$$= \frac{1}{2}\partial_\mu h\,\partial^\mu h + \frac{e^2q^2}{2}A_\mu A^\mu\,(v+h)^2$$

$$= \frac{1}{2}\partial_\mu h\,\partial^\mu h + \frac{1}{2}\underbrace{(eqv)^2}_{m_A^2}A_\mu A^\mu$$

$$+ v e^2 q^2 A_\mu A^\mu h + \frac{1}{2}e^2q^2 A_\mu A^\mu h^2$$

$$\mathcal{L} \xrightarrow{SSB} -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 A_\mu A^\mu \left(1+\frac{h}{v}\right)^2$$

$$+ \frac{1}{2} \partial_\mu h \, \partial^\mu h - V\left(\frac{v+h}{\sqrt{2}}\right)$$

$$V\left(\frac{v+h}{\sqrt{2}}\right) = \frac{r}{2} (v+h)^2 + \frac{\lambda}{4} (v+h)^4$$

$$= -\frac{\lambda v^2}{2} (v+h)^2 + \frac{\lambda}{4} (v+h)^4$$

$$= \frac{\lambda}{2} (v+h)^2 \left[\frac{1}{2} (v+h)^2 - v^2\right]$$

$$= \frac{\lambda}{4} (h^2 + 2vh + v^2)(h^2 + 2vh - v^2)$$

$$= \frac{\lambda}{4} \left[(2vh + h^2)^2 - v^4 \right]$$

$$= \lambda v^2 h^2 \left(1 + \frac{h}{2v}\right)^2 - \frac{\lambda v^4}{4}$$

$$M_h^2 = 2\lambda v^2$$

$$V\left(\frac{v+h}{\sqrt{2}}\right) = \frac{1}{2} M_h^2 h^2 \left(1+\frac{h}{2v}\right)^2 + V_0$$

irrelevant constant term

remark: physical degrees of freedom

unbroken U(1):	massless vector field	complex scalar
$(r>0)$	2	2
SSB of U(1):	massive vector field	real scalar
$(r<0)$	3	1

2) Nonabelian gauge theory

N real scalars (real representation of gauge group)

$$\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_N \end{bmatrix} \qquad R_a := -i\,T_a \text{ real and antisymmetric}$$

remark: S complex $\rightarrow$ $S = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

real fields

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^T D^\mu\phi - V(\phi)$$

$$D_\mu\phi = (\partial_\mu + igT_a A^a_\mu)\phi = (\partial_\mu - g R_a A^a_\mu)\phi$$

SSB $\langle 0|\phi(x)|0\rangle = v \neq 0$

$\rightarrow$ mass term for the gauge fields

$$\mathcal{L}_m(A) = \frac{1}{2} g^2 (R_a v)^T R_b v \; A^a_\mu A^\mu_b$$

$$M^2_{ab} = g^2 (R_a v)^T R_b v = -g^2 v^T R_a R_b v \quad \text{mass matrix}$$

real and nonnegative as $\xi^T M^2 \xi =$

$$= \xi_a M^2_{ab} \xi_b = g^2 (\xi_a R_a v)^T (\xi_b R_b v) \geq 0 \quad \forall \xi \in \mathbb{R}^n$$

ONB of eigenvectors of M^2:

$\xi_1, \ldots \xi_r$; $\quad M^2 \xi_i = M_i^2 \xi_i \quad , \; M_i^2 \neq 0$

$\eta_1, \ldots, \eta_{n-r}$; $\quad M^2 \eta_k = 0$

$$0 = \eta_k^T M^2 \eta_k = (\eta_k)_a M^2_{ab} (\eta_k)_b =$$

$$= g^2 \left[(\eta_k)_a R_a v\right]^T (\eta_k)_b R_b v \Rightarrow (\eta_k)_a R_a v = 0$$

$\Rightarrow \{ (\eta_k)_a R_a \mid k = 1, \ldots, n-r \}$ generates the unbroken subgroup $H \subset G$

the vectors $b_i := \frac{1}{M_i} g \, (\xi_i)_a R_a v \quad (i=1,\dots,r)$

form an ONS:

$$b_i^T b_j = \frac{g^2}{M_i M_j} \sum_{a,b} (\xi_i)_a (\xi_j)_b \; (R_a v)^T R_b v$$

$$= \frac{1}{M_i M_j} \sum_{a,b} (\xi_i)_a \; M^2_{ab} \; (\xi_j)_b$$

$$= \frac{1}{M_i M_j} \sum_a (\xi_i)_a \; M_j^2 \; (\xi_j)_a \; = \delta_{ij}$$

$\Rightarrow$ the vectors $\{ b_i \}_{i=1}^r$ are linearly independent, they span the r-dimensional space of the Goldstone bosons

<u>remark:</u> this implies that H is indeed a group:

$$[(\eta_k)_a R_a , (\eta_l)_b R_b] = (\eta_k)_a (\eta_l)_b f_{abc} R_c$$

$\xi_i \xi_i^T + \eta_k \eta_k^T = \mathbb{1}$ completeness relation

$$(\xi_i)_a (\xi_i)_b + (\eta_k)_a (\eta_k)_b = \delta_{ab}$$

$$\Rightarrow \quad (\eta_k)_a (\eta_l)_b f_{abc} R_c =$$

$$= (\eta_k)_a (\eta_l)_b f_{abc} \delta_{cd} R_d$$

$$= (\eta_k)_a (\eta_l)_b f_{abc} (\zeta_i)_c (\zeta_i)_d R_d$$

$$+ (\eta_k)_a (\eta_l)_b f_{abc} (\eta_m)_c (\eta_m)_d R_d$$

to be shown: $\quad (\eta_k)_a (\eta_l)_b f_{abc} (\zeta_i)_c = 0$

$$0 = [(\eta_k)_a R_a , (\eta_l)_b R_b] v =$$

$$= (\eta_k)_a (\eta_l)_b f_{abc} (\zeta_i)_c \underbrace{(\zeta_i)_d R_d v}_{\frac{M_i}{g} b_i}$$

the vectors $\{ M_i b_i \}_{i=1}^{r}$ are linearly independent

$$\Rightarrow \quad (\eta_k)_a (\eta_l)_b f_{abc} (\zeta_i)_c = 0$$

scalar mass matrix

$$(M_\phi^2)_{ij} = \left.\frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\right|_{\phi=v}$$

eigenvalue 0 is r-fold degenerate with eigenvectors b_i

further N−r eigenvectors denoted by c_ℓ

→ ONB $\{b_i, c_\ell\}$

possible field parametrization

$$\phi(x) = e^{\sum_{i=1}^{r} \beta_i(x)(\xi_i)_a R_a} \left(v + \sum_{\ell=1}^{N-r} c_\ell h_\ell(x)\right)$$

$$= v + \sum_{\ell=1}^{N-r} c_\ell h_\ell(x) + \sum_{i=1}^{r} \beta_i(x) \underbrace{(\xi_i)_a R_a v}_{\frac{M_i}{g} b_i}$$

gauge transformation → <u>unitary gauge</u>

$$\phi(x) \to v + \sum_{\ell=1}^{N-r} c_\ell h_\ell(x)$$

would-be-Goldstone bosons disappear from $\mathcal{L}$

<u>remark:</u>

$$
\begin{tikzcd}[column sep=large]
 & \text{\underline{global} symmetry} & \text{\underline{massless} Goldstone bosons} \\
\text{\underline{continuous} symmetry (Lie group)} \arrow[ur] \arrow[dr] & \arrow[r, "\text{SSB}"] & {} \\
 & \text{\underline{local} symmetry} & \text{\underline{massive} vector bosons}
\end{tikzcd}
$$

<u>remark:</u> gauge theories are <u>renormalizable</u> ('t Hooft, Veltman), SSB does <u>not</u> spoil renormalizability in spite of massive vector bosons

gauge boson propagator in <u>unitary</u> gauge:

$$\frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}}{k^2 - M^2}$$

in "<u>renormalizable</u>" gauges: $\frac{k_\mu k_\nu}{M^2} \longrightarrow \frac{k_\mu k_\nu}{k^2}$

+ would-be-Goldstone bosons + ghosts