Chapter 8: Path Integral Method

→ Alternative approach to quantum physics that does not rely on operators and is based a generalization of the concept of classical trajectories and the quantum mechanical superposition principle.
↳ Only based on complex valued functions (i.e. $\mathbb{C}$ numbers) for bosons and Grassmann-valued functions for fermions.

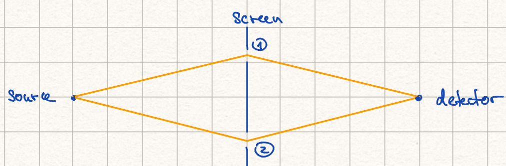
→ Very useful for quantum field theory for non-Abelian gauge theories (weak and strong interactions) ↗ "Standard Model"
where the canonical operator approach is too cumbersome and not useful anymore.

→ Literature: Feynman/Hibbs: QM and Path Integrals ← Method was pioneered by Feynman!
Rivers: Path Integrals in QFT
Roepsdorff: Path Integrals in Quantum Physics

8.1. Basic Idea

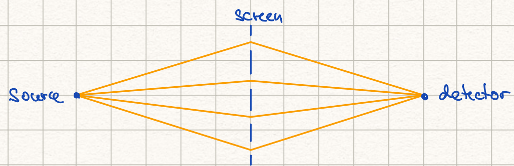
(a) Double slit experiment:

$$\phi = \phi_1 + \phi_2$$



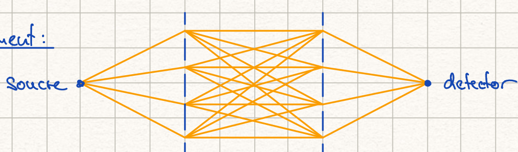
(b) Multi slit experiment:

$$\phi = \sum_i \phi_i$$



(c) Multi slit/screen experiment:

$$\phi = \sum_{i,j} \phi_{ij}$$



(d) Limit of ∞ slits and screens:
↳ $\hat{=}$ No screens

$$\phi = \sum_{\substack{\text{all paths} \\ \vec{x}(t)}} \phi(\vec{x}(t))$$

Questions: What exactly is ϕ?
What are the ϕ_i, $\phi(\vec{x}(t))$?

↳ We consider an application where we can derive everything based on results we already know from canonical quantum theory.

8.2. Green's Function of the Time-Dependent Free Schrödinger Equation

→ We know: $\left(i\frac{\partial}{\partial t} - H_0\right) G_0(t,\vec{x};t',\vec{x}') = i\,\delta(t-t')\,\delta^{(3)}(\vec{x}-\vec{x}')$, $H_0 = \frac{\vec{p}^2}{2\mu}$

$$G_0(t,\vec{x};t',\vec{x}') = \int \frac{d^4q}{(2\pi)^4}\frac{d^4q'}{(2\pi)^4}\, e^{-iq.x}\, \frac{i(2\pi)^4\delta^{(4)}(q-q')}{q_0 - \frac{\vec{q}^2}{2\mu} + i\epsilon}\, e^{+iq'.x'} \qquad \langle\vec{x}|\vec{q}\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\vec{q}\vec{x}}$$

$$= \Theta(t-t')\int d^3\vec{q}\, \langle\vec{x}|\vec{q}\rangle\, e^{-i\frac{\vec{q}^2}{2\mu}(t-t')}\langle\vec{q}|\vec{x}'\rangle \qquad q.x := q_0 t - \vec{q}\vec{x}$$

$$= \frac{\Theta(t-t')}{[2\pi i(t-t')/\mu]^{3/2}} \exp\left(\frac{i\mu(\vec{x}-\vec{x}')^2}{2(t-t')}\right)$$

$$= \Theta(t-t')\langle\vec{x}|e^{-iH_0(t-t')}|\vec{x}'\rangle \qquad |\vec{x},t\rangle = e^{-iH_0 t}|\vec{x}\rangle$$

$$= \langle\vec{x}|G_0(t,t')|\vec{x}'\rangle$$

$$= \Theta(t-t')\langle\vec{x},t|\vec{x}',t'\rangle$$

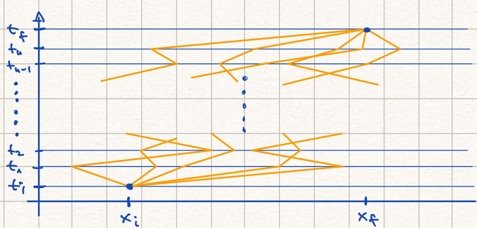
→ The Green's function contains the complete physics encoded in H_0 and can be used to derive everything related to H_0 (e.g. Schrödinger equation, energy eigenstates + eigenvalues, etc.)

In the following:
* We consider (for simplicity) 1 space-dimension.
* We define (for simplicity) $\langle\vec{x},t|\vec{x}',t'\rangle \neq 0$ only for $t>t'$.
  ↳ We can drop the Θ-fct.
* We include also a potential $V(x)$ in H $(H = H_0 + V(x))$. → More general.

→ We have:

$$\langle x_f,t_f|x_i,t_i\rangle = \int dx\, \langle x_f,t_f|x,t\rangle\langle x,t|x_i,t_i\rangle \qquad (t_i<t<t_f)$$

$$= \int dx_1\dots dx_n\, \langle x_f,t_f|x_n,t_n\rangle\langle x_n,t_n|x_{n-1},t_{n-1}\rangle\dots\langle x_1,t_1|x_i,t_i\rangle \qquad (t_i<t_1<\dots<t_n<t_f)$$



→ Limit $n \longrightarrow \infty$: $t_{i+1} - t_i = \frac{1}{n+1}(t_f - t_i) = \tau = dt \longrightarrow 0$

$$\langle x_{i+1},t_{i+1}|x_i,t_i\rangle = \langle x_{i+1}|e^{-iH\tau}|x_i\rangle = \langle x_{i+1}|1-iH\tau|x_i\rangle$$

$$= \delta(x_{i+1}-x_i) - i\tau\langle x_{i+1}|H|x_i\rangle$$

$$= \int\frac{dq}{2\pi}\, e^{iq(x_{i+1}-x_i)} - i\tau\langle x_{i+1}|H|x_i\rangle$$

$$H = \frac{p^2}{2\mu} + V(x) = \overset{\text{operators}}{H(p,x)}$$

$$\langle x_{i+1}| \frac{p^2}{2\mu} |x_i\rangle = \int dp\,dp' \underbrace{\langle x_{i+1}|p'\rangle}_{\frac{1}{(2\pi)^{1/2}} e^{-ip'x_{i+1}}} \underbrace{\langle p'| \frac{p^2}{2\mu}|p\rangle}_{\frac{p^2}{2\mu}\delta(p-p')} \langle p|x_i\rangle$$

$$= \int \frac{dp\,dp'}{2\pi} e^{-i(p'x_{i+1} - p x_i)} \frac{p^2}{2\mu} \delta(p-p') = \int \frac{dp}{2\pi} e^{ip(x_{i+1}-x_i)} \frac{p^2}{2\mu}$$

$$\langle x_{i+1}| V(x) |x_i\rangle = V(x_i)\,\delta(x_{i+1}-x_i)$$

$$= \int \frac{dp}{2\pi} e^{ip(x_{i+1}-x_i)} V(x_i)$$

number!

$$\Rightarrow \langle x_{i+1}| H |x_i\rangle = \int \frac{dp}{2\pi} e^{ip(x_{i+1}-x_i)} H(p, x_{i+1})$$

$$\Rightarrow \langle x_{i+1}, t_{i+1} | x_i, t_i\rangle = \int \frac{dp_{i+1}}{2\pi} e^{ip_{i+1}(x_{i+1}-x_i)} \left(1 - i\tau H(p_{i+1}, x_{i+1})\right)$$

$$\overset{n\to\infty}{=} \int \frac{dp_{i+1}}{2\pi} \exp\left(ip_{i+1}(x_{i+1}-x_i) - i\tau H(p_{i+1}, x_{i+1})\right)$$

$$\boxed{\begin{aligned} \langle x_f, t_f | x_i, t_i\rangle &= \lim_{n\to\infty} \int \frac{dp^{n+1}}{(2\pi)} \prod_{j=1}^{n} \frac{dp_j}{2\pi} dx_j \exp\left\{ i \sum_{j=0}^{n} \left[p_{j+1}(x_{j+1}-x_j) - \tau H(p_{j+1}, x_{j+1}) \right] \right\} \\ &(x_0 = x_i,\ x_{n+1} = x_f) \\ &= \int \frac{[\mathcal{D}p(t)][\mathcal{D}x(t)]}{2\pi} \exp\left\{ i \int_{t_i}^{t_f} dt \left[p(t)\dot{x}(t) - H(p(t), x(t)) \right] \right\} \end{aligned}}$$

"Sum" over <u>all</u> functions $p(t)$ and <u>all</u> functions $x(t)$ with $x(t_i) = x_i$, $x(t_f) = x_f$ } $p(t), x(t)$: possible classic paths in x-p phase space

↳ "Path integral representation for the propagator" → There are no operators! Only classic quantities appear!

→ <u>Problem:</u> Path integral is well-defined on a finite space-time lattice, but its systematic mathematical definition in continuous (and ∞-sized) space is not so clear. → (How is $[\mathcal{D}x]$ defined precisely?)
↳ We ignore this issue (as usual) and see how far we can go ...

→ If the Hamiltonian has the form $H = \frac{\vec{P}^2}{2\mu} + V(\vec{X})$ the path integral can be simplified further

$$\begin{aligned}
\langle x_f, t_f | x_i, t_i \rangle &= \lim_{n\to\infty} \int \frac{dp^{n+1}}{(2\pi)} \prod_{j=1}^{n} \frac{dp_j}{2\pi} dx_j \; \exp\left\{ i \sum_{j=0}^{n} \left[p_{j+1}(x_{j+1}-x_j) - \frac{p_{j+1}^2}{2\mu}\tau - V(x_{j+1})\tau \right] \right\} \\
&= \lim_{n\to\infty} \left(\frac{2\pi\mu}{i\tau}\right)^{\frac{n+1}{2}} \frac{1}{(2\pi)^{n+1}} \int \prod_{j=1}^{n} dx_j \; \exp\left\{ i \sum_{j=0}^{n} \left[\frac{\mu}{2\tau^2}(x_{j+1}-x_j)^2 - V(x_{j+1}) \right]\tau \right\} \\
&= N \int [\mathcal{D}x(t)] \exp\left\{ i \int_{t_i}^{t_f} dt \left[\frac{\mu}{2}(\dot{x}(t))^2 - V(x(t)) \right] \right\} \\
&= N \int [\mathcal{D}x(t)] \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \; L(x(t), \dot{x}(t)) \right\} \quad \text{classical Lagrangian } L \\
&= N \int_{(x(t_i)=x_i,\, x(t_f)=x_f)} [\mathcal{D}x(t)] \exp\left\{ \frac{i}{\hbar} S[x(t)] \right\} \quad \text{classical action } S
\end{aligned}$$

Some norm which we do not specify (as long as not absolutely needed).

We used: $p_{j+1}(x_{j+1}-x_j) - \frac{p_{j+1}^2}{2\mu}\tau - V(x_{j+1})\tau = -\frac{1}{2\mu}\left(p_{j+1} - \frac{\mu}{\tau}(x_{j+1}-x_j)\right)^2\tau + \frac{\mu}{2\tau}(x_{j+1}-x_j)^2 - V(x_{j+1})\tau$

$$\left.\begin{aligned}
\int_{-\infty}^{+\infty} dx \; \exp(-ax^2) &= \left(\frac{\pi}{a}\right)^{1/2} \\
\int_{-\infty}^{+\infty} dx \; \exp(-ax^2+bx+c) &= \left(\frac{\pi}{a}\right)^{1/2} \exp\left(\frac{b^2}{4a}+c\right)
\end{aligned}\right\} \text{ for } \mathrm{Re}\, a > 0$$

→ Classic physics: For the classic trajectory $x_{cl}(t)$ (= path the system goes in time) is the one for which the action is extremal

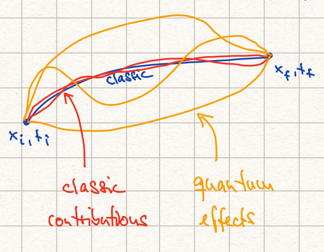
⇒ $\delta S[x_{cl}(t)] = S[x_{cl}(t) + \delta x(t)] - S[x_{cl}(t)] = 0 \quad , \quad \delta x(t_i) = \delta(t_f) = 0$

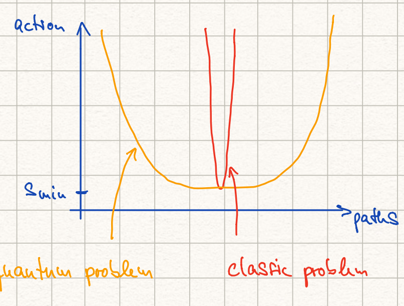
↳ gives Euler-Lagrange equations

The other paths lead to so much oscillations contributing to the path integral that the average out to zero and do not contribute.

→ Quantum physics: All in principle possible paths contribute, but the paths "close" to the classical path $x_{cl}(t)$ have a larger weight and the paths "far away" from the classic path have a small weight.

⇒ principle of the stationary phase.





$\hookrightarrow$ The path integral method allows to illustrate (maybe also "understand") the transition from systems that behave classical to the ones that behave quantum mechanical.

<u>Important conclusion:</u> <u>Every</u> classical problem starts having quantum mechanical behavior if one can just resolve paths $x(t)$ that are so close to $x_{cl}(t)$ that $S[x(t)] \approx S(x_{cl}(t))$.

$\hookrightarrow$ At some point (e.g. when experimental precision keep increasing) any real classical system follows the rules of quantum physics.

→ Explicit calculation for the free Green's function

In practice one has to refer back to discrete times when evaluating the path integral.

$$\langle x_f, t_f | x_i, t_i \rangle = \lim_{n\to\infty} \left(\frac{2\pi\mu}{i\tau}\right)^{\frac{n+1}{2}} \frac{1}{(2\pi)^{n+1}} \int \prod_{j=1}^{n} dx_j \; \exp\left\{ i \frac{\mu}{2\tau}\left[\sum_{j=0}^{n} (x_{j+1}-x_j)^2 + i\epsilon \right] \right\}$$

"regulator"

$$\left[\; (x_1-x_0)^2 + (x_2-x_1)^2 = 2\left(x_1 - \left(\frac{x_0+x_2}{2}\right)\right)^2 + \frac{1}{2}(x_2-x_0)^2 \right.$$

$$= \lim_{n\to\infty} \left(\frac{2\pi\mu}{i\tau}\right)^{\frac{n+1}{2}} \frac{1}{(2\pi)^{n+1}} \int \prod_{j=2}^{n} dx_j \left(\frac{i\pi 2\tau}{\mu}\right)^{1/2} \frac{1}{(2)^{1/2}} \exp\left\{ i\frac{\mu}{2\tau}\left[\frac{1}{2}(x_2-x_0)^2 + \sum_{j=1}^{n}(x_{j+1}-x_j)^2\right]\right\}$$

$$\left[\; \frac{1}{2}(x_2-x_0)^2 + (x_3-x_2)^2 = \frac{3}{2}\left(x_2 - \frac{2}{3}\left(\frac{x_0}{2}+x_3\right)\right)^2 + \frac{1}{3}(x_3-x_0)^2 \right.$$

$$= \lim_{n\to\infty} \left(\frac{2\pi\mu}{i\tau}\right)^{\frac{n+1}{2}} \frac{1}{(2\pi)^{n+1}} \int \prod_{j=3}^{n} dx_j \left(\frac{i\pi 2\tau}{\mu}\right)^{2/2} \frac{1}{(3)^{1/2}} \exp\left\{ i\frac{\mu}{2\tau}\left[\frac{1}{3}(x_3-x_0)^2 + \sum_{j=2}^{n}(x_{j+1}-x_j)^2\right]\right\}$$

$$\left[\; \frac{1}{j}(x_j-x_0)^2 + (x_{j+1}-x_j)^2 = \frac{j+1}{j}\left(x_j - \frac{j}{j+1}\left(\frac{x_0}{j}+x_{j+1}\right)\right)^2 + \frac{1}{j+1}(x_{j+1}-x_0) \right.$$

$$= \lim_{n\to\infty} \left(\frac{2\pi\mu}{i\tau}\right)^{\frac{n+1}{2}} \frac{1}{(2\pi)^{n+1}} \left(\frac{i\pi 2\tau}{\mu}\right)^{\frac{n}{2}} \frac{1}{(n+1)^{1/2}} \exp\left\{ i \frac{\mu}{2\tau(n+1)} (x_f-x_i)^2 \right\}$$

$$= \lim_{n\to\infty} \left(\frac{\mu}{2\pi i \tau (n+1)}\right)^{1/2} \exp\left\{ i \frac{\mu}{2\tau(n+1)} (x_f-x_i)^2 \right\} \qquad (n+1)\tau \longrightarrow t_f - t_i$$

$$= \frac{1}{(2\pi i (t_f-t_i)/\mu)^{1/2}} \exp\left\{ \frac{i\mu (x_f-x_i)^2}{2(t_f-t_i)} \right\}$$

Agrees precisely with result derived earlier! ✓

→ Connection to the classic action

Classic trajectory of a free particle with $x(t_i) = x_i$, $x(t_f) = x_f$: $x(t) = x_i + \frac{x_f - x_i}{t_f - t_i}(t - t_i)$

$\hookrightarrow$ $\dot{x}(t) = \frac{x_f - x_i}{t_f - t_i} \implies S[x(t)] = \frac{\mu}{2}\frac{(x_f-x_i)^2}{(t_f-t_i)}$

We see: $\langle x_f, t_f | x_i, t_i \rangle = \frac{1}{(2\pi i (t_f - t_i)/\mu)^{1/2}} \exp\left[i S_{classic}\right]$

In general one can show that for any Lagrangian of the form

$$L(x,\dot{x},t) = a(t)\dot{x}^2 + b(t)\dot{x}x + c(t)x^2 + d(t)\dot{x} + e(t)x + f(t)$$

we have $\langle x_f, t_f | x_i, t_i\rangle = A(t_f,t_i)\exp\left[i S_{classic}\right]$

8.3. Time-Dependent Perturbation Theory

→ We now consider the Green's function of a particle in a non-trivial time-dependent potential

$$\left(i\frac{\partial}{\partial t} - H\right) G(t_f,\vec{x}_f; t_i,\vec{x}_i) = i\,\delta(t_f - t_i)\,\delta^{(3)}(\vec{x}_f - \vec{x}_i) \quad , \quad H = \frac{\vec{p}^2}{2\mu} + V(\vec{X},t) = H_0 + V(\vec{X},t)$$

$$\Leftrightarrow \left(i\frac{\partial}{\partial t} - H_0\right) G(t_f,\vec{x}_f; t_i,\vec{x}_i) = i\,\delta(t_f - t_i)\,\delta^{(3)}(\vec{x}_f - \vec{x}_i) + V(\vec{x}_f,t_f)\, G(t_f,\vec{x}_f; t_i,\vec{x}_i)$$

↳ Use Green's function of H_0:

$$G(t_f,\vec{x}_f; t_i,\vec{x}_i) = G_0(t_f,\vec{x}_f; t_i,\vec{x}_i) - i\int_{t_i}^{t_f} dt \int d\vec{x}\; G_0(t_f,\vec{x}_f; t,\vec{x})\, V(\vec{x},t)\, G(t,\vec{x}; t_i,\vec{x}_i)$$

$$= G_0(x_f,x_i) - i\int d^4x\; G_0(x_f,x)V(x)G_0(x,x_i)$$

$$+ (-i)^2 \int d^4x\, d^4x'\; G_0(x_f,x')V(x')G_0(x',x)V(x)\,G_0(x,x_i)$$

$$+ \dots$$

$|\vec{x},t\rangle V(\vec{x},t)\langle\vec{x},t|$

$= e^{-iH_0 t}|\vec{x}\rangle V(\vec{x},t)\langle\vec{x}|e^{iH_0 t}$

$= V_H(\vec{X}_H(t),t)$

$\{|\vec{x},t\rangle\}$ is ONS for each t

$$= \langle\vec{x}_f,t_f|\vec{x}_i,t_i\rangle - i\int_{t_i}^{t_f} dt\,\langle\vec{x}_f,t_f|V_H(\vec{X}_H(t),t)|\vec{x}_i,t_i\rangle$$

operators

$$+ \frac{(-i)^2}{2}\int_{t_i}^{t_f} dt\, dt'\,\langle\vec{x}_f,t_f|\,T\; V_H(\vec{X}_H(t'),t')\,V_H(\vec{X}_H(t),t)\,|\vec{x}_i,t_i\rangle$$

$$+ \dots$$

time-ordering operator

→ Basic quantity that appears in perturbative computations with interactions:

$$\boxed{\langle\vec{x}_f,t_f|\underbrace{T\,V(\vec{X}_H(t_1),t_1)\,V(\vec{X}_H(t_2),t_2)\dots V(\vec{X}_H(t_n),t_n)}_{\text{time-ordered product of functions of the Heisenberg position operator}}|\vec{x}_i,t_i\rangle}$$

Path integral result for the matrix element involving the time-ordered product:

$$\langle \vec{x}_f, t_f | T\, V(\vec{X}_H(\tilde{t}_1), \tilde{t}_1) \ldots V(\vec{X}_H(\tilde{t}_n), \tilde{t}_n) | \vec{x}_i, t_i \rangle$$

choose t_j such that $\tilde{t}_k = t_{j_k}$

$$= \lim_{n\to\infty} \int \frac{dp^{n+1}}{(2\pi)} \prod_{j=1}^{n} \frac{dp_j}{2\pi} dx_j \; V(x_{j_1}, t_{j_1}) \ldots V(x_{j_n}, t_{j_n}) \exp\left\{ i \sum_{j=0}^{n} \left[p_{j+1}(x_{j+1} - x_j) - \tau H_0(p_{j+1}, x_{j+1}) \right] \right\}$$

$(x_0 = x_i, x_{n+1} = x_f)$

$$= \int \frac{[\mathcal{D}p(t)][\mathcal{D}x(t)]}{2\pi} V(\vec{x}(\tilde{t}_1), \tilde{t}_1) \ldots V(\vec{x}(\tilde{t}_n), \tilde{t}_n) \exp\left\{ i \int_{t_i}^{t_f} dt \left[p(t)\dot{x}(t) - H_0(p(t), x(t)) \right] \right\}$$

$$= N \int [\mathcal{D}x(t)] \, V(\vec{x}(\tilde{t}_1), \tilde{t}_1) \ldots V(\vec{x}(\tilde{t}_n), \tilde{t}_n) \exp\left\{ i \int_{t_i}^{t_f} dt \; L_0(x(t), \dot{x}(t)) \right\}$$

$\hookrightarrow$ We see: The time-ordering present in the operator language is <u>not</u> present any more in the path-integral approach

$\to$ We can define a generating functional from which all terms can be determined by functional derivatives:

$$\boxed{\begin{aligned} \langle x_f, t_f | x_i, t_i \rangle^J &:= \int \frac{[\mathcal{D}p(t)][\mathcal{D}x(t)]}{2\pi} \exp\left\{ i \int_{t_i}^{t_f} dt \left[\vec{p}(t)\dot{\vec{x}}(t) - H_0(\vec{p}(t), \vec{x}(t)) + J(t)\, V(\vec{x}(t), t) \right] \right\} \\ &= N \int [\mathcal{D}x(t)] \exp\left\{ i \int_{t_i}^{t_f} dt \; L_0(x(t), \dot{x}(t)) + J(t)\, V(\vec{x}(t), t) \right] \right\} \end{aligned}}$$

$\hookrightarrow$

$$\boxed{\begin{aligned} &\langle \vec{x}_f, t_f | T\, V(\vec{X}_H(\tilde{t}_1), \tilde{t}_1) \ldots V(\vec{X}_H(\tilde{t}_n), \tilde{t}_n) | \vec{x}_i, t_i \rangle \\ &= (-i)^n \frac{\delta^n}{\delta J(\tilde{t}_1) \ldots \delta J(\tilde{t}_n)} \langle x_f, t_f | x_i, t_i \rangle^J \Big|_{J=0} \end{aligned}}$$