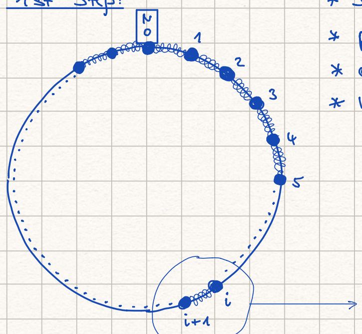
Chapter 7: Quantum Field Theory

→ We look at theoretical elements that connect a system of a finite number of particles that could be described by regular N-particle non-relativistic quantum mechanics to quantum field theory in the $N \to \infty$ limit.

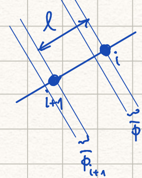
We consider only a non-relativistic theory of bosonic particles, which however contains all relevant elements that are also present in the context of a relativistic theory of bosonic particles.

7.1. One-dimensional "Classic" String

1st step:



* System of N mass points, connected by springs
* periodic boundary condition: $\bar{\Phi}_0 = \bar{\Phi}_N$, $\frac{d}{dt}\bar{\Phi}_0 = \frac{d}{dt}\bar{\Phi}_N$
* otherwise free oscillations
* non-relativistic kinematic



longitudinal elongation

$m_i = m$ mass
l : equilibrium distance
k : spring constant

kinetic energy $= \frac{1}{2} m \sum_{i=0}^{N-1} \left(\frac{d}{dt}\bar{\Phi}_i\right)^2 = KE$

potential energy $= \frac{1}{2} k \sum_{i=0}^{N-1} \left(\bar{\Phi}_{i+1} - \bar{\Phi}_i\right)^2 = PE$

2nd step: "continuum limit": $N \longrightarrow \infty$, $l \longrightarrow 0$, $\bar{\Phi}_i(t) = \bar{\Phi}(z_i, t) \longrightarrow \bar{\Phi}(z,t)$

where we have: total length $L = Nl = \text{const}$ $\qquad l \longrightarrow dz \longrightarrow 0$

mass density $\mu = \frac{m}{l} = \text{const}$

string tension $T = kl = \text{const}$ $\qquad$ Also useful: rescaling $\Phi(z,t) := \sqrt{T}\,\bar{\Phi}(z,t)$

$$\Rightarrow KE = \frac{1}{2}\frac{m}{l}\sum_{i=0}^{N-1} l \left(\frac{d}{dt}\bar{\Phi}_i\right)^2 \longrightarrow \frac{1}{2}\mu \int_0^L dz \left(\frac{\partial}{\partial t}\bar{\Phi}(z,t)\right)^2 = \frac{1}{2}\frac{\mu}{T}\int_0^L dz \left(\frac{\partial}{\partial t}\Phi(z,t)\right)^2$$

$$PE = \frac{kl}{2}\sum_{i=0}^{N-1} l \left(\frac{\bar{\Phi}_{i+1} - \bar{\Phi}_i}{l}\right)^2 \longrightarrow \frac{T}{2}\int_0^L dz \left(\frac{\partial}{\partial z}\bar{\Phi}(z,t)\right)^2 = \frac{1}{2}\int_0^L dz \left(\frac{\partial}{\partial z}\Phi(z,t)\right)^2$$

$$\boxed{\text{Phase velocity: } v = \sqrt{\frac{T}{\mu}}}$$

Lagrangian: $L = KE - PE = \frac{m}{2}\sum_{i=0}^{N-1}(\dot{\bar{\Phi}}_i)^2 - \frac{k}{2}\sum_{i=0}^{N-1}(\bar{\Phi}_{i+1} - \bar{\Phi}_i)^2 = L(\bar{\Phi}_i, \dot{\bar{\Phi}}_i)$

$\longrightarrow \int_0^L dz\, \mathcal{L}(z,t)$ $\qquad \mathcal{L}$: Lagrange density

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no dependence on ϕ in our case

Lagrange density: $\mathcal{L}(z,t) = \mathcal{L}\left(\phi, \frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial z}\right)$ $\qquad \phi = \phi(z,t)$

$$= \frac{1}{2}\left[\frac{1}{v^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2\right]$$

Hamiltonian: $H = KE + PE = \frac{m}{2}\sum_{i=0}^{N-1}(\dot{\bar\phi}_i)^2 + \frac{k}{2}\sum_{i=0}^{N-1}(\bar\phi_{i+1} - \bar\phi_i)^2$

$$\longrightarrow \int_0^L dz\, \mathcal{H}(z,t) \qquad \mathcal{H}: \text{Hamilton density}$$

$$\mathcal{H}(z,t) = \frac{1}{2}\left[\frac{1}{v^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]$$

Comments:

* $\bar\phi_i$ could also describe a transversal elongation

* String (1-dim: $\phi = \phi(z,t)$) can be generalized to a membrane (2-dim: $\phi = \phi(x,y,t)$) or a field (3-dim: $\phi = \phi(t,\vec{x})$)
* ϕ describes a scalar excitation (i.e. one value at each space-time point). It could be generalized to a multidimensional excitation
  $\longrightarrow$ e.g. vector field $\vec\phi(t,\vec x) = (\phi_x(t,\vec x), \phi_y(t,\vec x), \phi_z(t,\vec x))$

7.2. Equations of Motion and Normalmodes

Discrete case: Action: $S = \int dt\, L(\bar\phi_i ; \dot{\bar\phi}_i)$

$$\delta S = \sum_{i=1}^{N-1}\int dt\left(\frac{\partial L}{\partial\bar\phi_i}\delta\bar\phi_i + \frac{\partial L}{\partial\dot{\bar\phi}_i}\delta\dot{\bar\phi}_i\right)$$

$$= \sum_{i=1}^{N-1}\int dt\left(\frac{\partial L}{\partial\bar\phi_i} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\bar\phi}_i}\right)\delta\bar\phi_i \overset{!}{=} 0 \implies \boxed{\frac{d}{dt}\frac{\partial L}{\partial\dot{\bar\phi}} - \frac{\partial L}{\partial\bar\phi_i} = 0 \quad, i = 1,..,N}$$

Euler-Lagrange equations

$$\hookrightarrow m\ddot{\bar\phi}_i - k(\phi_i - \phi_{i+1}) - k(\phi_i - \phi_{i-1}) = 0$$

Continuum: Action: $S = \int d^4x\, \mathcal{L}(\phi(x), \partial_\mu\phi(x))$

$$\delta S = \int d^4x\left(\frac{\partial\mathcal{L}(x)}{\partial\phi(x)}\delta\phi(x) + \frac{\partial\mathcal{L}(x)}{\partial\partial_\mu\phi(x)}\delta(\partial_\mu\phi(x))\right)$$

for all x

$$= \int d^4x\left(\frac{\partial\mathcal{L}(x)}{\partial\phi(x)} - \partial_\mu\left(\frac{\partial\mathcal{L}(x)}{\partial\partial_\mu\phi(x)}\right)\right)\delta\phi(x) \overset{!}{=} 0 \implies \boxed{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi(x)}\right) - \frac{\partial\mathcal{L}}{\partial\phi(x)} = 0}$$

String: $\mathcal{L} = \frac{1}{2}\left[\frac{1}{v^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2\right] \implies \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial z^2} = 0$

Field: $\mathcal{L} = \frac{1}{2}\left[\frac{1}{v^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial\vec x}\right)^2\right] \implies \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} - \vec\nabla^2\phi = 0$

"classic" wave equations (because we started with a classic problem)

$\longrightarrow$ But we could as well start with a multi-particle quantum mechanical problem!

↳ Normal modes of the String: (with periodic boundary conditions) → see photon quantization

$$\boxed{\phi_n(z,t) = \frac{1}{\sqrt{L}} e^{-i(\omega_n t - k_n z)} \quad , \quad \omega_n = v|k_n| \qquad k_n = \frac{2\pi n}{L} \quad , \quad n = 0, \pm1, \pm2, \dots}$$

* Orthonormality: $\int_0^L dz\, \phi_n^*(z,t)\, \phi_m(z,t) = \delta_{nm}$ $\left(\text{also: } \int_0^L dz\, \phi_n(z,t)\, \phi_m(z,t) = \delta_{n,-m} e^{-2i\omega_n t}\right)$

* Completeness: $\{\phi_n,\, n \in \mathbb{Z}_0\}$ form a complete set of states
→ A general excitation (which is real) can be written as

$$\boxed{\begin{aligned}\phi(z,t) &= \sum_{n=-\infty}^{+\infty} c_n \left[a_n(0)\, \phi_n(z,t) + a_n^*(0)\, \phi_n^*(z,t)\right] \quad , \quad c_n: \text{norm to be fixed} \\ &= \sum_{n=-\infty}^{+\infty} \frac{c_n}{\sqrt{L}} \left[a_n(t) e^{ik_n z} + a_n^*(t) e^{-ik_n z}\right] \quad , \quad a_n(t) := a_n(0) e^{-i\omega_n t}\end{aligned}}$$

a_n: normal mode coefficient

↳ $\ddot{a}_n(t) = \omega_n^2 a_n(t)$ ⟶ We can consider each normal mode as an independent harmonic oscillator.

↳ We determine the norm c_n: We want the standard form for the Hamiltonian

$$H = \frac{1}{2} \int_0^L dz \left(\frac{1}{v^2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right)$$

$$= \sum_{n=-\infty}^{+\infty} \left[c_n^2 \left(\frac{\omega_n^2}{v^2} + k_n^2\right) a_n^*(t)\, a_n(t) \right.$$

$$\left. + \frac{1}{2} c_n c_{-n} \left(\cancel{-\frac{\omega_n^2}{v^2} + k_n^2}\right)\left(a_n(t)\, a_{-n}(t) + a_n^*(t)\, a_{-n}^*(t)\right)\right]$$

$$= \sum_{n=-\infty}^{+\infty} c_n^2 \frac{2\omega_n^2}{v^2} a_n^*(t)\, a_n(t)$$

$$\overset{!}{=} \sum_{n=-\infty}^{+\infty} \omega_n\, a_n^*(t)\, a_n(t) \quad \Rightarrow \quad c_n = \frac{v}{\sqrt{2\omega_n}} \quad \Rightarrow \quad a_n, a_n^* \text{ dimensionless}$$

↳ $$\boxed{H = \sum_{n=-\infty}^{+\infty} \omega_n\, a_n^*(t)\, a_n(t) = \sum_{n=-\infty}^{+\infty} \omega_n\, a_n^*(0)\, a_n(0)}$$

orange: make ħ explicit

→ If we consider the string as a classic string we can in analogy to the classical harmonic oscillator introduce for each normal mode canonical location and momentum variables:

$$\left.\begin{aligned} q_n(t) &= \frac{1}{\sqrt{2\omega_n}} \left(a_n(t) + a_n^*(t)\right) \times \sqrt{\hbar} \\ p_n(t) &= \frac{dq_n}{dt} = -\frac{i\omega_n}{\sqrt{2\omega_n}} \left(a_n(t) - a_n^*(t)\right) \times \sqrt{\hbar} \end{aligned}\right\} \Leftrightarrow \quad \begin{aligned} a_n &= \frac{1}{\sqrt{2\omega_n}} \left(i p_n + \omega_n q_n\right) \times \frac{1}{\sqrt{\hbar}} \\ a_n^* &= \frac{1}{\sqrt{2\omega_n}} \left(-i p_n + \omega_n q_n\right) \times \frac{1}{\sqrt{\hbar}} \end{aligned}$$

↳ $$\boxed{H = \sum_{n=-\infty}^{+\infty} \frac{1}{2}\left[p_n^2 + \omega_n^2 q_n^2\right]}$$ → Sum of independent harmonic oscillators with mass 1 and frequency ω_n

→ We check that q_n and p_n are indeed canonical by examining the Poisson bracket:

$$[u,v] := \sum_n \left(\frac{\partial u}{\partial q_n}\frac{\partial v}{\partial p_n} - \frac{\partial u}{\partial p_n}\frac{\partial u}{\partial q_n}\right)$$

↳ $[q_n, q_m] = [p_n, p_m] = 0$, $[q_n, p_m] = \delta_{nm}$ ✓

$$\frac{du}{dt} = [u, H] \Rightarrow \frac{dq_n}{dt} = [q_n, H] = \frac{\partial H}{\partial p_n} = p_n \quad ✓$$

$$\frac{dp_n}{dt} = [p_n, H] = -\frac{\partial H}{\partial q_n} = -\omega_n^2 q_n \quad ✓$$

↳ $[a_n, a_m^*] = i\frac{\omega_n}{\sqrt{2\omega_n}^2}\left([p_n, q_m] - [q_n, p_m]\right) = -i\delta_{nm}$

7.3. Quantization of the String

→ Up to now we have dealt with wave modes "in a box" where (up to the point how the wave functions have to interpreted physically) it didn't matter whether we consider quantum mechanics or a classic field theory.

From the classic point of view we can now quantize the string by promoting the canonical coordinates and momenta to operators and the Poisson brackets to commutators.

$$q_n, p_n, a_n, a_n^* \longrightarrow Q_n, P_n, a_n, a_n^+ \quad \left\{ \begin{array}{l} Q_n = \sqrt{\frac{\hbar}{2\omega_n}}\,(a_n + a_n^+) \\ P_n = -i\sqrt{\frac{\hbar\omega_n}{2}}\,(a_n - a_n^+) \end{array} \right.$$

$$\boxed{\left.\begin{array}{l} [Q_n, P_m] = i\hbar\delta_{nm} \\ [Q_n, Q_m] = [P_n, P_m] = 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} [a_n, a_m^+] = \delta_{nm} \\ [a_n, a_m] = [a_n^+, a_m^+] = 0 \end{array}\right.}$$

"canonical" commutation relations

↳ a_n^+ : Creation operator for a normal mode with momentum k_n
(≙ Particle ("phonon") with momentum k_n)

a_n : Annihilation operator for a normal mode with momentum k_n

↳ Quantized string allows to describe production/annihilation of classic normal modes according to quantum mechanical rules.

If we interpret the normal modes quantum mechanical as well (→ probability amplitudes) we arrive at quantum field theory (QFT).

↳ 2nd quantization

Quantum field operators

→ The canonical coordinate and momentum operators Q_n and P_n are not suitable variables for an efficient formulation of the theory.

It is more suitable to use the concept of the quantum field operators

$$\boxed{\phi(z,t) = \underbrace{\sum_{n=-\infty}^{+\infty}}_{\text{sum over modes}} \frac{v}{\sqrt{2\omega_n L}} \Big[\underbrace{a_n}_{\substack{\text{annihilator of} \\ \text{phonon with mom } k_n}} \underbrace{e^{-i(\omega_n t - k_n z)}}_{\substack{\text{incoming} \\ \text{phonon wave}}} + \underbrace{a_n^+}_{\substack{\text{creator of} \\ \text{phonon with mom } k_n}} \underbrace{e^{+i(\omega_n t - k_n z)}}_{\substack{\text{outgoing} \\ \text{phonon wave}}} \Big]}$$

Comments:

* Field operator $\phi(z,t)$ is the "2nd-quantized wave function"

* Field operators do in general not have to be Hermitian (but $\phi(z,t)$ treated here is). Only observables constructed from field operators have to be Hermitian.

* We ignore the infrared problem (i.e. divergence) for the mode $n=0$ ($\omega_n = 0$)
  ↳ Collider physics lecture.

* Field operators (must!) always satisfy the Euler-Lagrange equations, i.e. the classic equations of motion.

↳ $$\boxed{\frac{1}{v^2} \frac{\partial^2}{\partial t^2} \phi(z,t) - \frac{\partial^2}{\partial z^2} \phi(z,t) = 0}$$

Canonical Commutation Relations for Quantum Fields

$\phi(z,t)$: canonical "coordinate" field operator

$$\Pi(z,t) := \frac{\partial \mathcal{L}(\phi, \partial_t \phi)}{\partial(\partial_t \phi)} = \frac{1}{v^2} \partial_t \phi(z,t) = \frac{1}{v^2} \dot{\phi}(z,t) :$$

canonical "momentum" field operator

↳ $$\boxed{\Pi(z,t) = \sum_{n=-\infty}^{+\infty} \frac{i\omega_n}{v\sqrt{2\omega_n L}} \Big[-a_n e^{-i(\omega_n t - k_n z)} + a_n^+ e^{+i(\omega_n t - k_n z)} \Big]}$$

→ From the canonical commutation relations we can derive the equal-time canonical commutation relations for the field operators

$$\boxed{\begin{aligned}&[\phi(z,t),\Pi(z',t)] = i\,\delta(z-z')\\ &[\phi(z,t),\phi(z',t)] = [\Pi(z,t),\Pi(z',t)] = 0\end{aligned}}$$

↳ One can either use the commutation relations for the creation and annihilation operators (a_n, a_n^+) OR the equal-time commutation rules for the field operators as the starting point of QFT.

Proof of relations:

$$[\Pi(z,t),\phi(z',t)] = -\frac{i}{2L}\sum_{n,m}\sqrt{\frac{\omega_n}{\omega_m}}\Big\{\cancel{[a_n,a_m]}\,e^{-i(\omega_n+\omega_m)t+i(k_nz+k_mz')}$$
$$-\cancel{[a_n^+,a_m^+]}\,e^{i(\omega_n+\omega_m)t-i(k_nz+k_mz')} + \overbrace{[a_n,a_m^+]}^{\delta_{nm}}\,e^{-i(\omega_n-\omega_m)t+i(k_nz-k_mz')}$$
$$-\overbrace{[a_n^+,a_m]}^{-\delta_{nm}}\,e^{i(\omega_n-\omega_m)t-i(k_nz-k_mz')}\Big\}$$
$$= -\frac{i}{2L}\sum_{n=-\infty}^{+\infty}\left\{e^{ik_n(z-z')}+e^{-ik_n(z-z')}\right\}$$
$$= -\frac{i}{L}\sum_{n=-\infty}^{+\infty}e^{ik_n(z-z')}$$

completeness

$$= -i\,\delta(z-z')$$

$$[\phi(z,t),\phi(z',t)] = \frac{v^2}{2L}\sum_{n,m}\frac{1}{\sqrt{\omega_n\omega_m}}\Big\{[a_n,a_m^+]\,e^{-i(\omega_n-\omega_m)t+i(k_nz-k_mz')}$$
$$+[a_n^+,a_m]\,e^{i(\omega_n-\omega_m)t-i(k_nz-k_mz')}$$
$$= \frac{v^2}{2L}\sum_{n=-\infty}^{+\infty}\frac{1}{\omega_n}\left[e^{ik_n(z-z')}-e^{-ik_n(z-z')}\right] \qquad k_n = -k_{-n}$$
$$= 0$$

$$[\Pi(z,t),\Pi(z',t)] = 0 \qquad \text{(analogous computation)}$$

Physical Operators and Normal Ordering

→ We construct total energy and momentum operators from the corresponding classical quantities using the correspondence principle.

↳ We will find that the concept of normal ordering is required to avoid a divergence due to the harmonic oscillator zero-point energy / vacuum fluctuations.

Energy $\longleftrightarrow$ Hamilton Operator

↳ We apply the correspondence principle directly:

$$\int_0^L dz\, \mathcal{H}(\Pi,\phi) = \int_0^L dz\, \frac{1}{2}\left(\frac{1}{v^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right) = \int_0^L dz\, \frac{1}{2}\left(v^2\Pi^2(z,t) + \left(\frac{\partial}{\partial z}\phi(z,t)\right)^2\right)$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{2}\omega_n\left(a_n^\dagger a_n + a_n a_n^\dagger\right)$$

$$= \sum_{n=-\infty}^{+\infty} \omega_n\left(a_n^\dagger a_n + \frac{1}{2}\right)$$

$$= \sum_{n=-\infty}^{+\infty} \omega_n a_n^\dagger a_n + \frac{1}{2}\sum_{n=-\infty}^{+\infty}\omega_n$$

divergent zero-point energy

↳ The divergent zero-point energy is the <u>same for all states</u>.
We can therefor in principle ignore it for (most) practical applications.

↳ "We can set the zero-energy point by hand."

↳ In QFT this is achieved by the <u>normal ordering prescription</u>

Normal Ordering for Creation and Annihilation Operators

$$:(\text{arbitrary product of } a_n\text{'s and } a_m^\dagger\text{'s}): \;=\; \underbrace{(\text{all } a_m^\dagger\text{'s})(\text{all } a_n\text{'s})}$$

z.B.: $:a_n^\dagger a_m: \;=\; a_n^\dagger a_m$ All creators left ⊕ all annihilators right.

$:a_n a_m^\dagger: \;=\; a_m^\dagger a_n$

All physical quantum field operators (i.e. used to describe physical processes, quantities) are constructed from normal ordered functions of quantum field operators.

Hamilton Operator

$$H = \int_0^L dz : \mathcal{H}(\Pi(z,t),\phi(z,t)): \;=\; \sum_{n=-\infty}^{+\infty}\omega_n a_n^\dagger a_n$$

$a_n^\dagger a_n$: number operator

↳ We check that the Hamilton operator is the generator for time translations:

$$[\phi(z,t), H_{\text{time}=t}] = \int_0^L dz' \left[\phi(z,t), \frac{1}{2}v^2\Pi^2(z',t) + \frac{1}{2}\left(\frac{\partial}{\partial z}\phi(z',t)\right)^2\right]$$

$$= \frac{1}{2}v^2 \int_0^L dz' \left[\phi(z,t), \Pi^2(z',t)\right]$$

$$\Pi(z',t)\underbrace{[\phi(z,t),\Pi(z',t)]}_{i\delta(z-z')} + [\dots]\Pi(z',t)$$

$$= iv^2 \int_0^L dz'\, \Pi(z',t)\,\delta(z-z') = iv^2\Pi(z,t) = i\frac{\partial}{\partial t}\phi(z,t) \quad ✓$$

We can ignore the normal-ordering here because $:\mathcal{H}: = \mathcal{H} + \text{number}$

$$[\Pi(z,t), H] = \int_0^L dz' \left[\Pi(z,t), \frac{1}{2}\left(\frac{\partial}{\partial z'}\phi(z',t)\right)^2\right]$$

$$= \int_0^L dz' \left(\frac{\partial}{\partial z'}\phi(z',t)\right) \frac{\partial}{\partial z'} \underbrace{[\Pi(z,t), \phi(z',t)]}_{-i\delta(z-z')}$$

$$= -i \int_0^L dz' \left(\frac{\partial}{\partial z'}\phi(z',t)\right) \frac{\partial}{\partial z'} \delta(z-z')$$

$$= i \int_0^L dz' \, \delta(z-z') \left(\frac{\partial^2}{\partial z'^2}\phi(z',t)\right)$$

$$= i \frac{\partial^2}{\partial z^2}\phi(z,t)$$

Euler-Lagrange equ.

$$= i \frac{1}{v^2} \frac{\partial^2}{\partial t^2}\phi(z,t)$$

$$= i \frac{\partial}{\partial t}\Pi(z,t) \quad \checkmark$$

⇒ Heisenberg equations are satisfied.

Total Momentum Operator

⇒ We construct it from the E-p continuity equation (classic):

$$\frac{1}{v^2}\frac{\partial}{\partial t}\mathcal{H}(z,t) = -\frac{\partial}{\partial z}P(z,t) \qquad \longleftarrow \text{works only for waves!}$$

$$= \frac{1}{v^2}\frac{\partial}{\partial t}\frac{1}{2}\left[\frac{1}{v^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] = \frac{1}{v^2}\left[\frac{1}{v^2}\left(\frac{\partial\phi}{\partial t}\right)\left(\frac{\partial^2\phi}{\partial t^2}\right) + \left(\frac{\partial\phi}{\partial z}\right)\left(\frac{\partial^2\phi}{\partial z\partial t}\right)\right]$$

Euler-Lagrange

$$= \frac{1}{v^2}\left[\left(\frac{\partial\phi}{\partial t}\right)\left(\frac{\partial^2\phi}{\partial z^2}\right) + \left(\frac{\partial\phi}{\partial z}\right)\left(\frac{\partial^2\phi}{\partial z\partial t}\right)\right]$$

$$= \frac{1}{v^2}\frac{\partial}{\partial z}\left[\left(\frac{\partial\phi}{\partial t}\right)\left(\frac{\partial\phi}{\partial z}\right)\right]$$

$$\Rightarrow P = -\int_0^L dz \frac{1}{v^2} :\left(\frac{\partial\phi}{\partial t}\right)\left(\frac{\partial\phi}{\partial z}\right): \qquad \omega_n = \omega_{-n},\ k_{-n} = -k_n$$

$$= -\sum_{n,m}\frac{\omega_n k_m}{2L\sqrt{\omega_n\omega_m}} :\int_0^L dz \Big\{ a_n a_m e^{-i(\omega_n+\omega_m)t} e^{i(k_n+k_m)z} + a_n^+ a_m^+ e^{i(\omega_n+\omega_m)t - i(k_n+k_m)z}$$

$$- a_n^+ a_m e^{i(\omega_n-\omega_m)t} e^{-i(k_n-k_m)z} - a_n a_m^+ e^{-i(\omega_n-\omega_m)t+i(k_n-k_m)z} \Big\}:$$

$$= -\frac{1}{2}\sum_{n=-\infty}^{+\infty}\Big[\underbrace{\cancel{k_{-n} a_n a_{-n} e^{-2i\omega_n t}}}_{\text{sums to zero}} + \underbrace{\cancel{k_{-n} a_n^+ a_{-n}^+ e^{2i\omega_n t}}}_{\text{sums to zero}} - 2k_n a_n^+ a_n\Big]$$

$$= \sum_{n=-\infty}^{+\infty} k_n a_n^+ a_n \quad \checkmark \qquad \text{o.k.!}$$

Number Operator and Multi-Particle States

← analogous to photon quantization

→ $\boxed{N_n = a_n^\dagger a_n}$ Number operator that counts number of particles having momentum k_n

Properties:
* N_n Hermitian for all n
* $[N_n, N_m] = 0$ for all n, m
* $\{N_n\}$ build up a complete set of operators

↳ Set of eigenstates to $N_0, N_{\pm 1}, N_{\pm 2}, \ldots$ build a basis of the entire Hilbert space of multi-phonon states

↳ Every element of the basis is unambiguously determined by the occupation numbers $\{m_0, m_1, m_{-1}, m_2, m_{-2}, \ldots\}$ for phonon particles with momenta $(k_0, k_1, k_{-1}, k_2, k_{-2}, \ldots)$.
$\hat{=}$ Fock states

→ Multi-particle states (bosonic!)

$$|m_{n_1}, m_{n_2}, m_{n_3}, \ldots\rangle := \frac{(a_{n_1}^\dagger)^{m_{n_1}}}{\sqrt{m_{n_1}!}} \frac{(a_{n_2}^\dagger)^{m_{n_2}}}{\sqrt{m_{n_2}!}} \frac{(a_{n_3}^\dagger)^{m_{n_3}}}{\sqrt{m_{n_3}!}} \ldots |0\rangle$$

↳ $$H\, |m_{n_1}, m_{n_2}, m_{n_3}, \ldots\rangle = \left(\sum_i m_{n_i} \omega_{n_i}\right) |m_{n_1}, m_{n_2}, m_{n_3}, \ldots\rangle$$

Fock states with Bose-Einstein statistics