

17. Spontaneous symmetry breaking

Symmetries in QFT:

symmetry group G of QFT, i.e. $\{U(g) | g \in G\}$ is a representation of G in the Hilbert space of state vectors with $[U(g), H] = 0 \quad \forall g \in G$

Wigner-Weyl realization of a symmetry:

multiplets with respect to G

$$H |\alpha\rangle = E |\alpha\rangle$$

eigenspace associated with eigenvalue E spanned by vectors $\{|\alpha\rangle\}$

$$H U(g) |\alpha\rangle = U(g) \underbrace{H |\alpha\rangle}_{E |\alpha\rangle} = E U(g) |\alpha\rangle$$

$$U(g) |\alpha\rangle = \sum_{\beta} |\beta\rangle \underbrace{\langle \beta | U(g) |\alpha\rangle}_{U_{\beta\alpha}(g)}$$

\Rightarrow vectors $\{|\alpha\rangle\}$ form an invariant subspace under G (multiplet)

remark: this situation is well known from QM

potential invariant under rotations \rightarrow angular momentum conserved \rightarrow degenerate energy levels for given total angular momentum

if ground state unique $\Rightarrow U(g)|0\rangle = |0\rangle$

$\forall g \in G$ (ground state invariant under G)

basic assumption in QFT: unique vacuum state $|0\rangle$ ($P^\mu|0\rangle = 0$)

Nambu-Goldstone realization of a symmetry:

ground state not invariant under the symmetry group of the Hamiltonian

examples from solid state physics:

ferromagnet: Hamiltonian invariant under rotations
but: for $T < T_c$ (Curie temperature)
 ground state not rotation invariant;
 no orientation of the spins energetically
 preferred, but a specific one
singled out in the ground
 state \rightarrow rotational symmetry
spontaneously broken

infinite crystalline
solid:

translational symmetry
spontaneously broken

assumption: theory (Hamiltonian) invariant
 under Lie group G (continuous group)

$$\partial_\mu \mathcal{J}^\mu(x) = 0 \rightarrow Q = \int d^3x \mathcal{J}^0(x)$$

$$\|Q|0\rangle\|^2 = \langle 0|Q Q|0\rangle =$$

$$= \int d^3x \langle 0|Q \gamma^\mu(x)|0\rangle$$

$$= \int d^3x \langle 0|Q e^{iP_x} \gamma^\mu(0) e^{-iP_x}|0\rangle$$

$$= \int d^3x \underbrace{\langle 0|Q \gamma^\mu(0)|0\rangle}_{\text{independent of } x}$$



Goldstone alternative

$$Q|0\rangle = 0$$

$$\|Q|0\rangle\| = \infty$$

Wigner - Weyl

(i.e. Q not defined)

linear representation

Nambu - Goldstone

of the symmetry group

nonlinear realization

degenerate multiplets

of the symmetry
massless Goldstone bosons

unbroken (exact)

SSB

symmetry

Goldstone theorem

basic assumption: theory invariant under Lie group

G with conserved currents $J_a^\mu(x)$, $\partial_\mu J_a^\mu(x) = 0$

as seen previously, the associated charges

$Q_a = \int d^3x J_a^0(x)$ are formal objects

causality ensures, however, that

$$[J_a^0(x), O(y)] = 0 \quad \text{if } (x-y)^2 < 0$$

↑
local operator

space-like

$$\Rightarrow \int d^3x [J_a^0(x), O(y)] = [Q_a, O(y)]$$

receives only contributions from the finite volume $|\vec{x} - \vec{y}| \leq |x^0 - y^0|$

commutation relations $[Q_a, Q_b] = i f_{abc} Q_c$
 are also of formal nature, but both
 sides of

$$[Q_a, J_b^\mu(x)] = i f_{abc} J_c^\mu(x)$$

may be applied to physical states,
 with a perfectly meaningful result

consider all local operators $O_m(x)$ that
 may be formed with the dynamical variables
 of the theory, their derivatives and their
 products at a given point x of space-time,
 allowing for an arbitrary, but finite
 number of derivatives and fields

transform according to representation $D(g)$
 under G :

$$O_m \xrightarrow{g \in G} D_{mn}(g) O_n$$

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$$\begin{aligned}
 e^{i\alpha_a Q_a} O_m e^{-i\alpha_b Q_b} &= O_m + i\alpha_a [Q_a, O_m] + \mathcal{O}(\alpha^2) \\
 &= (e^{-i\alpha_a T_a})_{mn} O_n \\
 &= O_m - i\alpha_a (T_a)_{mn} O_n + \mathcal{O}(\alpha^2)
 \end{aligned}$$

$$[Q_a, O_m] = - (T_a)_{mn} O_n$$

$$[T_a, T_b] = i f_{abc} T_c$$

1) Wigner-Weyl : $|0\rangle$ invariant under G ,

$$\text{i.e. } Q_a |0\rangle = 0$$

$$\underbrace{\langle 0 | e^{i\alpha \cdot Q} O_m e^{-i\alpha \cdot Q} | 0 \rangle}_{Q_a |0\rangle = 0} = (e^{-i\alpha \cdot T})_{mn} \langle 0 | O_n | 0 \rangle$$

$$\Rightarrow \langle 0 | O_m | 0 \rangle$$

only those operators, which are invariant under G , may acquire a vacuum expectation value

2) Nambu-Goldstone (SSB):

$\langle 0 | O_m | 0 \rangle \neq 0$ possible, even if O_m transforms under G in a nontrivial manner

order parameters: vacuum expectation values of operators that do not commute with the charges (VEVs of operators that transform nontrivially under G)

order parameters are quantitative measures of spontaneous symmetry breaking - if the vacuum is invariant under G , they vanish

proposition: the occurrence of nonvanishing order parameters implies

$$\langle 0 | [Q_a, O_m(x)] | 0 \rangle \neq 0$$

for at least one charge Q_a and one operator O_m

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proof: we assume the existence of a local operator $O_n(x)$, which transforms nontrivially under G , with $\langle 0 | O_n(x) | 0 \rangle \neq 0$

O_n belongs to a certain irreducible representation contained in $D(g)$ \rightarrow we restrict our set of local operators to this irreducible representation:

$$v_m := \langle 0 | O_m(x) | 0 \rangle$$

$$\begin{aligned} & \langle 0 | e^{i\alpha \cdot Q} O_m(x) e^{-i\alpha \cdot Q} | 0 \rangle \\ &= \underbrace{\left(e^{-i\alpha \cdot T} \right)_{mn}}_{\text{irreducible representation}} \underbrace{\langle 0 | O_n(x) | 0 \rangle}_{v_n} \end{aligned}$$

$v \neq 0 \Rightarrow$ spans the whole representation space

\Rightarrow not all $T_\alpha v$ vanish

infinitesimal form:

$$i\alpha_a \langle 0 | [Q_a, O_m(\alpha)] | 0 \rangle = -i\alpha_a (T_a)_{mn} v_n$$

\Rightarrow not all $\langle 0 | [Q_a, O_m(x)] | 0 \rangle$ vanish,

i.e.

$$\langle 0 | [Q_a, O_m(x)] | 0 \rangle \neq 0$$

for at least one a and at least one m

\Rightarrow at least one entry of the
order parameter matrix

$$C_{am} := -i \langle 0 | [Q_a, O_m(x)] | 0 \rangle$$

is different from zero

-Goldstone theorem: Order parameters can take nonzero values only if the spectrum of the theory contains massless particles

remark 1: $G = \text{Lie group}$ is essential: SSB of discrete symmetries (like P, CP) does not give rise to Goldstone bosons

example:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{\kappa}{2} \varphi^2 - \frac{\lambda}{4} \varphi^4, \quad \lambda > 0$$

the discrete symmetry $\varphi \rightarrow -\varphi$ is spontaneously broken (for $\kappa > 0$) but

there is no massless state (exercise)

remark 2: proof of Goldstone Theorem given below makes use of Lorentz invariance; this is not essential, the Goldstone

theorem applies also to nonrelativistic systems; main difference: dispersion law

not necessarily $\omega(\vec{R}) = |\vec{R}| / v$ ($v=c$)

examples: magnons of a ferromagnet $\omega \sim \vec{R}^2$
antiferromagnet $\omega \sim |\vec{R}|$

remark 3: L Lorentz invariant $\not\Rightarrow$ $10>$ Lorentz inv.

in principle, Lorentz invariance might break down spontaneously, just as well as an internal symmetry (usual situation in condensed matter physics)

particle physics: presently no experimental evidence that the ground state would single out a preferred frame of reference

→ we restrict ourselves to theories with Lorentz invariant vacuum

$$\Rightarrow \langle 0 | j^\mu(x) | 0 \rangle = 0, \text{ otherwise}$$

$\langle 0 | j^\mu(x) | 0 \rangle$ would single out a certain direction in space-time

tensor fields: $\langle 0 | T^{\mu\nu}(x) | 0 \rangle = \frac{g^{\mu\nu}}{4} \langle 0 | T^\alpha_\alpha(x) | 0 \rangle$

reason: $T^{\mu\nu}$ does not transform irreducibly under the Lorentz group: T^α_α represents an invariant subspace, transforming like a scalar field

the vacuum expectation value of the traceless part, $T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T^\alpha_\alpha$ vanishes

all of the local operators may be decomposed in this manner \rightarrow only scalar or pseudo-scalar operators may give rise to order parameters

- proof of the Goldstone theorem

nonvanishing order parameter \Rightarrow

$$\langle 0 | [Q, \phi(x)] | 0 \rangle \neq 0$$

for at least one of the conserved charges
and one of the local Lorentz scalars

$$F_\mu^+(x) := \langle 0 | J_\mu(x) \phi(0) | 0 \rangle$$

$$Q = \int d^3x J_0(x)$$

↑
Wightman function
(no T symbol)

$$1 = \sum_n |n\rangle \langle n|$$

$\{|n\rangle\}$ complete set of
energy-momentum eigenstates

$$\Rightarrow F_\mu^+(x) = \sum_n \langle 0 | J_\mu(x) | n \rangle \langle n | \phi(0) | 0 \rangle$$

$\underbrace{e^{iP_x} J_\mu(0) e^{-iP_x}}$

$$= \sum_n e^{-ip_n x} \langle 0 | J_\mu(0) | n \rangle \langle n | \phi(0) | 0 \rangle$$

this expression contains only positive frequencies

$p_n^0 \geq 0 \Rightarrow F_\mu^+(x)$ analytic in the lower half of the x^0 -plane (real x^0 : $x^0 \rightarrow x^0 - i\varepsilon$)

Lorentz invariance $\Rightarrow F_\mu^+(x) = x_\mu F(x^2)$

$$\partial^\mu \gamma_\mu(x) = 0 \Rightarrow \partial^\mu F_\mu^+(x) = 0$$

$$\Rightarrow 4F(x^2) + x_\mu F'(x^2) 2x^\mu = 0$$

$$2F(x^2) + x^2 F'(x^2) = 0$$

ordinary first order differential equation

$$2F(u) + u \frac{dF(u)}{du} = 0 \quad | \cdot \frac{du}{uF}$$

$$\frac{2du}{u} + \frac{dF}{F} = 0$$

$$2\ln u + \ln F = \text{const.}$$

$$F(u) \sim \frac{1}{u^2}$$

$$\Rightarrow F_\mu^+(x) = \langle 0 | j_\mu(x) \Theta(0) | 0 \rangle$$

$$= -\frac{C}{2\pi^2} \frac{x_\mu}{x^4} = \frac{C}{4\pi^2} \partial^\mu \frac{1}{x^2}$$

remember: $\frac{1}{x^2} \rightarrow \frac{1}{(x^0 - i\varepsilon)^2 - \vec{x}^2}$

scalar propagator ($m=0$):

$$\Delta_0^+(x) = \Theta(x^0) i \underbrace{\int \frac{d^3 p}{(2\pi)^3 2|\vec{p}|} e^{-ipx}}_{\Delta_0^+(x)}$$

$$+ \Theta(-x^0) i \underbrace{\int d\mu(p) e^{+ipx}}_{\Delta_0^-(x)}$$

$$\begin{aligned} \Delta_0^+(x) &= i \int d\mu(p) e^{-ipx} \\ &= \frac{i}{(2\pi)^3} \int d^4 p \Theta(p^0) \delta(p^2) e^{-ipx} \\ &\stackrel{\text{ex.}}{=} \frac{1}{4\pi^2 i [(x^0 - i\varepsilon)^2 - \vec{x}^2]} \end{aligned}$$

analogously:

$$\bar{F}_\mu^-(x) = \langle 0 | \partial(0) J_\mu(x) | 0 \rangle = -\frac{C'}{2\pi^2} \frac{x_\mu}{x^4}$$

$$= \frac{C'}{4\pi^2} \partial_\mu \frac{1}{x^2} = i C' \partial_\mu \underbrace{\Delta_o^+(-x)}_{-\Delta_o^-(x)}$$

where $\frac{1}{x^2} \rightarrow \frac{1}{(x^0 + i\varepsilon)^2 - \vec{x}^2}$

causality $\Rightarrow \bar{F}_\mu^+(x) = \bar{F}_\mu^-(x)$ for $x^2 < 0$
 (spacelike)

$$\Rightarrow C = C'$$

$$\Rightarrow \langle 0 | [J_\mu(x), \partial(0)] | 0 \rangle = \bar{F}_\mu^+(x) - \bar{F}_\mu^-(x)$$

$$= \frac{C}{4\pi^2} \partial_\mu \left[\underbrace{\frac{1}{(x^0 - i\varepsilon)^2 - \vec{x}^2}}_{P \frac{1}{x^2} + i\pi\varepsilon(x^0)\delta(x^2)} - \underbrace{\frac{1}{(x^0 + i\varepsilon)^2 - \vec{x}^2}}_{P \frac{1}{x^2} - i\pi\varepsilon(x^0)\delta(x^2)} \right]$$

$$= \frac{iC}{2\pi} \partial_\mu [\varepsilon(x^0) \delta(x^2)]$$

where $\varepsilon(x^0) = \begin{cases} +1 & x^0 > 0 \\ -1 & x^0 < 0 \end{cases}$

integrate $\mu=0$ component over space at arbitrary $x^0 = t$

$$\delta(x^2) = \delta[(x^0)^2 - \vec{x}^2] =$$

$$= \delta[(x^0 - r)(x^0 + r)] , \quad r = |\vec{x}|$$

$$= \frac{1}{2r} \delta(x^0 - r) + \frac{1}{2r} \delta(x^0 + r)$$

$$\Rightarrow \int d^3x \langle 0 | [j_0(t, \vec{x}), \phi(0)] | 0 \rangle$$

$$= \frac{iC}{2\pi} \int d^3x \frac{\partial}{\partial x^0} [\varepsilon(x^0) \delta(x^2)] \Big|_{x^0=t}$$

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$$= \frac{iC}{2\pi} \frac{\partial}{\partial x^0} \varepsilon(x^0) 4\pi \int_0^\infty \frac{dr r^2}{2r} [\delta(x^0 - r) + \delta(x^0 + r)] \Big|_{x^0=t}$$

$$= iC \frac{\partial}{\partial x^0} \varepsilon(x^0) \int_{-\infty}^{+\infty} dr r \Theta(r) [\delta(x^0 - r) + \delta(x^0 + r)] \Big|_{x^0=t}$$

$$= iC \frac{\partial}{\partial x^0} \varepsilon(x^0) \underbrace{[x^0 \Theta(x^0) - x^0 \Theta(-x^0)]}_{\varepsilon(x^0) x^0} \Big|_{x^0=t}$$

$$= iC \frac{\partial}{\partial x^0} x^0 = iC$$

$$\Rightarrow \underbrace{\langle 0 | [Q, \delta(0)] | 0 \rangle}_{\neq 0} = iC$$

$$\Rightarrow C \neq 0$$

occurrence of the massless propagator in the above explicit expression for the Wightman function signals the presence of a massless particle:

$$F_\mu^+(x) = \langle 0 | \bar{J}_\mu(x) | 0 \rangle$$

$$= \sum_n e^{-ip_n x} \langle 0 | \bar{J}_\mu(0) | n \rangle \langle n | \bar{\phi}(0) | 0 \rangle$$

$$= i C \partial_\mu \Delta_0^+(x)$$

$$= \frac{iC}{(2\pi)^3} \int d^4 p \Theta(p^0) \delta(p^2) p_\mu e^{-ipx}$$

$$= i C \underbrace{\int d\mu(p) p_\mu}_{\frac{d^3 p}{(2\pi)^3 2p^0}}, \quad p^0 = |\vec{p}| e^{-ipx}$$

only intermediate states with $-p_n^2 = 0$ contribute

essential point: $C \neq 0 \Rightarrow$ spectrum contains massless particles

remark: $\bar{\phi}$ scalar $\Rightarrow \langle n | \bar{\phi}(0) | 0 \rangle \neq 0$ only if angular momentum of $|n\rangle$ vanishes $\Rightarrow \langle 0 | \bar{J}_\mu(0) | n \rangle \sim p_n^\mu$; current conservation $\Rightarrow p_n^\mu \langle 0 | \bar{J}_\mu(0) | n \rangle = 0$, both conditions can only be met if $p_n^2 = 0$

number of Goldstone bosons

so far, our proof showed that the spectrum of the theory contains one or more Goldstone bosons

denote the corresponding one-particle states by $|\pi_r(p)\rangle$, $r=1, \dots, N_\pi$

we want to determine the minimal number of massless particles required by the spontaneous breakdown of a continuous symmetry

of conserved currents $\gamma_a^\mu = \dim(G) =$
 $=$ # of the generators of the Lie group $G =$
 $=$ dimension of the Lie algebra
 $\mathcal{L} = \{Q \mid Q = \sum_a \alpha_a Q_a, \alpha_a \in \mathbb{R}\}$

$$\underbrace{\langle 0 | e^{i\alpha \cdot Q} O_m e^{-i\alpha \cdot Q} | 0 \rangle}_{\langle 0 | O_m | 0 \rangle + i\alpha_a \langle 0 | [Q_a, O_m] | 0 \rangle + \mathcal{O}(\alpha^2)} = \underbrace{(e^{-i\alpha \cdot T})_{mn}}_{\langle 0 | O_n | 0 \rangle - i\alpha_a (T_a)_{mn} \langle 0 | O_n | 0 \rangle + \mathcal{O}(\alpha^2)} \langle 0 | O_n | 0 \rangle$$

we consider those elements of \mathcal{L} for which

$$\langle 0 | e^{i\alpha \cdot Q} O_m e^{-i\alpha \cdot Q} | 0 \rangle = \langle 0 | O_m | 0 \rangle$$

forall local operators O_m

this is equivalent with the condition

$$\langle 0 | [Q, O_m] | 0 \rangle = 0 \quad \forall O_m$$

$$\mathcal{H} := \{ Q \in \mathcal{L} \mid \langle 0 | [Q, O_m] | 0 \rangle = 0 \quad \forall O_m \}$$

is a subalgebra of \mathcal{L} , which means

$$\text{that } Q, Q' \in \mathcal{H} \Rightarrow \alpha Q + \alpha' Q' \in \mathcal{H}$$

$$\text{and } [Q, Q'] \in \mathcal{H}$$

(proof is left as a simple home work problem)

$\Rightarrow \mathcal{H}$ generates a subgroup $H \subset G$

\Rightarrow decomposition of Lie algebra $L = \mathcal{H} + \mathcal{K}$

\mathcal{H} is spanned by those charges that annihilate the ground state (they represent symmetries of the vacuum)

\mathcal{K} exclusively contains elements with " $|Q|O\rangle \neq 0$ "

\mathcal{K} does not form a subalgebra!

SSB $\Leftrightarrow \dim(H) < \dim(G)$

$Q \in \mathcal{K} \Rightarrow \langle 0 | [Q, O_m] | 0 \rangle \neq 0$ for

at least one Lorentz scalar O_m

we choose a basis of L where

$H_1, \dots, H_{\dim(H)}$ spans the subalgebra \mathcal{H} , and

$K_1, \dots, K_{\dim(G) - \dim(H)}$ spans \mathcal{K}

$$\mathcal{H} = \left\langle H_i \right\rangle_{i=1}^{\dim(H)}, \quad \mathcal{K} = \left\langle K_s \right\rangle_{s=1}^{\dim(G) - \dim(H)}$$

order parameter matrix

$$C_{am} = -i \langle 0 | [Q_a, O_m] | 0 \rangle$$

has two different types of matrix elements
with respect to our new basis:

$$C_{im} = -i \langle 0 | [H_i, O_m] | 0 \rangle = 0 \quad \forall m$$

$$C_{sm} = -i \langle 0 | [K_s, O_m] | 0 \rangle$$



for given s different from zero
for at least one index $m = m(s)$

furthermore, a linear combination $\sum_s \alpha_s K_s$
with $\sum_s \alpha_s C_{sm} = 0 \quad \forall m$ is excluded,
as this would imply $\sum_s \alpha_s K_s \in \mathcal{H}$

form of the order parameter matrix
 $(C_{im} = 0 \text{ not shown})$

$$\begin{matrix} & O_1 & O_2 & \dots \\ K_1 & & \ddots & \\ K_2 & & \ddots & \\ \vdots & & & \\ K_{d(G)-d(H)} & & \ddots & \end{matrix} \quad \left. \right\} \begin{matrix} d(G) - d(H) \\ \text{linearly independent} \\ \text{rows} \end{matrix}$$

\Rightarrow the order parameter matrix C_{am} contains $d(G) - d(H)$ linearly independent rows

order parameter C_{sm} determines the Wightman function

$$\langle 0 | \bar{J}_S^\mu(x) O_m(0) | 0 \rangle = i C_{sm} \int \frac{d^3 p}{(2\pi)^3 2p^0} p^\mu e^{-ipx}$$

$$p^0 = |\vec{p}|$$

$$1 = \sum_n |n\rangle\langle n| = \sum_{r=1}^{N_\pi} \int \frac{dp^3}{(2\pi)^3 2p^0} |\pi_r(p)\rangle\langle\pi_r(p)| + \dots$$

↑
massless scalar states

$$\Rightarrow \langle 0 | J_s^\mu(x) O_m(0) | 0 \rangle = \underbrace{\sum_{r=1}^{N_\pi} \int dp \mu(p)}_{\text{only massless scalar states}} \underbrace{\langle 0 | J_s^\mu(x) | \pi_r(p) \rangle}_{\downarrow \text{contribute}} \langle \pi_r(p) | O_m(0) | 0 \rangle$$

$$= \sum_{r=1}^{N_\pi} \int dp \mu(p) \underbrace{\langle 0 | J_s^\mu(x) | \pi_r(p) \rangle}_{i F_{sr} p^\mu e^{-ipx}} \underbrace{\langle \pi_r(p) | O_m(0) | 0 \rangle}_{G_{rm}}$$

$$\Rightarrow C_{sm} = \sum_{r=1}^{N_\pi} F_{sr} G_{rm}$$

C_{sm} contains $\dim(G) - \dim(H)$ linearly independent rows $\Rightarrow N_\pi \geq \dim(G) - \dim(H)$

remark: the above argument leads to an inequality ; reason: symmetry arguments alone do not suffice to exclude the occurrence of additional massless particles;

trivial example: imagine that we had been looking at a subgroup of the full symmetry group without noticing and that some of the extra generators do not annihilate the vacuum \rightarrow entire discussion would have gone through, but we would have missed some of the Goldstone bosons

example for SSB: vacuum expectation value of a scalar field

order parameter $\langle 0 | \phi(x) | 0 \rangle \neq 0$ for scalar field $\phi(x)$

we consider the following model:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^T \partial^\mu \phi - V(\phi)$$

$$\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_N \end{bmatrix}$$

N real scalar fields

\mathcal{L} invariant under Lie group G with representation $e^{-i\alpha_a T_a}$ acting on ϕ (restricts potential $V(\phi)$)

remark: $T_a^T = -T_a = T_a^*$ (ϕ real)

Noether current: $J_a^\mu = \frac{i}{2} \phi^T T_a \overleftrightarrow{\partial}^\mu \phi$

$$Q_a = \int d^3x J_a^0$$

infinitesimal form of $e^{i\alpha \cdot Q} \phi e^{-i\alpha \cdot Q} = e^{-i\alpha \cdot T} \phi$:

$$[Q_a, \phi_m] = - (T_a)_{mn} \phi_n$$

we assume: $\langle 0 | \phi_m(x) | 0 \rangle = v_m$

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} \neq 0$$

$$\underbrace{\langle 0 | [Q_a, \phi_m] | 0 \rangle}_{i C_{am}} = - (T_a)_{mn} v_n$$

explicit determination of the Goldstone bosons of our scalar model (tree approximation)

a) vacuum expectation value (in the tree approximation) determined by the minimum of $V(\phi)$:

$$\frac{\partial V}{\partial \phi_m} \Big|_{\phi=v} = 0 \quad \forall m$$

b) mass matrix

$$V(\phi) = V(v) + \frac{1}{2!} \frac{\partial^2 V}{\partial \phi_m \partial \phi_n} \Big|_{\phi=v} (\phi - v)_m (\phi - v)_n + \dots$$

$$\phi' := \phi - v$$

$$\rightarrow \text{mass matrix } (M^2)_{mn} = \frac{\partial^2 V}{\partial \phi_m \partial \phi_n} \Big|_{\phi=v}$$

c) invariance of $V(\phi)$

$$V(\phi) = V(e^{-i\alpha_a T_a} \phi)$$

$$= V(\phi) - i\alpha_a (T_a \phi)_n \frac{\partial V}{\partial \phi_n} + \dots$$

$$\Rightarrow \frac{\partial V}{\partial \phi_n} (T_a \phi)_n = 0$$

d) massless degrees of freedom

differentiate previous equation $\left(\frac{\partial}{\partial \phi_m} \dots \right)$

$$\rightarrow \frac{\partial^2 V}{\partial \phi_m \partial \phi_n} (T_a \phi)_n + \frac{\partial V}{\partial \phi_n} (T_a)_{nm} = 0$$

$$\text{insert } \phi = v \rightarrow (M^2)_{mn} (T_a v)_n = 0 \quad \forall a$$

e) number of Goldstone Bosons

dimension of space spanned by $\{T_a v\}_a$

= # of Goldstone bosons = "# of generators with $T_a v \neq 0$ "

$\{b_1, \dots, b_{N_\pi}\}$ ONB of $\ll\{T_a v\}_a\gg$

$$\Rightarrow \pi_r = b_r^T \phi' , \quad r=1, \dots, N_\pi = \dim(G) - \dim(H)$$

f) unbroken subgroup H generated by

those generators $T = \sum_a \alpha_a T_a$ with

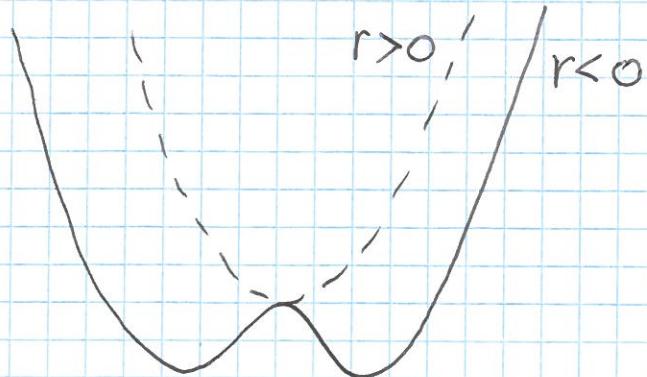
$$T v = 0$$

example: $G = SO(N)$, $N \geq 2$

$$\dim(G) = \frac{N(N-1)}{2} = \binom{N}{2}$$

ϕ transforming according N -dim. (defining)
repr. of $SO(N)$ (linear σ -model)

$$V(\phi) = \frac{1}{2} r \phi^T \phi + \frac{\lambda}{4} (\phi^T \phi)^2$$



$r > 0$ no SSB

$\rightarrow N$ scalars with
 $M^2 = r$

$r < 0$ SSB

$$\|\phi\| = \sqrt{\phi^\top \phi}$$

$$V = \frac{r}{2} \|\phi\|^2 + \frac{\lambda}{4} \|\phi\|^4$$

$$\frac{\partial V}{\partial \|\phi\|} = r \|\phi\| + \lambda \|\phi\|^3 = \|\phi\| (r + \lambda \|\phi\|^2) = 0$$

$$r < 0 \rightarrow \text{minimum at } \|\phi\|^2 = -\frac{r}{\lambda}$$

choose $v = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \|\phi\| \end{bmatrix}, \quad \|v\| = \sqrt{-\frac{r}{\lambda}}$

$H = SO(N-1)$ unbroken subgroup

$$\Rightarrow N_{\bar{\pi}} = \dim(G) - \dim(H) = \\ = \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2}$$

= $N-1$ Goldstone bosons

$\Rightarrow \exists N-1$ generators which do not annihilate v :

$$iT_1 = \left[\begin{array}{c|c} & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \textcircled{0} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline -1 & 0 \dots 0 \end{array} \right], \quad iT_2 = \left[\begin{array}{c|c} & \begin{matrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \textcircled{0} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 & 1 0 \dots 0 \end{array} \right], \dots$$

$$\langle\langle \{iT_\alpha v | \alpha=1, \dots, N-1\} \rangle\rangle =$$

$$= \langle\langle \{e_1, \dots, e_{N-1}\} \rangle\rangle$$

$\pi_1 = \phi_1, \dots, \pi_{N-1} = \phi_{N-1}$ massless Goldstone fields

only $\phi_N' = \phi_N - \|v\|$ is massive

check by inspection of $V(\phi)$:

$$\begin{aligned} V(\phi) &= V(v + \phi') = \\ &= \frac{1}{2} r \left[\phi_1^2 + \dots + \phi_{N-1}^2 + (\phi_N' + \|v\|)^2 \right] \\ &\quad + \frac{\lambda}{4} \left[\dots - \|v\|^2 \right]^2 = \end{aligned}$$

$$= \frac{r}{2} [\phi_1^2 + \dots + \phi_{N-1}^2 + (\phi_N' + \|v\|)^2]$$

$$+ \frac{\lambda}{4} [\phi_1^2 + \dots + \phi_{N-1}^2]^2$$

$$+ \frac{\lambda}{2} [\phi_1^2 + \dots + \phi_{N-1}^2] (\phi_N' + \|v\|)^2$$

$$+ \frac{\lambda}{4} (\phi_N' + \|v\|)^4$$

$$= \underbrace{\frac{r}{2} \|v\|^2 + \frac{\lambda}{4} \|v\|^4}_{V(v) = \text{const.}}$$

$$+ \underbrace{(r\|v\| + \lambda\|v\|^3)}_{\circ} \phi_N'$$

$$+ \underbrace{\frac{1}{2} (r + \lambda\|v\|^2)}_{\circ} (\phi_1^2 + \dots + \phi_{N-1}^2)$$

$$+ \underbrace{\frac{1}{2} (r + 3\lambda\|v\|^2)}_{2\lambda\|v\|^2 = M_N^2} (\phi_N')^2$$

+ cubic and quartic interaction terms

example: spontaneous breaking of chiral symmetry
in QCD

$$\mathcal{L}_{QCD} = -\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} + \sum_f (\bar{q}_f i \gamma^\mu D_\mu q_f - m_f \bar{q}_f q_f)$$

$$f = u, d, c, s, t, b$$

quark mass hierarchy in QCD:

$$m_u, m_d \ll m_s \ll m_c, b, t$$

→ study chiral limit of

a) $N_f = 2$ massless quark flavours ($m_u = m_d = 0$)

b) $N_f = 3$ -II- -II- -II- ($m_u = m_d = m_s = 0$)

$$\mathcal{L}_{QCD}^{(0)} = \bar{q}_L i \gamma^\mu D_\mu q_L + \bar{q}_R i \gamma^\mu D_\mu q_R$$

$$-\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} + \mathcal{L}_{\text{heavy quarks}}, \quad q_{L,R} = \frac{1 \mp \gamma_5}{2} q$$

$q = \begin{pmatrix} u \\ d \end{pmatrix}$ for $N_f = 2$ massless quark flavours

or

$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$ for $N_f = 3$ -II- -II- -II-

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$\mathcal{L}_{QCD}^{(0)}$ is invariant under global chiral transformations

$$q_R \rightarrow U_R q_R, \quad q_L \rightarrow U_L q_L$$

$$U_{R,L} \in U(N_f)$$

Noether currents:

$$V_i^\mu = \bar{q} \gamma^\mu \frac{\lambda_i}{2} q, \quad A_i^\mu = \bar{q} \gamma^\mu \gamma_5 \frac{\lambda_i}{2} q, \quad i=1, \dots, N^2-1$$

$$V_o^\mu = \bar{q} \gamma^\mu q, \quad A_o^\mu = \bar{q} \gamma^\mu \gamma_5 q$$

$$U(1)_A \text{ anomaly: } \partial_\mu A_o^\mu = \frac{N_f \alpha_s}{4\pi} G_{\mu\nu}^a \tilde{G}_a^{\mu\nu}$$

→ actual symmetry group of massless QCD:

$$\underbrace{SU(N_f)_L \times SU(N_f)_R \times U(1)_V}_{\text{chiral group (chiral symmetry)}}$$

$U(1)_V$: $\int d^3x V_o^\mu(x)$ counts #quarks - #antiquarks

→ $3 V_o^\mu$ baryon number current

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QCD vacuum \rightarrow chiral symmetry spontaneously broken

$$SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V$$

order parameters: non-vanishing quark condensates

$$\langle 0 | \bar{u} u | 0 \rangle = \langle 0 | \bar{d} d | 0 \rangle (= \langle 0 | \bar{s} s | 0 \rangle) \neq 0$$

vectorial subgroup $SU(N_f)_V$ remains unbroken
(symmetry of the vacuum)

of Goldstone bosons = # of broken generators

$N_f=2$: 3 Goldstone bosons π^\pm, π^0

$N_f=3$: 8 Goldstone bosons $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$

pseudoscalar mesons

real world: chiral symmetry also explicitly broken by
quark mass term $\bar{q}_L M q_R + \bar{q}_R M q_L$, M = mass matrix
of light quarks \rightarrow pseudoscalars become massive
(But masses still much smaller than masses of other mesons)
 \rightarrow effects of light quark masses can be treated perturbatively