Chapter 3 : Quantization of the Free Electromagnetic Field

→ Analogue to quantum mechanics of a free particle. ⟶ free means in the vaccum
Makes particle / wave character of photons / photon field manifest.

But: Photon is a massless particle ⟶ relativistic physics
Particle number ~~not conserved~~ ⟶ probability conservation more subtle

Gauss units ⟶ Heariside Lorentz
1) Multiply all fields by $\sqrt{4\pi}$
2) Divide all Sources (charge) by $\sqrt{4\pi}$

3.1. Eigenmodes of the EM field

→ At the point when we go beyond static electromagnetic systems and deal with propagating electromagnetic waves (= photons), the electromagnetic field already has quantum mechanical character.
↳ Wave equations : Maxwell equations ⟶ solutions: photon wave functions

But, because the number of photons is not conserved, and we – in addition – have to deal with the quantum physics of creation / annihilation of photons
→ "quantum field theory", "second quantization"

Both aspects together constitute the (full) quantum theory for the EM field.

→ This section: We analyze the eigenmodes of the free photon field.
They are the analogues of the free (massive) particle solutions the Schrödinge equation $(i\frac{\partial}{\partial t} - \frac{\vec{p}^2}{2\mu})\psi(x) = 0$.
↳ 2nd quantization : Sec. 3.2.

→ We keep factors of c (= speed of light) although not required.

Maxwell equations in vacuum = free photon wave equations

$$(i)\quad \vec{\nabla}\vec{E} = 0 \qquad\qquad (iii)\quad \vec{\nabla}\vec{B} = 0$$

$$(ii)\quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{B} = 0 \qquad\qquad (iv)\quad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = 0$$

In the presence of an electric charge distribution $\rho(x)$ and an electric current $j(x)$: (Gaussian units)

$$(i)\quad \vec{\nabla}\vec{E} = 4\pi\rho$$

$$(iv)\quad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}j$$

no factors (4π) in Heariside Lorentz units

↳ Set of 8 equations : Indicates that the photon waves are much more complicated that the spinless particle Schrödinger equation

To proceed it is useful to introduce : Vector potential $\vec{A}$ ⊕ Scalar potential ϕ

$(iii)\ \vec{\nabla}\vec{B} = 0 \longleftarrow$ Automatically satisfied if we require : $\vec{B} = \vec{\nabla}\times\vec{A}$ $\quad(\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0)$

$(ii)\ \vec{\nabla}\times(\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{A}) = 0 \longleftarrow$ Automatically satisfied if we require : $\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{A} = -\vec{\nabla}\phi$

$(\vec{\nabla}\times\vec{\nabla}\cdot\phi = 0)$

So after introducing $\vec{A}$ and ϕ (ii) and (iii) are trivially satisfied and we do not need to worry about them any more.

But: $\vec{A}$ and ϕ are not unique $\rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\phi$
$\vec{B}$ and $\vec{E}$ are unchanged under the

$$\boxed{\text{Gauge transformations: } \vec{A} \longrightarrow \vec{A} - \vec{\nabla}\Lambda \; , \; \phi \longrightarrow \phi + \frac{1}{c}\frac{\partial}{\partial t}\Lambda \; , \; \Lambda = \Lambda(x) \text{ arbitrary scalar fct}}$$

Proof: $\vec{B} = \vec{\nabla}\times\vec{A} \longrightarrow \vec{\nabla}\times\vec{A} - \overbrace{\vec{\nabla}\times\vec{\nabla}\Lambda}^{=0}$ ✓

$$\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\phi \longrightarrow -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} + \frac{1}{c}\cancel{\vec{\nabla}\frac{\partial}{\partial t}\Lambda} - \vec{\nabla}\phi - \frac{1}{c}\cancel{\vec{\nabla}\frac{\partial}{\partial t}\Lambda} \quad ✓$$

This gauge-freedom allows us to impose an (in principle ad hoc but very useful) constraint, called a "gauge" (= Eichung)

$$\Lambda = \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\vec{\nabla}_{x'}\vec{A}(t,\vec{x}')$$

$$\boxed{\text{Coulomb gauge: } \vec{\nabla}\vec{A} = 0}$$

$\longleftarrow$ It is possible to pick Λ such that this is satisfied.
We can still make gauge transformations as long as $\vec{\nabla}^2\Lambda = 0$.

Now look at (iv) $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \vec{\nabla}\times\vec{\nabla}\times A + \frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2\vec{A} + \frac{1}{c}\vec{\nabla}\frac{\partial}{\partial t}\phi$

$$= \left(\frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \vec{\nabla}^2\right)\vec{A} + \frac{1}{c}\vec{\nabla}\frac{\partial}{\partial t}\phi = 0$$

$$(\vec{\nabla}\times\vec{\nabla}\times\vec{A})_i = \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l A_m = (\vec{\nabla}\cdot\underbrace{(\vec{\nabla}\vec{A})}_{=0} - \vec{\nabla}^2\vec{A})_i$$

$$\text{(i)} \quad \vec{\nabla}\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\underbrace{\vec{\nabla}\vec{A}}_{=0} - \vec{\nabla}^2\phi = -\vec{\nabla}^2\phi = 0$$

Aha!

$\Rightarrow$ We can now impose a final gauge fixing such that $\boxed{\phi = 0}$

↳ $\boxed{\text{We call } \vec{A} \text{ the "photon field"}}$

Wave equation for the photon field

"relativistic Schrödinger eq." for the photon

$$\boxed{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{A}(x) =: \Box\vec{A}(x) = 0 \quad , \quad \vec{\nabla}\vec{A}(x) = 0 \quad \text{(Coulomb gauge condition)}}$$

D'Alembert operator $\Longleftrightarrow -\frac{1}{c^2}E^2 + \vec{p}^2$ (correspondence principle)
$(E^2 - c^2\vec{p}^2)\vec{A} = 0$

↳ This is the "relativistic Schrödinger equation" for the photon field in Coulomb gauge. We see that it is more complicated than the massive spinless particle Schrödinger equation
→ photon is a vector field (⤳ What about the polarizations?)
How does the Hamilton operator look?

Ansatz for 1st equation: $\vec{A} \sim e^{-ikx} = e^{-i(\omega t - \vec{k}\vec{x})}$

ω: energy $\quad \vec{q}$: momentum
$k^\mu = (\omega/c, \vec{k}) \quad x^\mu = (ct, \vec{x})$

↳ $\Box\vec{A} = \left(-\frac{1}{c^2}\omega^2 + \vec{k}^2\right)\vec{A} \stackrel{!}{=} 0 \Rightarrow \boxed{\omega = c|\vec{k}|}$

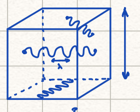
$E^2 = (kc)^2 + (mc^2)^2$

photon energy-momentum relation

3.2. Finite box quantization

1st Quantization

To quantify more easily the eigenmodes use again a finite size box with periodic boundary conditions and take the limit $V \to \infty$ later.



$V = L^3 \longrightarrow \infty$

$\Rightarrow$ Possible momenta : $\vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$, $n_i \in \mathbb{Z}$ $\rightsquigarrow$ $\sim e^{\pm i\vec{k}\vec{x}}$

Number of states in momentum volume element $d^3\vec{k}$: $\frac{V}{(2\pi)^3}\, d^3\vec{k}$

For the momentum eigenmodes we adopt the norm 1: $\frac{e^{i\vec{k}\vec{x}}}{\sqrt{V}}$ $\longleftarrow$ Relativistic standard choice has no $1/(2\pi)^{3/2}$ factor.

Ansatz for the photon field: $\vec{A}(t,\vec{x}) = \sum_{\vec{k}} \frac{e^{i\vec{k}\vec{x}}}{\sqrt{V}}\, \vec{c}_{\vec{k}}(t)$

$\hookrightarrow$ Conditions : $\vec{A}^* = \vec{A}$ ($\vec{B}$ field is real) , $\vec{\nabla}\vec{A} = 0$, $\Box\vec{A} = 0$

$\vec{A}^* = \vec{A} \quad \Rightarrow \quad \vec{c}_{-\vec{k}} = \vec{c}^{\,*}_{\vec{k}} \qquad$ (a)

$\vec{\nabla}\vec{A} = 0 \quad \Rightarrow \quad \vec{k}\cdot\vec{c}_{\vec{k}} = 0 \qquad$ (b)

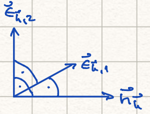
$\Box\vec{A} = 0 \quad \Rightarrow \quad \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{c}_{\vec{k}} + \vec{k}^2\vec{c}_{\vec{k}} = 0 \qquad$ (c)

photon energy $\Rightarrow$ real photon

$$\overset{(a)}{\Rightarrow} \quad \vec{c}_{\vec{k}}(t) = e^{-i\omega_{\vec{k}}t}\,\vec{b}_{\vec{k}} + e^{+i\omega_{\vec{k}}t}\,\vec{b}^{\,*}_{-\vec{k}} \; , \quad \boxed{\omega_{\vec{k}} \equiv c|\vec{k}|} \; , \quad \vec{k}\cdot\vec{b}_{\vec{k}} = 0$$

$$\hookrightarrow \vec{A}(t,\vec{x}) = \sum_{\vec{k}} \frac{e^{i\vec{k}\vec{x}}}{\sqrt{V}}\left(e^{-i\omega_{\vec{k}}t}\,\vec{b}_{\vec{k}} + e^{+i\omega_{\vec{k}}t}\,\vec{b}^{\,*}_{-\vec{k}}\right)$$

$$= \sum_{\vec{k}} \frac{1}{\sqrt{V}}\left(e^{-i\omega_{\vec{k}}t}\,e^{i\vec{k}\vec{x}}\,\vec{b}_{\vec{k}} + e^{+i\omega_{\vec{k}}t}\,e^{-i\vec{k}\vec{x}}\,\vec{b}^{\,*}_{\vec{k}}\right)$$



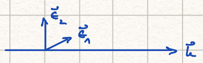
$|\vec{\epsilon}_{k,1,2}| = 1$

$\vec{\epsilon}_{\vec{k},i}\cdot\vec{\epsilon}_{\vec{k},j} = \delta_{ij}$

Polarization vectors:

We now rewrite the $\vec{b}_{\vec{k}}$ in the basis of the normalized real vectors $\vec{n}_{\vec{k}} = \frac{\vec{k}}{|\vec{k}|}$, $\vec{\epsilon}_{\vec{k},1}$, $\vec{\epsilon}_{\vec{k},2}$

$\hookrightarrow$ Linear polarizations: $\vec{b}_{\vec{k}} = b_{\vec{k},1}\cdot\vec{\epsilon}_{\vec{k},1} + b_{\vec{k},2}\,\vec{\epsilon}_{\vec{k},2}$



$b_{\vec{k},i}$: amplitude for each polarization ($\in \mathbb{R}$)

$$\Rightarrow \quad e^{-i\omega_{\vec{k}}t}\,e^{i\vec{k}\vec{x}}\,\vec{b}_{\vec{k}} + e^{+i\omega_{\vec{k}}t}\,e^{-i\vec{k}\vec{x}}\,\vec{b}^{\,*}_{\vec{k}} = 2\cos(\omega_{\vec{k}}t - \vec{k}\vec{x})\left(b_{\vec{k},1}\cdot\vec{\epsilon}_{\vec{k},1} + b_{\vec{k},2}\,\vec{\epsilon}_{\vec{k},2}\right)$$

both polarizations oscillate with the same phase

04|23|2018 ④

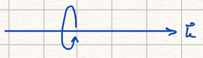
↳ Circular polarizations:

$$\vec{\epsilon}_{\vec{k},\pm} = \frac{1}{\sqrt{2}}\left(\vec{\epsilon}_{\vec{k},1} \overset{!}{\pm} i\, \vec{\epsilon}_{\vec{k},2}\right) \leftarrow \text{complex vectors} \quad , \quad \vec{\epsilon}_{\vec{k},\pm}\cdot \vec{n}_{\vec{k}} = 0 \quad , \quad \vec{\epsilon}^{\,*}_{\vec{k},\lambda}\cdot\vec{\epsilon}_{\vec{k},\lambda'} = \delta_{\lambda\lambda'} \quad , \quad \lambda,\lambda' = \pm$$

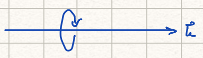
↳ $\vec{b}_{\vec{k}} = b_{\vec{k}+}\,\vec{\epsilon}_{\vec{k},+} + b_{\vec{k}-}\,\vec{\epsilon}_{\vec{k},-} \qquad (b_{\vec{k}\pm} \in \mathbb{R})$

$$\Rightarrow \; e^{-i\omega_{\vec{k}}t}\, e^{i\vec{k}\vec{x}}\, \vec{b}_{\vec{k}} + e^{+i\omega_{\vec{k}}t}\, e^{-i\vec{k}\vec{x}}\, \vec{b}^{\,*}_{\vec{k}} =$$

$$= 2b_{\vec{k}+}\left[\frac{1}{\sqrt{2}}\left(\cos(\omega_{\vec{k}}t-\vec{k}\vec{x})\,\vec{\epsilon}_{\vec{k},1} + \sin(\omega_{\vec{k}}t-\vec{k}\vec{x})\,\vec{\epsilon}_{\vec{k},2}\right]\right. \leftarrow \text{right-handed oscillation}$$



$$+2b_{\vec{k}-}\left[\frac{1}{\sqrt{2}}\left(\cos(\omega_{\vec{k}}t-\vec{k}\vec{x})\,\vec{\epsilon}_{\vec{k},1} - \sin(\omega_{\vec{k}}t-\vec{k}\vec{x})\,\vec{\epsilon}_{\vec{k},2}\right]\right. \leftarrow \text{left handed oscillation}$$



$$\Rightarrow \boxed{\begin{aligned} \vec{A}(t,\vec{x}) &= \sum_{\vec{k}} \sum_{\substack{\lambda=\pm \\ \text{or } 1,2}} \left(e^{-i\omega_{\vec{k}}t}\, \vec{u}_{\vec{k},\lambda}(\vec{x})\, b_{\vec{k},\lambda} + e^{+i\omega_{\vec{k}}t}\, \vec{u}^{\,*}_{\vec{k},\lambda}\, b^{*}_{\vec{k},\lambda}\right) \\ \vec{u}_{\vec{k},\lambda}(\vec{x}) &= \frac{1}{\sqrt{V}}\, e^{i\vec{k}\vec{x}}\, \vec{\epsilon}_{\vec{k},\lambda} \end{aligned}}$$

$\vec{k}_i = \frac{2\pi}{L} n_i \quad (i=1,2,3)$

$n_i \in \mathbb{Z} \quad , \quad \lambda = \pm \ (\text{or } 1,2)$

We adopt the convention : $\vec{\epsilon}_{-\vec{k},1} = -\vec{\epsilon}_{\vec{k},1} \; , \; \vec{\epsilon}_{-\vec{k},2} = \vec{\epsilon}_{\vec{k},2}$ (only one changes sign for $\vec{k} \to -\vec{k}$)

$\Rightarrow \vec{\epsilon}_{-\vec{k},\pm} = -\vec{\epsilon}^{\,*}_{\vec{k},\pm}$

Useful properties for later :

$$\int_V d^3\vec{x}\; \vec{u}_{\vec{k},\lambda}(\vec{x})\cdot\vec{u}_{\vec{k}',\lambda'}(\vec{x}) = \frac{1}{V}\int d^3\vec{x}\; e^{i(\vec{k}+\vec{k}')\vec{x}}\; \vec{\epsilon}_{\vec{k},\lambda}\cdot\vec{\epsilon}_{\vec{k}',\lambda'}$$

$$= \delta_{\vec{k},-\vec{k}'}\; \vec{\epsilon}_{\vec{k},\lambda}\cdot\vec{\epsilon}_{-\vec{k},\lambda'} = -\delta_{\vec{k},-\vec{k}'}\; \vec{\epsilon}_{\vec{k},\lambda}\vec{\epsilon}^{\,*}_{\vec{k},\lambda'} = -\delta_{\vec{k},-\vec{k}'}\;\delta_{\lambda\lambda'}$$

$$\int_V d^3\vec{x}\; (\vec{u}_{\vec{k},\lambda}(\vec{x}))^{*}\, \vec{u}_{\vec{k}',\lambda'}(\vec{x}) = \frac{1}{V}\int d^3\vec{x}\; e^{-i(\vec{k}-\vec{k}')\vec{x}}\; \vec{\epsilon}^{\,*}_{\vec{k},\lambda}\cdot\vec{\epsilon}_{\vec{k}',\lambda'} = \delta_{\vec{k},\vec{k}'}\; \vec{\epsilon}^{\,*}_{\vec{k},\lambda}\cdot\vec{\epsilon}_{\vec{k},\lambda'} = \delta_{\vec{k},\vec{k}'}\;\delta_{\lambda\lambda'}$$

Interpretation of $\vec{A}(\vec{x},t)$:

This is the general form of the quantum mechanical wave function of a single real photon.
It has in principle the same interpretation as the wave function of a massive particle.

↳ $|A(\vec{x},t)|^2$: probability density to find a photon at point $\vec{x}$ (at time t) in a location measurement.

→ $\boxed{\text{The Maxwell equations are quantum mechanical.}}$

This may sound very surprising, because one usually thinks of the Maxwell equations being classical field equations.

The resolution is achieved by considering the difference between real and virtual photons.

Real photons: → Wave functions with $\omega_{\vec k} = c|\vec k|$

These solutions of the Maxwell equations have quantum mechanical (i.e. wave ⊕ particle) character

⇝ Real photons can hit a photo plate / be seen by eye. ⊕

⊕ Interference pattern visible (i.e. double slit)

Virtual photons: → Wave functions with $\omega_{\vec k} \neq c|\vec k|$

These solutions of the Maxwell equations have only classical character, simply because they do not carry the correct amount of energy and momentum that a particle needs to have. So only their wave character remains!

↳ Important conclusion: Also massive particles can be either real $\left(E = \frac{\vec k^2}{2\mu}\right)$ or or virtual $\left(E \neq \frac{\vec k^2}{2\mu}\right)$.

|| → The virtual massive particles are making up the Green's function away from poles and cuts!

Toward 2nd quantization

We still have to extend our formalism concerning the multi-particle issue. To proceed let's see how the total energy (H) and momentum operators ($\vec P$) may be constructed. For them the multi-particle issue must be resolved, because otherwise they cannot be written down.

Let's have a closer look at the classical energy and momentum for electromagnetic fields.

Heaviside Lorentz:

$$E_{\text{field}} = \frac{1}{2}\int d^3\vec x\,(\vec E^2 + \vec B^2)$$

Energy of the classic electromagnetic field

Recall: $\vec B = \vec\nabla \times \vec A$

$\vec E = -\frac{1}{c}\frac{\partial}{\partial t}\vec A$

acts only once

$$E_{\text{field}} = \frac{1}{8\pi}\int_V d^3\vec x\,(\vec E^2 + \vec B^2) = \frac{1}{8\pi}\int_V d^3\vec x\left[\frac{1}{c^2}\dot{\vec A}^2 + (\vec\nabla\times\vec A)^2\right]$$

$$\epsilon^{ijl}\epsilon^{ink}(\nabla^j A^l)(\nabla^n A^k) = (\delta^{jn}\delta^{lk} - \delta^{jk}\delta^{ln})(\nabla^j A^l)(\nabla^n A^k)$$

$$= (\nabla^j A^k)(\nabla^j A^k) - (\nabla^j A^k)(\nabla^k A^j)$$

$$\int d^3\vec x\,(\nabla^j A^k)(\nabla^j A^k) = -\int d^3\vec x\, A^k(\vec\nabla^2 A^k) \overset{\Box\vec A = 0}{=} -\int d^3\vec x\,\frac{1}{c^2}\vec A\,\ddot{\vec A}$$

$$\int d^3\vec x\,(\nabla^j A^k)(\nabla^k A^j) = -\int d^3\vec x\, A^k(\nabla^k\vec\nabla\vec A) \overset{\vec\nabla\vec A = 0}{=} 0$$

← Surface terms for partial integration vanish due to periodic boundary conditions ←

$$= \frac{1}{8\pi c^2}\int_V d^3\vec x\,(\dot{\vec A}^2 - \vec A\,\ddot{\vec A})$$

$$\begin{aligned}= \frac{1}{8\pi c^2}\sum_{\vec k,\vec k'}\sum_{\lambda,\lambda'}\int d^3\vec x\Big[&\left(-i\omega_k e^{-i\omega_k t}\vec u_{\vec k,\lambda}(\vec x)\, b_{\vec k,\lambda} + i\omega_k e^{i\omega_k t}\vec u^*_{\vec k,\lambda}(\vec x)\, b^*_{\vec k,\lambda}\right)\\ &\times\left(-i\omega_{k'} e^{-i\omega_{k'} t}\vec u_{\vec k',\lambda'}(\vec x)\, b_{\vec k',\lambda'} + i\omega_{k'} e^{i\omega_{k'} t}\vec u^*_{\vec k',\lambda'}(\vec x)\, b^*_{\vec k',\lambda'}\right)\\ &-\left(e^{-i\omega_k t}\vec u_{\vec k,\lambda}(\vec x)\, b_{\vec k,\lambda} + e^{i\omega_k t}\vec u^*_{\vec k,\lambda}(\vec x)\, b^*_{\vec k,\lambda}\right)\\ &\times\left(-\omega_{k'}^2 e^{-i\omega_{k'} t}\vec u_{\vec k',\lambda'}(\vec x)\, b_{\vec k',\lambda'} - \omega_{k'}^2 e^{i\omega_{k'} t}\vec u^*_{\vec k',\lambda'}(\vec x)\, b^*_{\vec k',\lambda'}\right)\end{aligned}$$

$\omega_k = \omega_{-k}$

$$\begin{aligned}= \frac{1}{8\pi c^2}\sum_{\vec k}\sum_{\lambda}\omega_{\vec k}^2\Big[&e^{-2i\omega_k t}b_{\vec k,\lambda}b_{-\vec k,\lambda} + e^{2i\omega_k t}b_{\vec k,\lambda}b_{-\vec k,\lambda} + |b_{\vec k,\lambda}|^2 + |b_{\vec k,\lambda}|^2\\ &- e^{-2i\omega_k t}b_{\vec k,\lambda}b_{-\vec k,\lambda} - e^{2i\omega_k t}b_{\vec k,\lambda}b_{-\vec k,\lambda} + |b_{\vec k,\lambda}|^2 + |b_{\vec k,\lambda}|^2\Big]\end{aligned}$$

04/23/2018 ⑥

"classical" amplitude

$$= \frac{1}{2\pi c^2} \sum_{\vec{k},\lambda} \omega_{\vec{k}}^2 \, |b_{\vec{k},\lambda}|^2 = \sum_{\vec{k},\lambda} \frac{\vec{k}^2}{2\pi} |b_{\vec{k},\lambda}|^2$$

Heaviside Lorentz: $\vec{P} = \frac{1}{c} \int d^3\vec{x} \, \vec{E} \times \vec{B}$

Momentum of the classic electromagnetic field

→ Pointing vector

$$\left(\vec{B} \times (\vec{\nabla} \times \vec{A})\right)^i$$
$$= \epsilon^{ijn} B^j \epsilon^{kln} \nabla^l A^n$$
$$= (\delta^{il}\delta^{jn} - \delta^{in}\delta^{jl}) B^j \nabla^l A^n$$
$$= B^n \nabla^i A^n - B^n \nabla^n A^i$$

$$(\vec{P}_{\text{field}})^i = \frac{1}{4\pi c} \int_V d^3\vec{x} \, (\vec{E} \times \vec{B})^i = -\frac{1}{4\pi c^2} \int d^3\vec{x} \left(\dot{\vec{A}} \times (\vec{\nabla} \times \vec{A})\right)^i$$

$$= -\frac{1}{4\pi c^2} \int d^3\vec{x} \left(\dot{A}^n \nabla^i A^n - \dot{A}^n \nabla^n A^i \right)$$

$$= -\frac{1}{4\pi c^2} \int d^3\vec{x} \left(\dot{A}^n \nabla^i A^n + \frac{\partial}{\partial t} (\vec{\nabla}\vec{A}) A^i \right) \overset{\vec{\nabla}\vec{A}=0}{=} -\frac{1}{4\pi c^2} \int d^3\vec{x} \, \dot{A}^n \nabla^i A^n$$

$$= \frac{1}{2\pi c^2} \sum_{\vec{k},\lambda} k^i \omega_{\vec{k}} \, |b_{\vec{k},\lambda}|^2$$

2nd quantization

(We write explicit factors of $\hbar$ for a little while)

The expression derived for E_{field} and $\vec{P}_{\text{field}}$ do have a structure similar to the Hamilton (E) operator of a harmonic oscillator:

$$H_{\text{h.o.}} = \hbar \frac{\omega}{2} (a^+ a + a a^+) = \hbar \omega (a^+ a + \tfrac{1}{2}) \quad , \; [a, a^+] = \mathbb{1}$$

a^+ : raising operator , a : lowering operator , $N = a^+ a$: number operator

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \quad , \quad a|n\rangle = \sqrt{n}\,|n-1\rangle \quad , \quad N|n\rangle = n|n\rangle$$

$$[H, a^+] = \hbar \omega a^+ \quad , \quad [H, a] = -\hbar \omega a$$

Idea of 2nd quantization

For each photon state $|\vec{k}, \lambda\rangle$ there is a harmonic oscillator with frequency $\omega_k = c|\vec{k}|$ where the excitation n ($\rightarrow |n\rangle_{\text{h.o.}}$) corresponds to the number of photons in the state $|\vec{k}, \lambda\rangle$.

→ We are here using the algebraic structures of the harmonic oscillator $(a, a^+, |n\rangle)$ and not the explicit configuration space wave functions.

→ The 2nd quantization is combining the quantum physics of single (or a finite number of particles) with the quantum physics of creating and annihilating any number of particles using the structure of the harmonic oscillator.

↳ Is it obvious that that describes nature? ⟶ No! Not at all.

↑

But it works and describes nature to extremely high precision. (Checked by experiment.)

2nd quantization is not derived, but imposed (just like quantum mechanics itself).

Still it satisfies non-trivial consistency checks.

04/23/2018 ⑦

We impose:

Hamilton operator for the e.m.-field:

$$\boxed{\begin{array}{l} H_{EM} = \sum_{\vec{k},\lambda} \hbar \omega_{\vec{k}}\, a^{+}_{\vec{k},\lambda} a_{\vec{k},\lambda} \\ a^{+}_{\vec{k},\lambda} : \text{creation operator that generates 1 photon with } (\vec{k},\lambda). \\ a_{\vec{k},\lambda} : \text{annihilation operator that destroys 1 photon with } (\vec{k},\lambda) \\ N_{\vec{k},\lambda} = a^{+}_{\vec{k},\lambda} a_{\vec{k},\lambda} : \text{number operator that counts the number of } (\vec{k},\lambda) \text{ photons.} \end{array}}$$

$\longleftarrow$ We drop the 0-point energy which is $\sum_{\vec{k},\lambda} \frac{\hbar}{2}\omega_{\vec{k}} = \sum_{\vec{k},\lambda} \frac{\hbar}{2} c|\vec{k}| = \infty$.

"Argument": The vacuum energy being ∞ is irrelevant because only relative energies are physically important.

Canonical commutation relations:

$$\boxed{\begin{array}{l} [a_{\vec{k},\lambda}, a^{+}_{\vec{k}',\lambda'}] = \delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'}\,, \\ [a_{\vec{k},\lambda}, a_{\vec{k}',\lambda'}] = [a^{+}_{\vec{k},\lambda}, a^{+}_{\vec{k}',\lambda'}] = 0 \end{array}}$$

$\Longrightarrow$

$$\boxed{\begin{array}{l} [N_{\vec{k},\lambda}, a^{+}_{\vec{k}',\lambda'}] = \delta_{\vec{k},\vec{k}'}\delta_{\lambda\lambda'} a^{+}_{\vec{k},\lambda} \\ [N_{\vec{k},\lambda}, a_{\vec{k}',\lambda'}] = -\delta_{\vec{k},\vec{k}'}\delta_{\lambda\lambda'} a_{\vec{k},\lambda} \\ [H_{EM}, a^{+}_{\vec{k},\lambda}] = \hbar\omega_{\vec{k}} a^{+}_{\vec{k},\lambda} \\ [H_{EM}, a_{\vec{k},\lambda}] = -\hbar\omega_{\vec{k}} a_{\vec{k},\lambda} \end{array}}$$

↳ as usual

↳ From comparing classic field energy and Hamilton operator we get the association

$$E_{field} \longleftrightarrow H_{EM}$$

HL: $4\pi\times$ $\frac{\omega^2_{\vec{k},\lambda}}{2\pi c^2}|b_{\vec{k},\lambda}|^2 \longleftrightarrow \hbar\omega_{\vec{k},\lambda} a^{+}_{\vec{k},\lambda} a_{\vec{k},\lambda}$

HL: $\sqrt{\frac{\hbar c^2}{2\omega_{\vec{k}}}}$

$$b_{\vec{k},\lambda} \longleftrightarrow \sqrt{\frac{2\pi\hbar c^2}{\omega_{\vec{k}}}}\, a_{\vec{k},\lambda}$$

(standard convention)

Total momentum operator of the e.m. field:

$$\boxed{\vec{P}_{EM} = \sum_{\vec{k},\lambda} \hbar\vec{k}\, a^{+}_{\vec{k},\lambda} a_{\vec{k},\lambda}}$$

Operator that counts the number of all photons

$$\boxed{N = \sum_{\vec{k},\lambda} N_{\vec{k},\lambda} = \sum_{\vec{k},\lambda} a^{+}_{\vec{k},\lambda} a_{\vec{k},\lambda}}$$

Photon field operator

HL: $\sqrt{\frac{\hbar c^2}{V 2\omega_{\vec{k}}}}$

$$\boxed{\vec{A}(t,\vec{x}) = \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar c^2}{V\omega_{\vec{k}}}} \left(e^{-ikx}\vec{\varepsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} + e^{+ikx}\vec{\varepsilon}^{\,*}_{\vec{k},\lambda} a^{+}_{\vec{k},\lambda} \right)}$$

$k^\mu = (\omega_{\vec{k}}/c, \vec{k})$ wave number 4-vector

$p^\mu = (\hbar\omega_k/c, \hbar\vec{k})$ 4-momentum

$x^\mu = (ct, \vec{x})$ 4-event-vector

$k.x = k_\mu x^\mu = \omega_{\vec{k}} t - \vec{k}\vec{x} = \frac{p.x}{\hbar}$

05/06/2018 ①

Consistency check: $\dot{\vec{A}} = \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar c^2}{V\omega_{\vec{k}}}} \left(-i\omega_{\vec{k}} e^{-ikx} \vec{\varepsilon}_{\vec{k},\lambda} a_{\vec{k},\lambda} + i\omega_{\vec{k}} e^{+ikx} \vec{\varepsilon}^{\,*}_{\vec{k},\lambda} a^{+}_{\vec{k},\lambda}\right) = \frac{i}{\hbar}[H_{EM}, \vec{A}(t,\vec{x})]$ ✓

$$\boxed{\dot{\vec{A}}(t,\vec{x}) = \frac{i}{\hbar}[H_{EM}, \vec{A}(t,\vec{x})]}$$ ← Heisenberg equations are satisfied!

Multi-Photon States

Vacuum state: $|0\rangle$ ⟶ contains no (real) photon

Defined by the state for which
$$\boxed{\begin{array}{l} a_{\vec{k},\lambda}|0\rangle = 0 \quad \text{for all } (\vec{k},\lambda), \text{ and } \langle 0|0\rangle = 1 \\ \Longrightarrow \quad H|0\rangle = 0, \quad \vec{P}|0\rangle = 0, \quad N|0\rangle = 0 \end{array}}$$

1-particle states: $|\vec{k},\lambda\rangle = |1_{\vec{k},\lambda}\rangle := a^{+}_{\vec{k},\lambda}|0\rangle$ ⟶ State describing 1 photon with $(\vec{k},\lambda)$.

↳ $N_{\vec{k}',\lambda'}|\vec{k},\lambda\rangle = \delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'}|\vec{k},\lambda\rangle$

$N|\vec{k},\lambda\rangle = \sum_{\vec{k}',\lambda'} \delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'}|\vec{k},\lambda\rangle = 1\cdot|\vec{k},\lambda\rangle$

$H|\vec{k},\lambda\rangle = \hbar\omega_{\vec{k}}|\vec{k},\lambda\rangle = E_{\vec{k}}|\vec{k},\lambda\rangle$ $\quad$ $E = \hbar\omega = c\hbar|\vec{k}| = c|\vec{p}|$

$\vec{P}|\vec{k},\lambda\rangle = \hbar\vec{k}|\vec{k},\lambda\rangle = \vec{p}|\vec{k},\lambda\rangle$ $\quad$ → Energy-momentum relation of a real photon

$P^\mu|\vec{k},\lambda\rangle = p^\mu|\vec{k},\lambda\rangle$ $\quad$ $P^\mu = (H,\vec{P})$, $\quad p^\mu = (E,\vec{p})$,

Multi-particle states containing several photons with $(\vec{k},\lambda)$: $|n_{\vec{k},\lambda}\rangle = \frac{1}{\sqrt{n!}}(a^{+}_{\vec{k},\lambda})^n|0\rangle$

↑ State describing n photons with $(\vec{k},\lambda)$.

↳ $\langle n_{\vec{k},\lambda}|n_{\vec{k}',\lambda'}\rangle = \delta_{\vec{k},\vec{k}'}\delta_{\lambda\lambda'}$

$N_{\vec{k}',\lambda'}|n_{\vec{k},\lambda}\rangle = \delta_{\vec{k},\vec{k}'}\delta_{\lambda\lambda'} n_{\vec{k},\lambda}|n_{\vec{k},\lambda}\rangle$ ↖ keep index for bookkeeping

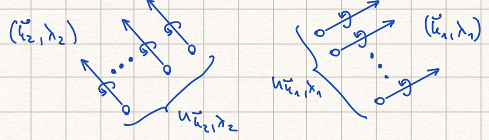
$N|n_{\vec{k},\lambda}\rangle = n_{\vec{k},\lambda}|n_{\vec{k},\lambda}\rangle$

$H|n_{\vec{k},\lambda}\rangle = n_{\vec{k},\lambda}\hbar\omega_{\vec{k}}|n_{\vec{k},\lambda}\rangle$

$\vec{P}|n_{\vec{k},\lambda}\rangle = n_{\vec{k},\lambda}\hbar\vec{k}|n_{\vec{k},\lambda}\rangle$

Multi-particle states containing several photons of different kind:

$$|n_{\vec{k}_1,\lambda_1}; n_{\vec{k}_2,\lambda}; \ldots\rangle = \frac{(a^{+}_{\vec{k}_1,\lambda_1})^{n_{\vec{k}_1,\lambda_1}}}{\sqrt{n_{\vec{k}_1,\lambda_1}!}} \frac{(a^{+}_{\vec{k}_2,\lambda_2})^{n_{\vec{k}_2,\lambda_2}}}{\sqrt{n_{\vec{k}_2,\lambda_2}!}} \cdots |0\rangle$$ ← State describing $n_{\vec{k}_1,\lambda_1}$ photons with $(\vec{k}_1,\lambda_1)$ and $n_{\vec{k}_2,\lambda_2}$ photons with $(\vec{k}_2,\lambda_2)$ and ...



05/06/2018 ②

$$N_{\vec k,\lambda}\,|\dots,n_{\vec k_i,\lambda_i},\dots\rangle = \hbar\,(\dots + n_{\vec k_i,\lambda_i}\,\delta_{\vec k_i,\vec k}\,\delta_{\lambda_i\lambda} + \dots)\,|\dots,n_{\vec k_i,\lambda_i},\dots\rangle$$

$$H\,|\dots,n_{\vec k_i,\lambda_i},\dots\rangle = \hbar\,(\dots + n_{\vec k_i,\lambda_i}\,\omega_{\vec k_i} + \dots)\,|\dots,n_{\vec k_i,\lambda_i},\dots\rangle$$

$$\vec P\,|\dots,n_{\vec k_i,\lambda_i},\dots\rangle = \hbar\,(\dots + n_{\vec k_i,\lambda_i}\,\vec k_i + \dots)\,|\dots,n_{\vec k_i,\lambda_i},\dots\rangle$$

$$N\,|\dots,n_{\vec k_i,\lambda_i},\dots\rangle = (\dots + n_{\vec k_i,\lambda_i} + \dots)\,|\dots,n_{\vec k_i,\lambda_i},\dots\rangle$$

Multi-particle Hilbert Space

$$\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(20} \oplus \dots = \sum_{n=0}^{\infty} \mathcal{H}^{(n)}$$

"Foch space"

$\mathcal{H}^{(0)} = \{|0\rangle\}$ ← 1-dim vacuum space

$\mathcal{H}^{(n>0)}$: n-particle space spanned by $\prod_{i=1}^{n} a^{+}_{\vec k_i,\lambda_i}\,|0\rangle$

$N|\phi\rangle = n|\phi\rangle$ for all $|\phi\rangle \in \mathcal{H}^{(n)}$

3.3. Continuum / infinite box limit

→ We take the limit of infinite volume $V \longrightarrow \infty$

$$k_i = \frac{2\pi}{L}\,n_i$$

↑ wave number; ↖ counts descrete states

$$\leadsto \boxed{d^3\vec n = \frac{V}{(2\pi)^3}\,d^3\vec k = \frac{V}{(2\pi\hbar)^3}\,d^3\vec p}$$

state volume element; # of states incrses with volume V; momentum volume element

$$\sum_{\vec k} \text{ (Sum over states)} \longrightarrow \frac{V}{(2\pi\hbar)^3}\int d^3\vec p = \frac{V}{(2\pi)^3}\int d^3\vec k$$

$$[a_{\vec k,\lambda}, a^{+}_{\vec k',\lambda'}] = \delta_{\vec k,\vec k'}\,\delta_{\lambda\lambda'} = \frac{1}{V}\int d^3\vec x\, e^{i(\vec k-\vec k')\vec x}\,\delta_{\lambda\lambda'} \longrightarrow \frac{(2\pi)^3}{V}\,\delta^{(3)}(\vec k-\vec k')\,\delta_{\lambda\lambda'} = \frac{(2\pi\hbar)^3}{V}\,\delta^{(3)}(\vec p-\vec p')\,\delta_{\lambda\lambda'}$$

Therfore: $a_{\vec k,\lambda} \longrightarrow \frac{(2\pi)^{3/2}}{V^{1/2}}\,a(\vec k,\lambda) = \frac{(2\pi\hbar)^{3/2}}{V^{1/2}}\,a(\vec p,\lambda)$

$$\leadsto \boxed{\begin{aligned}&[a(\vec p,\lambda), a^{+}(\vec p',\lambda')] = \delta^{(3)}(\vec p-\vec p')\,\delta_{\lambda\lambda'}\\ &[a(\vec p,\lambda), a(\vec p',\lambda')] = [a^{+}(\vec p,\lambda), a^{+}(\vec p',\lambda')] = 0\end{aligned}}$$

Standard QFT convention:

→ $[a(\vec p,\lambda), a^{+}(\vec p',\lambda')] = (2\pi)^3\,\delta^{(3)}(\vec p-\vec p')\,\delta_{\lambda\lambda'}$

$$\vec A(x) \longrightarrow \frac{V}{(2\pi\hbar)^3}\sum_{\lambda}\int d^3\vec p\left(\frac{2\pi\hbar c^2}{V\omega_{\vec k}}\right)^{1/2}\frac{(2\pi\hbar)^{3/2}}{V^{1/2}}\left(e^{-i\frac{p\cdot x}{\hbar}}\,\vec\epsilon(\vec p,\lambda)\,a(\vec p,\lambda) + e^{+i\frac{p\cdot x}{\hbar}}\,\vec\epsilon^{\,*}(\vec p,\lambda)\,a^{+}(\vec p,\lambda)\right)$$

$$\boxed{\vec A(x) = \sum_{\lambda}\int d^3\vec p\,\frac{c}{(2\pi\hbar)}\,\frac{1}{\sqrt{\omega_{\vec k}}}\left(e^{-i\frac{p\cdot x}{\hbar}}\,\vec\epsilon(\vec p,\lambda)\,a(\vec p,\lambda) + e^{+i\frac{p\cdot x}{\hbar}}\,\vec\epsilon^{\,*}(\vec p,\lambda)\,a^{+}(\vec p,\lambda)\right)}$$

$$\omega_{\vec k} = \frac{E_{\vec k}}{\hbar} = \frac{c\,|\vec p|}{\hbar}$$

Standard QFT convention ⊕ HL units:

$$\vec A(x) = \frac{1}{\sqrt{\hbar}}\sum_{\lambda}\int\frac{d^3\vec p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec k}}}\left(e^{-i\frac{p\cdot x}{\hbar}}\,\vec\epsilon(\vec p,\lambda)\,a(\vec p,\lambda) + e^{+i\frac{p\cdot x}{\hbar}}\,\vec\epsilon^{\,*}(\vec p,\lambda)\,a^{+}(\vec p,\lambda)\right)$$

05/06/2018 ③

Energy, momentum operators

$$H = \sum_{\vec{k},\lambda} \hbar \omega_{\vec{k}} \, a^{+}_{\vec{k},\lambda} a_{\vec{k},\lambda} \longrightarrow \frac{V}{(2\pi\hbar)^3} \int d^3\vec{p} \, \hbar\omega_{\vec{k}} \frac{(2\pi\hbar)^3}{V} \overset{+}{a}(\vec{p},\lambda) \, a(\vec{p},\lambda)$$

Standard QFT:

$$\boxed{\begin{aligned} H &= \sum_{\lambda} \int d^3\vec{p} \; E_{\vec{p}} \, \overset{+}{a}(\vec{p},\lambda) a(\vec{p},\lambda) \\ \vec{P} &= \sum_{\lambda} \int d^3\vec{p} \; \vec{p} \; a^{+}(\vec{p},\lambda) a(\vec{p},\lambda) \end{aligned}}$$

$$H = \sum_{\lambda} \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} \, \overset{+}{a}(\vec{p},\lambda) a(\vec{p},\lambda)$$

$$\vec{P} = \sum_{\lambda} \int \frac{d^3\vec{p}}{(2\pi)^2} \vec{p} \; a^{+}(\vec{p},\lambda) a(\vec{p},\lambda)$$

Particle number density op.: $N(\vec{p},\lambda) = a^{+}(\vec{p},\lambda) a(\vec{p},\lambda)$

Particle number operator: $N = \sum_{\lambda} \int d^3\vec{p} \, N(\vec{p},\lambda)$ $\qquad N = \sum_{\lambda} \int \frac{d^3\vec{p}}{(2\pi)^3} a^{+}(\vec{p},\lambda) a(\vec{p},\lambda)$

Vacuum state: $|0\rangle$ with $a(\vec{p},\lambda)|0\rangle = 0$ for all $(\vec{p},\lambda)$

↳ Norm: $\langle 0|0\rangle = 1$

momentum eigenstates

1-particle states: $|1_{\vec{p},\lambda}\rangle = |\vec{p},\lambda\rangle = a^{+}(\vec{p},\lambda)|0\rangle$ $\qquad |\vec{p},\lambda\rangle = \sqrt{2E_{\vec{p}}} \, a^{+}(\vec{p},\lambda)|0\rangle$

↳ e.g. $P^{\mu}|\vec{p},\lambda\rangle = p^{\mu}|\vec{p},\lambda\rangle$, $\quad N|\vec{p},\lambda\rangle = 1 \cdot |\vec{p},\lambda\rangle$

Norm: $\langle \vec{p},\lambda|\vec{p}',\lambda'\rangle = \langle 0|a(\vec{p},\lambda) a^{+}(\vec{p}',\lambda')|0\rangle = \langle 0|\overbrace{[a(\vec{p},\lambda), a^{+}(\vec{p},\lambda)]}^{\delta^{(3)}(\vec{p}-\vec{p}')\delta_{\lambda\lambda'}}|0\rangle$

$= \delta^{(3)}(\vec{p}-\vec{p}')\delta_{\lambda\lambda'} \overbrace{\langle 0|0\rangle}^{=1}$ $\qquad \langle \vec{p},\lambda|\vec{p}',\lambda'\rangle = 2E_{\vec{p}}(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')\delta_{\lambda\lambda'}$

$\left[\langle \vec{x},\lambda'|\vec{p},\lambda\rangle_t = \frac{\delta_{\lambda\lambda'}}{(2\pi)^{3/2}} e^{-i\omega_{\vec{p}}t} e^{i\vec{k}\vec{x}}|\lambda\rangle = \delta_{\lambda\lambda'} \frac{e^{-i\frac{p.x}{\hbar}}}{(2\pi)^{3/2}}|\lambda\rangle \right]$ ← NOT NEEDED ANYWHERE?

$\langle \vec{x},\lambda'|\vec{p},\lambda\rangle_t = \delta_{\lambda\lambda'} \sqrt{2E_{\vec{p}}} \, e^{-i\frac{p.x}{\hbar}}|\lambda\rangle$

General 1 particle state: $|\psi^{(1)}\rangle = \sum_{\lambda} \int d^3\vec{p} \; \tilde{\psi}^{(1)}(\vec{p},\lambda)|\vec{p},\lambda\rangle$ $\qquad |\psi^{(1)}\rangle = \sum_{\lambda} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \tilde{\psi}^{(1)}(\vec{p},\lambda)|\vec{p},\lambda\rangle$

↳ $\langle \psi^{(1)}|\psi^{(1)}\rangle = \sum_{\lambda} \int d^3\vec{p} \, |\tilde{\psi}(\vec{p},\lambda)|^2 \overset{!}{=} 1$ $\qquad \langle \psi^{(1)}|\psi^{(1)}\rangle = \sum_{\lambda} \int \frac{d^3\vec{p}}{(2\pi)^3} |\tilde{\psi}(\vec{p},\lambda)|^2 \overset{!}{=} 1$

$\tilde{\psi}^{(1)}(\vec{p},\lambda) = \langle \vec{p},\lambda|\psi^{(1)}\rangle$ $\qquad \tilde{\psi}^{(1)}(\vec{p},\lambda) = \frac{1}{\sqrt{2E_p}} \langle \vec{p},\lambda|\psi^{(1)}\rangle$

$d^3\vec{p} \, |\tilde{\psi}^{(1)}(\vec{p},\lambda)|^2$ probability to find particle with polarization λ in momentum space element $d^3\vec{p}$ (at $\vec{p}$).

Projector on 1-particle space $\mathcal{H}^{(1)}$: $P^{(1)} = \sum_{\lambda} \int d^3\vec{p} \, |\vec{p},\lambda\rangle\langle\vec{p},\lambda|$ $\qquad P^{(1)} = \sum_{\lambda} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} |\vec{p},\lambda\rangle\langle\vec{p},\lambda|$

↳ $P^{(1)+} = P^{(1)}$, $\quad (P^{(1)})^2 = P^{(1)}$

$P^{(1)} \underbrace{|\psi^{(1)}\rangle}_{\in \mathcal{H}^{(1)}} = |\psi^{(1)}\rangle$, $\quad P^{(1)} \underbrace{|\psi\rangle}_{\notin \mathcal{H}^{(1)}} = 0$

2-particle states: $|\vec{p}_1\lambda_1;\vec{p}_2\lambda_2\rangle := a^\dagger(\vec{p}_1,\lambda_1)\,a^\dagger(\vec{p}_2,\lambda)\,|0\rangle$

$$\hookrightarrow P^\mu|\vec{p}_1,\lambda;\vec{p}_2,\lambda_2\rangle = (p_1+p_2)^\mu|\vec{p}_1,\lambda;\vec{p}_2,\lambda_2\rangle$$

$$N|\vec{p}_1,\lambda;\vec{p}_2,\lambda_2\rangle = 2|\vec{p}_1,\lambda;\vec{p}_2,\lambda_2\rangle$$

Norm: $\langle\vec{p}_1{}',\lambda_1';\vec{p}_2{}'\lambda_2'|\vec{p}_1,\lambda_1;\vec{p}_2\lambda_2\rangle$

$$= \delta^{(3)}(\vec{p}_1-\vec{p}_1{}')\,\delta^{(3)}(\vec{p}_2-\vec{p}_2{}')\,\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} + \delta^{(3)}(\vec{p}_1-\vec{p}_2)\,\delta^{(3)}(\vec{p}_2-\vec{p}_1{}')\,\delta_{\lambda_1\lambda_2'}\delta_{\lambda_2\lambda_1'}$$

Projector on $\mathcal{H}^{(2)}$: $P^{(2)} = \frac{1}{2}\sum_{\lambda_1\lambda_2}\int d^3\vec{p}_1\,d^3\vec{p}_2\,|\vec{p}_1,\lambda_1;\vec{p}_2\lambda_2\rangle\langle\vec{p}_1,\lambda_1;\vec{p}_2,\lambda_2|$

$$\hookrightarrow P^{(2)\dagger} = P^{(2)}\,,\quad (P^{(2)})^2 = P^{(2)}\,,\quad P^{(2)}\mathcal{H}^{(2)} = \mathcal{H}^{(2)}$$

General 2-particle state $|\psi^{(2)}\rangle \in \mathcal{H}^{(2)}$

$$\hookrightarrow |\psi^{(2)}\rangle = P^{(2)}|\psi^{(2)}\rangle = \frac{1}{2}\sum_{\lambda_1\lambda_2}\int d^3\vec{p}_1\,d^3\vec{p}_2\,|\vec{p}_1,\lambda_1;\vec{p}_2\lambda_2\rangle\underbrace{\langle\vec{p}_1,\lambda_1;\vec{p}_2,\lambda_2|\psi^{(2)}\rangle}_{\tilde{\psi}^{(2)}(\vec{p}_1,\lambda_1;\vec{p}_2,\lambda_2)}$$

Bose-Symmetry: Because $[a^\dagger(\vec{p}_1,\lambda_1),a^\dagger(\vec{p}_2,\lambda_2)]=0$, the 2-particle wave function $\psi^{(2)}$ is symmetric under the exchange $(\vec{p}_1,\lambda_1)\longleftrightarrow(\vec{p}_2,\lambda_2)$

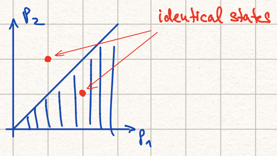
$$\hookrightarrow \tilde{\psi}^{(2)}(\vec{p}_1,\lambda_1;\vec{p}_2,\lambda_2) = \psi^{(2)}(\vec{p}_2,\lambda_2;\vec{p}_1,\lambda_1)$$

Norm: $\langle\psi^{(2)}|\psi^{(2)}\rangle = \langle\psi^{(2)}|P^{(2)}|\psi^{(2)}\rangle = \frac{1}{2}\sum_{\lambda_1\lambda_2}\int d^3\vec{p}_1\,d^3\vec{p}_2\,\langle\psi^{(2)}|\vec{p}_1,\lambda_1;\vec{p}_2\lambda_2\rangle\langle\vec{p}_1,\lambda_1;\vec{p}_2,\lambda_2|\psi^{(2)}\rangle$

$$= \frac{1}{2}\sum_{\lambda_1\lambda_2}\int d^3\vec{p}_1\,d^3\vec{p}_2\,|\tilde{\psi}^{(2)}(\vec{p}_1,\lambda_1;\vec{p}_2,\lambda_2)|^2 = 1 \quad \text{(if normalized)}$$

↑ Why factor $\frac{1}{2}$?

→ Due to Bose symmetry the states $|\vec{p}_1,\lambda_1;\vec{p}_2,\lambda_2\rangle$ and $|\vec{p}_2,\lambda_2;\vec{p}_1,\lambda_1\rangle$ are <u>physically identical</u>. This state must only be counted once. The factor $\frac{1}{2}$ is needed to avoid double counting.



One can drop the factor $\frac{1}{2}$, if one restricts the $\vec{p}_1$ and $\vec{p}_2$ integrations accordingly.

So $|\tilde{\psi}^{(2)}(\vec{p}_1,\lambda_1;\vec{p}_2,\lambda_2)|^2\,d^3\vec{p}_1\,d^3\vec{p}_2$ is the probability to find a photon with polarization λ_1 in vol. element $d^3\vec{p}_1$ around $\vec{p}_1$ <u>and</u> a photon with polarization λ_1 in vol. element $d^3\vec{p}_2$ around $\vec{p}_2$.

05/06/2018 ⑤

<u>n-particle states:</u> $|\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n\rangle := a^+(\vec{p}_1,\lambda_1)\dots a^+(\vec{p}_n,\lambda_n)|0\rangle$

↳ $P^\mu|\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n\rangle = (p_1^\mu+\dots+p_n^\mu)|\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n\rangle$

$N|\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n\rangle = n|\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n\rangle$

Norm: $\langle\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n|\vec{p}\,'_1,\lambda'_1;\dots;\vec{p}\,'_n,\lambda'_n\rangle$

$$= \sum_{\sigma\in S_n}\prod_{i=1}^{n}\delta^{(3)}(\vec{p}_i-\vec{p}\,'_{\sigma(i)})\,\delta_{\lambda_i\lambda'_{\sigma(i)}}$$

↳ All permutations of $\{1,\dots,n\}$ (# of elements = $n!$)

Projector on $\mathcal{H}^{(n)}$: $P^{(n)} = \frac{1}{n!}\sum_{\lambda_1,\dots,\lambda_n}\int d^3\vec{p}_1\dots d^3\vec{p}_n\,|\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n\rangle\langle\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n|$

↳ $P^{(n)+} = P^{(n)}$, $(P^{(n)})^2 = P^{(n)}$, $P^{(n)}\mathcal{H}^{(n)} = \mathcal{H}^{(n)}$

general n-particle state $|\psi^{(n)}\rangle\in\mathcal{H}^{(n)}$

$$↳ |\psi^{(n)}\rangle = P^{(n)}|\psi^{(n)}\rangle = \frac{1}{n!}\sum_{\lambda_1,\dots,\lambda_n}\int d^3\vec{p}_1\dots d^3\vec{p}_n\,|\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n\rangle\overbrace{\langle\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n|\psi^{(n)}\rangle}^{\tilde{\psi}^{(n)}(\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n)}$$

Norm: $\langle\psi^{(2)}|\psi^{(2)}\rangle = \frac{1}{n!}\sum_{\lambda_1,\dots,\lambda_n}\int d^3\vec{p}_1\dots d^3\vec{p}_n\,|\tilde{\psi}^{(n)}(\vec{p}_1,\lambda_1;\dots;\vec{p}_n,\lambda_n)|^2$

↳ Bose symmetry factor

<u>Complete Fock space unit operator:</u> $\mathbb{1} = \sum_{n=0}^{\infty}P^{(n)}$, $P^{(0)} = |0\rangle\langle 0|$

3.4. Helicity of the photon

↳ Angular momentum of the electromagnetic field:

$$\vec{J} = \frac{1}{4\pi c}\int d^3\vec{x}\;\vec{x}\times(\vec{E}\times\vec{B})$$

$$J^i = \underbrace{\frac{\epsilon_{ijm}}{4\pi c}\int d^3x^j\,E^n(\nabla^m A^n)}_{\vec{L}} + \underbrace{\frac{\epsilon_{ijm}}{4\pi c}\int d^3\vec{x}\;E^j A^m}_{\vec{S}}$$

$\vec{L}$: spatial angular momentum — $\vec{S}$: intrinsic angular momentum (spin)

$$(\vec{E}\times(\vec{\nabla}\times\vec{A}))^m$$
$$= \epsilon^{mjk}E^j\epsilon^{kln}\nabla^l A^n$$
$$= (\delta^{ml}\delta^{jn}-\delta^{mn}\delta^{jl})E^j\nabla^l A^n$$
$$= E^n\nabla^m A^n - E^n\nabla^n A^m$$

⊕ partial integration $\vec{\nabla}\vec{E}=0$ ↳

$$\epsilon^{ijm}x^j(E^n\nabla^m A^n - E^n\nabla^n A^m)$$
$$= \epsilon^{ijm}x^jE^n\nabla^m A^n + \epsilon^{ijm}(\nabla^n x^j \overset{=0}{E^n})A^m$$
$$= \epsilon^{ijm}x^jE^n\nabla^m A^n + \epsilon^{inm}\overset{\delta_{nj}}{}E^nA^m$$

→ Consider momentum eigenstate $|\vec{p},\lambda\rangle = a^+(\vec{p},\lambda)|0\rangle$.
Because $[J^k,P^l]\neq 0$ the state $|\vec{p},\lambda\rangle$ is in general not an angular momentum eigenstate (see QM 1).

<u>But:</u> $|\vec{p},\lambda\rangle$ for circular polarizations $\lambda=\pm$ are eigenstates for the angular momentum projection <u>on the momentum direction</u> of the photon

05/06/2018 ⑥

We can separate the spatial angular momentum and the intrinsic one by multiplying with the momentum $\vec{p}$ ($\neq$ operator!)

↳ $\vec{L}\cdot\vec{p}\,|\vec{p},\lambda\rangle = 0$ (exercise) $\Rightarrow \vec{J}\cdot\vec{p}\,|\vec{p},\lambda\rangle = \vec{S}\cdot\vec{p}\,|\vec{p},\lambda\rangle$

$$\vec{E} = -\frac{1}{c}\dot{\vec{A}}$$

$$\vec{S} = -\frac{1}{4\pi c^2}\int d^3\vec{x}\;\dot{\vec{A}}\times\vec{A}$$

$$= -\frac{1}{4\pi c^2}\sum_{\lambda\lambda'}\frac{c^2}{(2\pi\hbar)^2}\int d^3\vec{p}\,d^3\vec{p}\,'\frac{1}{\sqrt{\omega_p}}\frac{1}{\sqrt{\omega_{p'}}}\int d^3\vec{x}\left(-i\omega_p e^{-\frac{ipx}{\hbar}}\vec{\epsilon}(\vec{p},\lambda)a(\vec{p},\lambda) + i\omega_p e^{\frac{ipx}{\hbar}}\vec{\epsilon}^{\,*}(\vec{p},\lambda)a^{+}(\vec{p},\lambda)\right)$$

$$\times\left(e^{-\frac{ip'x}{\hbar}}\vec{\epsilon}(\vec{p}\,',\lambda')a(\vec{p}\,',\lambda') + e^{\frac{ip'x}{\hbar}}\vec{\epsilon}^{\,*}(\vec{p}\,',\lambda')a^{+}(\vec{p}\,',\lambda')\right)$$

$$= \frac{i}{4\pi c^2}\frac{c^2}{(2\pi\hbar)^2}(2\pi\hbar)^3\sum_{\lambda\lambda'}\int d^3\vec{p}\left(\vec{\epsilon}(\vec{p},\lambda)\times\overset{\text{antisymmetric!}}{\cancel{\vec{\epsilon}(-\vec{p},\lambda')}}a(\vec{p},\lambda)a(-\vec{p},\lambda') + \vec{\epsilon}(\vec{p},\lambda)\times\vec{\epsilon}^{\,*}(\vec{p},\lambda')a(\vec{p},\lambda)a^{+}(\vec{p},\lambda')\right.$$

can be swapped $\vec{\epsilon}(\vec{p},\pm)\times\vec{\epsilon}(\vec{p},\pm) = 0$

$$\left.-\vec{\epsilon}^{\,*}(\vec{p},\lambda)\times\vec{\epsilon}(\vec{p},\lambda')a^{+}(\vec{p},\lambda)a(\vec{p},\lambda') - \vec{\epsilon}^{\,*}(\vec{p},\lambda)\times\cancel{\vec{\epsilon}^{\,*}(-\vec{p},\lambda')}a^{+}(\vec{p},\lambda)a^{+}(-\vec{p},\lambda')\right)$$

$$= i\hbar\sum_{\lambda\lambda'}\int d^3\vec{p}\;\vec{\epsilon}^{\,*}(\vec{p},\lambda)\times\vec{\epsilon}(\vec{p},\lambda')\;a^{+}(\vec{p},\lambda)a(\vec{p},\lambda')$$

$$\boxed{\text{Helicity operator}:\quad\frac{\vec{J}\cdot\vec{p}}{|\vec{p}|} = \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}}$$

Projection of angular momentum on direction of movement

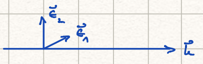
Note: $\vec{P} = \frac{1}{4\pi c}\int d^3\vec{x}\,(\vec{E}\times\vec{B})$, but this does not mean that $\vec{J}\cdot\vec{P}$ is automatically zero because of the volume integrations that connect $\vec{E}$ and $\vec{B}$ fields at different locations!

$$\hookrightarrow\; \vec{S}\cdot\vec{P}\,|\vec{p},\lambda\rangle = -i\hbar\sum_{\alpha,\beta}\int d^3\vec{q}\,\left[\vec{\epsilon}^{\,*}(\vec{q},\alpha)\times\vec{\epsilon}(\vec{q},\beta)\right]\cdot\vec{p}\;a^{+}(\vec{q},\alpha)\overbrace{a(\vec{q},\beta)\,a^{+}(\vec{p},\lambda)}^{\delta^{(3)}(\vec{p}-\vec{q})\,\delta_{\lambda\beta}}|0\rangle$$

$$= -i\hbar\sum_{\alpha}\left[\vec{\epsilon}^{\,*}(\vec{p},\alpha)\times\vec{\epsilon}(\vec{p},\lambda)\right]\cdot\vec{p}\,|\vec{p},\alpha\rangle$$

$$\hookrightarrow\;\boxed{\frac{\vec{J}\cdot\vec{P}}{|\vec{P}|}|\vec{p},\lambda\rangle = -i\hbar\sum_{\alpha}\left[\vec{\epsilon}^{\,*}(\vec{p},\alpha)\times\vec{\epsilon}(\vec{p},\lambda)\right]\cdot\frac{\vec{p}}{|\vec{p}|}|\vec{p},\alpha\rangle}$$

Apply to linear polarizations: ($\vec{\epsilon}^{\,*} = \vec{\epsilon}$)



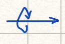
$$\frac{\vec{J}\cdot\vec{P}}{|\vec{P}|}|\vec{p},1\rangle = +i\hbar|\vec{p},2\rangle,\quad\frac{\vec{J}\cdot\vec{P}}{|\vec{P}|}|\vec{p},2\rangle = -i\hbar|\vec{p},1\rangle$$

$$\Rightarrow\langle\vec{p},\alpha|\frac{\vec{J}\cdot\vec{P}}{|\vec{P}|}|\vec{p},\beta\rangle = \hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\rightarrow\text{eigenvalues }\pm 1$$

eigenvectors: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$, EV $= +1$ | $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$, EV $= -1$

$$\hookrightarrow\;\boxed{\begin{aligned}|\vec{p},+\rangle &= \frac{1}{\sqrt{2}}\left(|\vec{p},1\rangle + i\,|\vec{p},2\rangle\right)\\ |\vec{p},-\rangle &= \frac{1}{\sqrt{2}}\left(|\vec{p},1\rangle - i\,|\vec{p},2\rangle\right)\\ \frac{\vec{J}\cdot\vec{P}}{|\vec{P}|}|\vec{p},\pm\rangle &= \pm\hbar\,|\vec{p},\pm\rangle\end{aligned}}\;\rightarrow$$

RH polarized photon has angular momentum $+\hbar$
LH polarized photon has angular momentum $-\hbar$
with respect to direction of motion

RH:  helicity $+\hbar$, LH:  helicity $-\hbar$

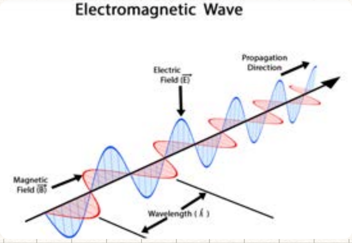
05/06/2018 ⑦

→ $\boxed{\text{The photon is a particle that has spin 1.}}$

The photon is a spin-1 particle, but it does not have a state with helicity $= 0$. This is related to the gauge invariance of the Maxwell equations and eventually related to the representation theory of the Poincaré group for massless spin 1 particles.

⇒ Massless spin-1 particles can only have 2 transverse polarizations with helicities $\pm\hbar$ and no longitudinal polarization with helicity 0. Only massive spin 1 particles can have all 3 helicities.

3.5. Coherent States



→ What kind of quantum state corresponds to a classic electromagnetic wave?

We consider "classic" a state that has a measurable $\vec{E}$ and/or $\vec{B}$ field.

Assume linear polarization and finit volume

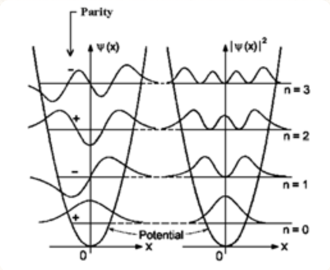
$$\vec{E}(t,\vec{x}) = -\frac{1}{c}\dot{\vec{A}}(t,\vec{x}) = i\sum_{k,\lambda}\sqrt{\frac{2\pi\hbar}{V}}\sqrt{\omega_k}\,\vec{\epsilon}_{k,\lambda}\left[e^{i\vec{k}\vec{x}}a_{\vec{k},\lambda} - e^{-i\vec{k}\vec{x}}a^{\dagger}_{\vec{k},\lambda}\right]$$

What about $\langle n|\vec{E}^2|n\rangle$

Consider n-photon state $|n\rangle$: $\Rightarrow \langle n|\vec{E}|n\rangle = 0$ because $\langle n|n\pm 1\rangle = 0$

⇒ A classic electromagnetic wave cannot be a state with a fixed number of photons, but must be related to a linear combination of states with different particle number.

Recall harmonic oscillator:



→ A usual eigenstate $|n\rangle$ does not at all behave classically.

But a coherent state corresponds to a gaussian wave packet that oscillates with ω and does not disperse
⇝ classic behavior

Coherent state satisfies: $a|z\rangle = z|z\rangle$, $z\in\mathbb{C}$ $\qquad \langle z|z\rangle = 1$

$$\xrightarrow{\text{QM I}} |z\rangle = e^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{(za^{\dagger})^n}{n!}|0\rangle = e^{-\frac{|z|^2}{2}}e^{za^{\dagger}}|0\rangle = e^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle$$

$$|z(t)\rangle = e^{-\frac{|z|^2}{2}}e^{-\frac{i\omega t}{2}}\sum_{n=0}^{\infty}\frac{(ze^{-i\omega t})^n}{\sqrt{n!}}|n\rangle$$

→ We apply this to construct a coherent photon state:

$$|z\rangle := \prod_{\vec{k},\lambda} \left(e^{-\frac{|z_{\vec{k},\lambda}|^2}{2}} e^{z_{\vec{k},\lambda} a^+_{\vec{k},\lambda}} \right) |0\rangle \longleftarrow$$

coherent state made of different photon types (linear combination of different coherent states for various $(\vec{k},\lambda)$)

Average number of $(\vec{k},\lambda)$ photons:

$$\langle z | N_{\vec{k},\lambda} | z \rangle = \langle z | a^+_{\vec{k},\lambda} a_{\vec{k},\lambda} | z \rangle = |z_{\vec{k},\lambda}|^2$$

Possible measurement values for $N_{\vec{k},\lambda}$: $0, 1, 2, \ldots$

Poisson distribution ↓

Probability to measure n $(\vec{k},\lambda)$ photons: $|\langle n_{\vec{k},\lambda} | z \rangle|^2 = e^{-|z_{\vec{k},\lambda}|^2} \frac{|z_{\vec{k},\lambda}|^{2n}}{n!}$

Uncertainty in a measurement of the number of $(\vec{k},\lambda)$ photons:

$$\langle z | N^2_{\vec{k},\lambda} | z \rangle = \langle z | a^+_{\vec{k},\lambda} a_{\vec{k},\lambda} a^+_{\vec{k},\lambda} a_{\vec{k},\lambda} | z \rangle = |z_{\vec{k},\lambda}|^2 \langle z | a_{\vec{k},\lambda} a^+_{\vec{k},\lambda} | z \rangle$$

$$= |z_{\vec{k},\lambda}|^2 \langle z | 1 + a^+_{\vec{k},\lambda} a_{\vec{k},\lambda} | z \rangle = |z_{\vec{k},\lambda}|^2 (1 + |z_{\vec{k}}|^2)$$

$$(\Delta N_{\vec{k},\lambda})^2 = \langle z | N^2_{\vec{k},\lambda} | z \rangle - \langle z | N_{\vec{k},\lambda} | z \rangle^2 = |z_{\vec{k},\lambda}|^2$$

⤳ $\Delta N_{\vec{k},\lambda} = |z_{\vec{k},\lambda}|$ Standard deviation

$$\Rightarrow \frac{\Delta N_{\vec{k},\lambda}}{\langle N_{\vec{k},\lambda} \rangle} = \frac{1}{|z_{\vec{k},\lambda}|} = \frac{1}{\langle N_{\vec{k},\lambda} \rangle^{1/2}}$$

⤳ classic limit corresponds to $\langle N_{\vec{k},\lambda} \rangle \to \infty$, $|z_{\vec{k},\lambda}| \to \infty$ (limit of ∞ many particles)

→ Average $\vec{E}$ field:

$$\langle z | \vec{E}(t,\vec{x}) | z \rangle = i \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar}{V}} \sqrt{\omega_k} \, \vec{\epsilon}_{\vec{k},\lambda} \left[e^{-ikx} z_{\vec{k},\lambda} - e^{ikx} z^*_{\vec{k},\lambda} \right] \longleftarrow$$

grows with $|z_{\vec{k},\lambda}|$ but divergent in general

Uncertainty in $\vec{E}$ field measurement:

$$\langle z | \vec{E}^2(0,\vec{x}) | z \rangle = -\frac{2\pi\hbar}{V} \sum_{\vec{k},\lambda} \sum_{\vec{k}',\lambda'} \sqrt{\omega_{\vec{k}} \omega_{\vec{k}'}} \, \vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k}',\lambda'}$$

$$\times \underbrace{\langle z | (e^{+i\vec{k}\vec{x}} a_{\vec{k},\lambda} - e^{-i\vec{k}\vec{x}} a^+_{\vec{k},\lambda})(e^{+i\vec{k}'\vec{x}} a_{\vec{k}',\lambda'} - e^{-i\vec{k}'\vec{x}} a^+_{\vec{k}',\lambda'}) | z \rangle}_{\langle z | e^{+i(\vec{k}+\vec{k}')} a_{k,\lambda} a_{k',\lambda'} - e^{+i(\vec{k}-\vec{k}')x} \underset{= a^+a + \delta_{kk'}\delta_{\lambda\lambda'}}{a_{k,\lambda} a^+_{k',\lambda'}} - e^{-i(\vec{k}-\vec{k}')x} a^+_{k,\lambda} a_{k',\lambda'} + e^{-i(\vec{k}+\vec{k}')x} a^+_{k,\lambda} a^+_{k',\lambda'} | z \rangle}$$

$$= \langle z | \vec{E}(0,\vec{x}) | z \rangle^2 + \frac{2\pi\hbar}{V} \sum_{\vec{k},\lambda} \omega_{\vec{k}}$$

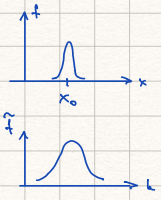
$$(\Delta E(0,\vec{x}))^2 = \frac{2\pi\hbar}{V} \sum_{\vec{k},\lambda} \omega_{\vec{k}} \longleftarrow$$

Does not grow with $|z|$, but divergent also! What's going on?

⤳ <u>Resolution:</u> Our assumption of a $\vec{E}$ field measurement exactly(!) at $\vec{x}$ is unrealistic. Such a sharp measurement involves infinitely large momenta, which cannot be realized.

05/06/2018

↳ Realistic $\vec{E}$ field operator : $\vec{E}_{\substack{\text{finite}\\\text{resol.}}}(0,\vec{x}_0) = \int d^3\vec{x}\ \underset{\text{localized at } \vec{x}_0}{f(\vec{x})}\ E(0,\vec{x})$



$$\vec{E}_{\substack{\text{finite}\\\text{resol.}}}(0,\vec{x}_0) = i \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar}{V}} \sqrt{\omega_k}\ \vec{\varepsilon}_{\vec{k},\lambda} \int d^3\vec{x}\ f(\vec{x}) \left[e^{i\vec{k}\vec{x}} a_{\vec{k},\lambda} - e^{-i\vec{k}\vec{x}} a^{+}_{\vec{k},\lambda} \right]$$

$$= i \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar}{V}} \sqrt{\omega_k}\ \vec{\varepsilon}_{\vec{k},\lambda} \left[\tilde{f}(\vec{k})\, a_{\vec{k},\lambda} - \tilde{f}^{*}(\vec{k})\, a^{+}_{\vec{k},\lambda} \right]$$

↑ makes sum over $\vec{k}$ finite

$$\langle z | \vec{E}_{fr}(0,\vec{x}_0) | z \rangle = i \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar}{V}} \sqrt{\omega_k}\ \vec{\varepsilon}_{\vec{k},\lambda} \left[\tilde{f}(\vec{k})\, z_{\vec{k},\lambda} - \tilde{f}^{*}(\vec{k})\, z^{*}_{\vec{k},\lambda} \right] \longleftarrow \text{finite + grows with } |z_{\vec{k},\lambda}|$$

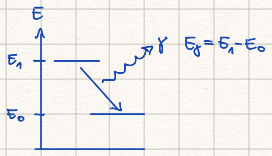
$$\langle z | \vec{E}^2_{fr}(0,x_0) | z \rangle = \langle z | \vec{E}_{fr}(0,x_0) | z \rangle^2 + \frac{2\pi\hbar}{V} \sum_{\vec{k},\lambda} |\tilde{f}(\vec{k})|^2 \omega_{\vec{k}} \longleftarrow \text{finite, does not grow with } |z_{\vec{k},\lambda}|$$

✓ o.k.

3.6. Interaction of radiation with matter

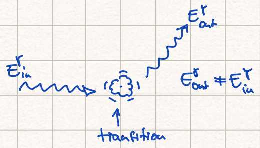
(Gauß units !)

→ We consider the interaction of a single electron bound in an atom with the radiation field.



We may consider :

- Spontaneous emission of photons from an excited state
- Induced transition : inelastic scattering of a photon off the atom



→ Radiation – matter coupling

$$H_{tot} = H_{atom} + H_{EM} + H_{inter}$$

We use the classical minimal coupling of charged particles to the e.m. field :

$$\frac{\vec{p}^2}{2\mu} \longrightarrow \frac{1}{2\mu}(\vec{p} - q\vec{A})^2 + q\Phi \qquad q_{e^-} = -e$$

↳ $H_{atom} = \dfrac{\vec{p}^2}{2\mu} - \dfrac{e^2}{|\vec{x}|}$ ⟵ bound state problem already solved (QM I), $m_{p^+} = \infty$

$H_{EM} = \sum_{\lambda} \int d^3\vec{p}\ E_{\vec{p}}\ a^{+}(\vec{p},\lambda)\, a(\vec{p},\lambda)$ ⟵ free photon problem solved

$$H_{inter}(t) = \frac{e}{2\mu}\left(\underset{\substack{\uparrow\\\text{mom. operator}\\\text{for electron}}}{\vec{P}} \cdot \vec{A}(t,\underset{\substack{\nwarrow\\\text{location operator}\\\text{for electron}}}{\vec{X}}) + \vec{A}(t,\vec{X})\vec{P} \right) + \frac{e^2}{2\mu}\vec{A}^2(t,\vec{X}) \longleftarrow \text{interaction Hamiltonian to be treated perturbatively}$$

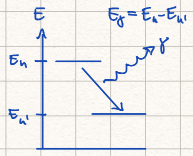
↳ Description : – Photon field already falls in Heisenberg picture
– Electron : we take the interaction picture

05/13/2018

Spontaneous Emission – Decay rate $\Gamma = \frac{1}{\tau}$ of an excited state

($\hbar = c = 1$, blue Gauß convention!)

We proceed exactly as we have learned:



incoming state: $|in\rangle = |0_\gamma\rangle \otimes |\psi_n\rangle$ ⟵ excited state ⊕ 0 photons

outgoing state: $|out\rangle = |\vec{p},\lambda\rangle \otimes |\psi_{n'}\rangle$ ⟵ lower exc. state ⊕ 1 photon

Interaction picture perturbative Hamiltonian (Photon field is in the Heisenberg picture!)

$$H^I_{inter}(t) = U_{atom}(t_0,t)\, H_{inter}(t)\, U_{atom}(t,t_0) \qquad \text{We pick } t_0 = 0$$

$$= e^{iH_{atom}t}\, H_{inter}(t)\, e^{-iH_{atom}t}$$

S-matrix element: $S_{fi}(T) = \langle out | T \exp\left[-i\int_{-T/2}^{T/2} dt\, H^I_{inter}(t)\right] | in\rangle$ ⟵ We keep T finite for now and let T → ∞ at the end.

↳ The dominant contribution arises from 1 insertion of H_{inter}: one $\vec{A}$ needed.

$$S_{fi} = -i\int_{-T/2}^{T/2} dt\, \langle \vec{p},\lambda; \psi_{n'}| e^{iH_{atom}t} H_{inter}(t)\, e^{-iH_{atom}t} |0_\gamma; \psi_n\rangle \qquad \mathbb{1} = \int d^3\vec{x}\, |\vec{x}\rangle\langle\vec{x}|$$

acts on everything to the right

$$= -i\int_{-T/2}^{T/2} dt \int d^3\vec{x}\, \langle\vec{p},\lambda| e^{i(E_{n'}-E_n)t}\, \psi^*_{n'}(\vec{x}) \left(-i\frac{e}{2\mu}\right)\left(\overset{\downarrow}{\vec{\nabla}}\cdot\vec{A}(t,\vec{x}) + \vec{A}(t,\vec{x})\cdot\vec{\nabla}\right)\psi_n(\vec{x})\, |0_\gamma\rangle \qquad \overset{\rightarrow}{\vec{\nabla}}A(t,\vec{x}) = 0$$

$$= -i\int_{-T/2}^{T/2} dt \int d^3\vec{x}\, \langle\vec{p},\lambda| e^{i(E_{n'}-E_n)t}\, \psi^*_{n'}(\vec{x}) \left(-i\frac{e}{\mu}\right)\vec{A}(t,\vec{x})\,\vec{\nabla}\psi_n(\vec{x})\, |0_\gamma\rangle$$

$$= -\frac{e}{\mu}\int_{-T/2}^{T/2} dt\, e^{i(E_{n'}-E_n)t} \int d^3\vec{x}\left(\psi^*_{n'}(\vec{x})\,\overset{\rightarrow}{\vec{\nabla}}\psi_n(\vec{x})\right)\langle\vec{p},\lambda|\vec{A}(t,\vec{x})|0_\gamma\rangle$$

only this term survives

$$\langle\vec{p},\lambda|\vec{A}(t,\vec{x})|0_\gamma\rangle = \sum_{\lambda'}\int d^3\vec{q}\,\frac{1}{2\pi}\frac{1}{\sqrt{E_{\vec{q}}}}\,\langle\vec{p},\lambda|\Big(\cancel{e^{-iq.x}\,\vec{\varepsilon}(\vec{q},\lambda')\,a(\vec{q},\lambda')} + e^{iq.x}\,\vec{\varepsilon}^*(\vec{q},\lambda')\underbrace{a^+(\vec{q},\lambda')\Big)|0_\gamma\rangle}_{|\vec{q},\lambda'\rangle}$$

$$= \sum_{\lambda'}\int d^3\vec{q}\,\frac{1}{2\pi}\frac{1}{\sqrt{E_{\vec{q}}}}\, e^{iE_{\vec{q}}t}\, e^{-i\vec{q}\vec{x}}\,\vec{\varepsilon}^*(\vec{q},\lambda')\,\overbrace{\langle\vec{p},\lambda|\vec{q},\lambda'\rangle}^{\delta^{(3)}(\vec{p}-\vec{q})\,\delta_{\lambda\lambda'}}$$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{E_{\vec{p}}}}\, e^{iE_{\vec{p}}t}\, e^{-i\vec{p}\vec{x}}\,\vec{\varepsilon}^*(\vec{p},\lambda)$$

$$S_{fi} = -i\frac{e}{2\pi\mu}\frac{1}{\sqrt{E_{\vec{p}}}}\,\vec{\varepsilon}^*(\vec{p},\lambda)\underbrace{\int_{-T/2}^{T/2} dt\, e^{i(E_{n'}+E_{\vec{p}}-E_n)t}}_{\xrightarrow{T\to\infty}\ 2\pi\,\delta(E_{n'}+E_{\vec{p}}-E_n)}\ \underbrace{\int d^3\vec{x}\, e^{-i\vec{p}\vec{x}}\left(\psi^*_{n'}(\vec{x})(-i)\vec{\nabla}\psi_n(\vec{x})\right)}_{=\, i\vec{d}_{n'n}(\vec{p})\ =\ ?}$$

energy conservation

↳ H-atom: $|\psi_{nlm}\rangle$ has energy $E_n = -\frac{m_e\alpha^2}{2n^2}$, $\alpha \approx \frac{1}{137} \ll 1$

⟹ Energy of the emitted photon is $E_\gamma = |\vec{p}| = E_n - E_{n'} \lesssim m_e\alpha^2$

But $\langle|\vec{x}|\rangle_{nlm} = \int d^3\vec{x}\, |\vec{x}|\, |\langle\vec{x}|\psi_{nlm}\rangle|^2 \sim (n\, m_e\alpha)^{-1} \ll (m_e\alpha^2)^{-1}$

Also true for most other atoms!

→ $\langle\vec{p}\vec{x}\rangle \sim \alpha$

⟹ We can expand $e^{-i\vec{p}\vec{x}} = 1 - i\vec{p}\vec{x} - \frac{1}{2}(\vec{p}\vec{x})^2 + \dots$

"Dipole" transitions

"Quadrupole" transitions

Implies that E_γ and $\vec{p}_\gamma$ are so small that the atom remains at rest.

05/13/2018 ②

e.g. dipole transitions:

$$\vec{d}_{n'n}(\vec{p}) = -i\int d^3\vec{x}\; e^{-i\vec{p}\vec{x}} \left(\psi^*_{n'}(\vec{x})\vec{\nabla}\psi_n(\vec{x})\right) \approx -i\int d^3\vec{x}\; \left(\psi^*_{n'}(\vec{x})\vec{\nabla}\psi_n(\vec{x})\right) = \langle \psi_{n'}|\vec{P}|\psi_n\rangle = \vec{d}_{n'n}(0)$$

↳ Heisenberg equation : $\vec{P} = \mu\dot{\vec{X}} = i\mu[H_{atom}, \vec{X}]$

dipole matrix element

$$\vec{d}_{n'n}(0) = i\mu\,\langle n'|[H_{atom}, \vec{X}]|n\rangle = i\mu\,(E_{n'} - E_n)\,\langle n'|\vec{X}|n\rangle$$

⟶ For the H-atom one finds that $\langle n'\ell' m'|\vec{X}|n\ell m\rangle \neq 0$ only for $m' = m$ or $m' = m \pm 1$ } Consequence of angular momentum conservation (photon has spin 1)

↳ In general: $\vec{\epsilon}^*(\vec{p},\lambda)\cdot\vec{d}_{n'n}(\vec{p}) \neq 0$ only if allowed by angular momentum and momentum conservation.

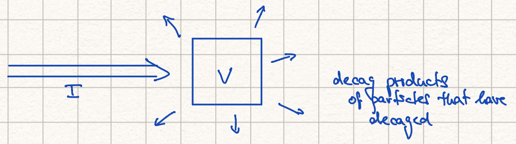
⟶ Note: Usual momentum conservation can be correctly accounted for here! Here $m_{atom} \to \infty$, so that the atom can absorb any amount of momentum

Decay Rate Formula

(Γ: "decay rate" or "width")

⟶ Decay rate $= \Gamma =$ Probability that the particle decays (in unit time) per unit time

↳ The "unit time" is much larger than the average life time of the particle and we think of a potential experimental setup as a stream of incoming particles (= beam with some particle intensity I) such that the number of particles that are entering volume V in time interval T is constant in time. This number of initial particles grows linearly with T.



The decay rate is the fraction of particles that have decayed in the time interval T. This fraction is not depending on the size of the time interval T (as long as T ≫ average life time τ)

$$\Rightarrow \boxed{\Gamma = \lim_{T\to\infty}\frac{1}{T}\sum_f |S_{fi}(T)|^2, \quad \sum_f : \text{sum over all possible final state} \qquad S_{fi}(T) = \langle out|T\exp\left[-i\int_{-T/2}^{T/2}dt\, H^I_{inter}(t)\right]|in\rangle}$$

⟵ General formula for any kind of decay!

$$S_{fi}(T) = -i\frac{e}{2\pi\mu}\frac{1}{\sqrt{E_{\vec{p}}}}\,\vec{\epsilon}^*(\vec{p},\lambda)\cdot\vec{d}(\vec{p},\lambda)\int_{-T/2}^{T/2}dt\, e^{i(E_{n'}+E_{\vec{p}}-E_n)t}$$

$$\lim_{T\to\infty}\frac{1}{T}\left|\int_{-T/2}^{T/2}dt\, e^{i(E_{n'}+E_{\vec{p}}-E_n)t}\right|^2 = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt\, e^{i(E_{n'}+E_{\vec{p}}-E_n)t}\int_{-T/2}^{T/2}dt'\, e^{-i(E_{n'}+E_{\vec{p}}-E_n)t'}$$

$$= \lim_{T\to\infty}\frac{1}{T}(2\pi)\delta(E_{n'}+E_{\vec{p}}-E_n)\int_{-T/2}^{T/2}dt\,\underbrace{e^{i(E_{n'}+E_{\vec{p}}-E_n)t}}_{=1} = (2\pi)\delta(E_{n'}+E_{\vec{p}}-E_n)$$

05/13/2018 ③

↳ Final formula for $\psi_n \longrightarrow \psi_{n'} + \gamma$ decay rate

$$\Gamma(\psi_n \longrightarrow \psi_{n'} + \gamma) = \frac{e^2}{2\pi\mu^2} \sum_\lambda \int \frac{d^3\vec{p}}{E_{\vec{p}}} \left| \vec{\varepsilon}^{\,*}(\vec{p},\lambda) \cdot \vec{d}_{n'n}(\vec{p}) \right|^2 \overbrace{\delta(E_{n'} + E_{\vec{p}} - E_n)}^{\text{fixes } |\vec{p}| = E_n - E_{n'}}$$

↳ We can now do a few more calculations without assuming any specific expression for $\vec{d}_{n'n}(\vec{p})$

$$\int \frac{d^3\vec{p}}{E_{\vec{p}}} \delta(E_{n'} + E_{\vec{p}} - E_n) \overset{E_p = |\vec{p}|}{=} (E_n - E_{n'}) \int d\Omega$$

$$\sum_\lambda \left| \vec{\varepsilon}^{\,*}(\vec{p},\lambda) \cdot \vec{d}_{n'n} \right|^2 = d^i_{n'n} \, d^{*j}_{n'n} \underbrace{\sum_\lambda \varepsilon^{*i}(\vec{p},\lambda)\, \varepsilon^j(\vec{p},\lambda)}$$

completeness relation in photon polarization space } = projector on plane ⊥ to $\vec{p}_\gamma$

$$= \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2} = \Lambda^{ij}(\vec{p})$$

↳ $\Lambda^{ij}(\vec{p}) \cdot p^j = 0$

$\Lambda^{ij}(\vec{p}) \cdot n^j = n^j$ if $\vec{n}\vec{p} = 0$

$$= d^i_{n'n} \, d^{*j}_{n'n} \left(\delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2} \right)$$

all angles of $\vec{p}$
↓

$$\int d\Omega \left(\delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2} \right) \longrightarrow \text{We use the standard trick!}$$

↳ After the integration the result cannot depend any more on $\vec{p}$. The integral is obviously invariant under arbitrary rotations
⟹ It is proportional to δ^{ij}

↳ $A^{ij} = \int d\Omega \left(\delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2} \right) = a\,\delta^{ij}$ $\quad | \times \delta^{ij}$ on both sides

$$\Longleftrightarrow \quad 3a = \int d\Omega\,(3-1) \quad \Longrightarrow \quad a = \frac{8\pi}{3}$$

↳ $$\Gamma(\psi_n \longrightarrow \psi_{n'} + \gamma) = \frac{e^2}{2\pi\mu^2}(E_n - E_{n'}) \cdot \frac{8\pi}{3} |\vec{d}_{n'n}(\vec{p})|^2$$

$$= e^2 \left(\frac{4(E_n - E_{n'})}{3\mu^2} \right) |\vec{d}_{n'n}(\vec{p})|^2, \quad \alpha_{em} = e^2$$

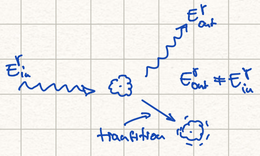
HL units: additional factor $\frac{1}{4\pi}$ $\quad \alpha_{em} = \frac{e^2}{4\pi}$

$$= \alpha_{em} \left(\frac{4(E_n - E_{n'})}{3\mu^2} \right) |\vec{d}_{n'n}(\vec{p})|^2$$

← Same in HL units!

Inelastic scattering of a photon off an atom

($\hbar = c = 1$)



We proceed exactly as we have learned:

incoming state: $|in\rangle = |p,\lambda\rangle \otimes |\psi_n\rangle$ ⟵ State n ⊕ 1 photon

outgoing state: $|out\rangle = |p',\lambda'\rangle \otimes |\psi_{n'}\rangle$ ⟵ State n' ⊕ 1 photon

→ We only look at the S-matrix element since we have already derived the cross section formula in Chapter 2.

We assume that the atom remains stationary: $m_{atom} \longrightarrow \infty$, so the atom can absorb any amount of momentum.

05/13/2018 ④

S-matrix element: $S_{fi}(T) = \langle out | T \exp\left[-i\int_{-\infty}^{+\infty} dt\, H^{I}_{inter}(t)\right] | in \rangle$ ← We can let go $T \to \infty$ for scattering → Chap. 2

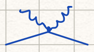
↳ We need two insertions of $\vec{A}$ to get the dominant contribution:

$\mathbb{1} = \int d^3\vec{x}'\, |\vec{x}'\rangle\langle\vec{x}'|$

$$S_{fi} = -i\int_{-\infty}^{+\infty} dt\, \langle \vec{p}',\lambda'; \psi_{n'}| e^{iH_{atom}t} H_{intr}(t)\, e^{-iH_{atom}t} |\vec{p},\lambda;\psi_n\rangle$$

$$+ (-i)^2 \int_{-\infty}^{+\infty} dt \int_{-\infty}^{t} dt'\, \langle \vec{p}',\lambda'; \psi_{n'}| e^{iH_{atom}t} H_{intr}(t)\, e^{-iH_{atom}t} \overset{\mathbb{1}}{\searrow} e^{iH_{atom}t'} H_{intr}(t')\, e^{-iH_{atom}t'} |\vec{p},\lambda;\psi_n\rangle$$

$$\simeq -i\int_{-\infty}^{+\infty} dt \int d^3\vec{x}\, \langle \vec{p}',\lambda'| e^{i(E_{n'}-E_n)t} \left(\frac{e^2}{2\mu}\right) \psi^*_{n'}(\vec{x})\, \vec{A}^2(t,\vec{x})\, \psi_n(\vec{x})\, |\vec{p},\lambda\rangle \leftarrow$$


local interaction

$$+ (-i)^2 \int_{-\infty}^{-\infty} dt \int_{-\infty}^{t} dt' \int d^3\vec{x}\, d^3\vec{x}'\, \langle \vec{p}',\lambda'| \psi^*_{n'}(\vec{x})\, e^{iE_{n'}t} \left(-i\frac{e}{2\mu}\right) \vec{A}(t,\vec{x})\, \vec{\nabla}_x\, G_{atom}(\vec{x},\vec{x}') \leftarrow$$

$$\times \left(-i\frac{e}{2\mu}\right) \vec{A}(t,\vec{x}')\, \vec{\nabla}_{x'}\, e^{-iE_nt'} \psi_n(\vec{x}')\, |\vec{p},\lambda\rangle$$


non-local interaction

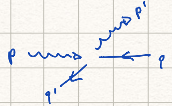
Green's function of the electron bound in the atom:

$$(i\partial_t - H_{atom})\, G_{atom}(x,x') = i\,\delta^{(4)}(x-x') = i\,\delta(t-t')\,\delta^{(3)}(\vec{x}-\vec{x}')$$

charge q, free particle

Scattering of a photon and a (non-relativistic and spinless) charged particle

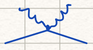
$(\hbar = c = 1)$



incoming state: $|in\rangle = |(\vec{p},\lambda)_\gamma, \vec{q}\rangle$

outgoing state: $|out\rangle = |(\vec{p}',\lambda')_\gamma, \vec{q}'\rangle$ $\qquad E_{\vec{q}} = \frac{\vec{q}^2}{2\mu}$

$$↳\ S_{fi} = -i\int_{-\infty}^{+\infty} dt \int d^3\vec{x}\, \langle \vec{p}',\lambda'| e^{i(E_{q'}-E_q)t} \frac{e^{-i\vec{q}'\vec{x}}}{(2\pi)^{3/2}} \left(\frac{q^2}{2\mu}\right) \vec{A}^2(t,\vec{x}) \frac{e^{i\vec{q}\vec{x}}}{(2\pi)^{3/2}} |\vec{p},\lambda\rangle \leftarrow$$


local interaction

$$+ (-i)^2 \int_{-\infty}^{-\infty} dt \int_{-\infty}^{t} dt' \int d^3\vec{x}\, d^3\vec{x}'\, \langle \vec{p}',\lambda'| \frac{e^{-i\vec{q}'\vec{x}}}{(2\pi)^{3/2}} e^{iE_{q'}t} \left(i\frac{q}{2\mu}\right) \vec{A}(t,\vec{x})\, \vec{\nabla}_x\, G_0(t,\vec{x};t',\vec{x}') \leftarrow$$

$$\times \left(i\frac{q}{2\mu}\right) \vec{A}(t,\vec{x}')\, \vec{\nabla}_{x'}\, e^{-iE_qt'} \frac{e^{i\vec{q}\vec{x}'}}{(2\pi)^{3/2}} |\vec{p},\lambda\rangle$$


non-local interaction

$$= -i\left(\frac{q^2}{2\mu}\right)\frac{1}{(2\pi)^3}\int d^4x\, e^{-i(q-q')x} \langle \vec{p}',\lambda'| \vec{A}^2(x) |\vec{p},\lambda\rangle$$

$$+ \left(\frac{q^2}{4\mu^2}\right)\frac{1}{(2\pi)^3}\int d^4x \int^{t'<t} d^4x'\, e^{iq'x} \langle \vec{p}',\lambda'| A^i(x)\, \nabla^i_{\vec{x}}\, G_0(t,\vec{x};t',\vec{x}')\, A^j(x')\, \nabla^j_{\vec{x}'}\, e^{-iqx'} |\vec{p},\lambda\rangle$$

We have to consider 4 operators:

$$\langle \vec{p}',\lambda'| \mathcal{O} |\vec{p},\lambda\rangle = \langle 0| a(\vec{p}',\lambda')\, \mathcal{O}\, a^\dagger(\vec{p},\lambda) |0\rangle$$

$$\text{with } \mathcal{O} = a(\vec{k},\alpha)\,a(\vec{k}',\alpha'),\ \overbrace{a^\dagger(\vec{k},\alpha)\,a(\vec{k}',\alpha'),\ a(\,)\,a^\dagger(\,)}^{\text{only these can give } \neq 0 \text{ result}},\ a^\dagger(\,)\,a^\dagger(\,)$$

↳ Only matrix elements with # of a's = # of a⁺'s give non-zero results

"nonzero contractions"

$$\langle \vec p',\lambda' | a(\vec k,\alpha)\, a^+(\vec k',\alpha') | \vec p,\lambda\rangle = \langle 0| a_{p'}\, a_k\, a^+_{k'}\, a^+_p |0\rangle = \langle 0| a_{p'}\, a^+_{k'}\, a_k\, a^+_p |0\rangle + \delta_{kk'}\langle 0| a_{p'} a^+_p |0\rangle$$

$$= \delta_{kp}\langle 0| a_{p'}\, a_{k'} |0\rangle + \delta_{kk'}\langle 0| a_{p'} a^+_p |0\rangle = \delta^{(3)}(\vec k-\vec p)\,\delta_{\alpha\lambda}\,\delta^{(3)}(k'-p')\,\delta_{\alpha'\lambda'} + \underbrace{\delta^{(3)}(k-k')\,\delta_{\alpha\alpha'}\,\delta^{(3)}(p-p')\,\delta_{\lambda\lambda'}}$$

"non-zero contractions"

$$\langle \vec p',\lambda' | a^+(\vec k,\alpha)\, a(\vec k',\alpha') | \vec p,\lambda\rangle = \langle 0| a_{p'}\, a^+_k\, a_{k'}\, a^+_p |0\rangle = \delta^{(3)}(k'-p)\,\delta_{\alpha'\lambda}\,\delta^{(3)}(k-p')\,\delta_{\alpha\lambda'}$$

photon not scattered!
⟹ neglect!

↳ $\langle \vec p',\lambda' | A^i(t,\vec x)\, A^j(t',\vec x') | \vec p,\lambda\rangle$

$$= \sum_{\alpha\alpha'} \int \frac{d^3k}{(2\pi)|\vec k|^{1/2}} \frac{d^3k'}{(2\pi)|\vec k'|^{1/2}} \langle \vec p',\lambda'| \left(e^{-ikx}\epsilon^i(\vec k,\alpha)\, a(\vec k,\alpha) + e^{ikx}\epsilon^{*i}(\vec k,\alpha)\, a^+(\vec k,\alpha)\right)\left(e^{-ik'x'}\epsilon^j(\vec k',\alpha')\, a(\vec k',\alpha') + e^{ik'x'}\epsilon^{*j}(\vec k',\alpha')\, a^+(\vec k',\alpha')\right)|\vec p,\lambda\rangle$$

$$= \sum_{\alpha\alpha'} \int \frac{d^3k}{(2\pi)|\vec k|^{1/2}} \frac{d^3k'}{(2\pi)|\vec k'|^{1/2}} \Big(\epsilon^i(\vec k,\alpha)\,\epsilon^{*j}(\vec k',\alpha')\, e^{-ikx}\, e^{ik'x'}\, \delta^{(3)}(\vec k-\vec p)\,\delta_{\alpha\lambda}\,\delta^{(3)}(\vec k'-\vec p')\,\delta_{\alpha'\lambda'}$$

$$+ \epsilon^{*i}(\vec k,\alpha)\,\epsilon^j(\vec k',\alpha')\, e^{ikx}\, e^{-ik'x'}\, \delta^{(3)}(\vec k-\vec p')\,\delta_{\alpha\lambda'}\,\delta^{(3)}(\vec k'-\vec p)\,\delta_{\alpha'\lambda} \Big)$$

$$= \frac{1}{(2\pi)^2}\frac{1}{|\vec p|^{1/2}}\frac{1}{|\vec p'|^{1/2}} \left(\epsilon^i(\vec p,\lambda)\,\epsilon^{*j}(\vec p',\lambda')\, e^{-ipx}\, e^{ip'x'} + \epsilon^{*i}(\vec p',\lambda')\,\epsilon^j(\vec p,\lambda)\, e^{ip'x}\, e^{-ipx'} \right)$$

↳ $-i\left(\frac{q^2}{2\mu}\right)\frac{1}{(2\pi)^3}\int d^4x\; e^{-i(q-q')x}\langle \vec p',\lambda'|\vec A^2(x)|\vec p,\lambda\rangle$

$$= -i\left(\frac{q^2}{2\mu}\right)\frac{2}{(2\pi)^5}\left(\vec\epsilon^{\,*}(\vec p',\lambda')\vec\epsilon(\vec p,\lambda)\right)\frac{1}{|\vec p|^{1/2}}\frac{1}{|\vec p'|^{1/2}}\int d^4x\; e^{-i(q+p-q'-p')}$$

$$= -i\left(\frac{e^2}{2\mu}\right)\frac{2}{(2\pi)^2}\left(\vec\epsilon^{\,*}(\vec p',\lambda')\vec\epsilon(\vec p,\lambda)\right)\frac{1}{|\vec p|}\;(2\pi)\;\delta(E_{q'}+|\vec p'|-E_q-|\vec p|)\,\delta^{(3)}(\vec q'+\vec p'-\vec q-\vec p)$$

↳ $+\left(\frac{q^2}{4\mu^2}\right)\frac{1}{(2\pi)^3}\int d^4x \int^{t'<t} d^4x'\; e^{iq'x}\langle \vec p',\lambda'| A^i(x)\,\nabla^i_{\vec x}\, G_0(t,\vec x,t',\vec x')\, A^j(x')\,\nabla^j_{\vec x'}\, e^{-iqx'}|\vec p,\lambda\rangle$

$$= +\left(\frac{q^2}{4\mu^2}\right)\frac{1}{(2\pi)^5}\frac{1}{|\vec p|^{1/2}}\frac{1}{|\vec p'|^{1/2}}\int d^4x\int^{t'<t} d^4x' \Big[\epsilon^i(\vec p,\lambda)\,\epsilon^{*j}(\vec p',\lambda')\, e^{-i(p-q')x}\,\nabla^i_{\vec x}\, G_0(t,\vec x;t',\vec x')\,(iq^j)\, e^{-i(q-p')x'}$$

$$+ \epsilon^{*i}(\vec p',\lambda')\,\epsilon^j(\vec p,\lambda)\, e^{+i(p'+q')}\underbrace{\nabla^i_{\vec x}\, G_0(t,\vec x;t',\vec x')}_{= \int \frac{d^4k}{(2\pi)^4}(ik^i)\, e^{-ik(x-x')}\frac{i}{k_0-\frac{k^2}{2\mu}+i\epsilon}}(iq^j)\, e^{-i(q+p)x'}\Big]$$

$$= -\left(\frac{q^2}{4\mu^2}\right)\frac{1}{(2\pi)^5}\frac{1}{|\vec p|^{1/2}}\frac{1}{|\vec p'|^{1/2}}\int\frac{d^4k}{(2\pi)^4}\int d^4x\int^{t'<t} d^4x'\Big[(\vec k\,\vec\epsilon(\vec p,\lambda))(\vec q\cdot\vec\epsilon(\vec p',\lambda'))^*\, e^{-i(p-q'+k)x}\frac{i}{k_0-\frac{k^2}{2\mu}+i\epsilon}\, e^{-i(q-p'-k)x'}$$

$$+ (\vec q\,\vec\epsilon(\vec p,\lambda))(\vec k\,\vec\epsilon(\vec p',\lambda'))^*\, e^{i(p'+q'-k)x}\frac{i}{k_0-\frac{k^2}{2\mu}+i\epsilon}\, e^{-i(q+p-k)x'}\Big]$$

$$\theta(t-t') = \frac{1}{2\pi i}\int s_0\,\frac{e^{i(t-t')s_0}}{s_0-i\epsilon}$$

$p = |\vec{p}|\,,\ p' = |\vec{p}'|$ 05|13|2018 ⑥

$$= -\left(\frac{e^2}{4\mu^2}\right)\frac{1}{(2\pi)^6}\,\frac{1}{|\vec{p}|^{1/2}}\,\frac{1}{|\vec{p}'|^{1/2}}\int\frac{d^4k}{(2\pi)^4}\int\frac{ds_0}{s_0-i\epsilon}\,(2\pi)^8\left[\frac{(\vec{k}\,\vec{\epsilon}(\vec{p},\lambda))(\vec{q}\cdot\vec{\epsilon}(\vec{p}',\lambda'))^*}{k_0-\frac{\vec{k}^2}{2\mu}+i\epsilon}\,\delta(p-E_{q'}+k_0-s_0)\,\delta^{(3)}(\vec{p}-\vec{q}'+\vec{k})\,\delta(E_q-p'-k_0+s_0)\,\delta^{(3)}(\vec{q}-\vec{p}'-\vec{k})\right.$$

$$\left.+\frac{(\vec{q}\,\vec{\epsilon}(\vec{p},\lambda))(\vec{k}\,\vec{\epsilon}(\vec{p}',\lambda'))^*}{k_0-\frac{\vec{k}^2}{2\mu}+i\epsilon}\,\delta(p'+E_{q'}-k_0+s_0)\,\delta^{(3)}(\vec{p}+\vec{q}'-\vec{k})\,\delta(E_q+p-k_0+s_0)\,\delta^{(3)}(\vec{q}+\vec{p}-\vec{k})\right]$$

$$= -\left(\frac{e^2}{4\mu^2}\right)\frac{1}{(2\pi)^6}\,\frac{1}{|\vec{p}|^{1/2}}\,\frac{1}{|\vec{p}'|^{1/2}}\int\frac{ds_0}{s_0-i\epsilon}\,(2\pi)^4\left[((\vec{q}-\vec{p}')\vec{\epsilon}(\vec{p},\lambda))(\vec{q}\cdot\vec{\epsilon}(\vec{p}',\lambda'))^*\frac{1}{E_q-p'+s_0-(\vec{q}-\vec{p}')^2/2\mu+i\epsilon}\right.$$

implies $|\vec{p}'| = |\vec{p}|$

$$\left.+(\vec{q}\,\vec{\epsilon}(\vec{p},\lambda))((\vec{q}+\vec{p})\vec{\epsilon}(\vec{p}',\lambda'))^*\frac{1}{E_q+p+s_0-(\vec{q}+\vec{p})^2/2\mu+i\epsilon}\right]\delta(E_{q'}+|\vec{p}'|-E_q-|\vec{p}|)\,\delta^{(3)}(\vec{q}'+\vec{p}'-\vec{q}-\vec{p})$$

$$= -\left(\frac{e^2}{4\mu^2}\right)\frac{1}{(2\pi)^2}\,\frac{1}{|\vec{p}|}\int\frac{ds_0}{(2\pi)}\,\frac{1}{s_0-i\epsilon}\left[((\vec{q}-\vec{p}')\vec{\epsilon}(\vec{p},\lambda))(\vec{q}\cdot\vec{\epsilon}(\vec{p}',\lambda'))^*\frac{1}{E_q-p'+s_0-(\vec{q}-\vec{p}')^2/2\mu+i\epsilon}\right.$$

residue at $s=+i\epsilon$

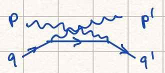
$$\left.+(\vec{q}\,\vec{\epsilon}(\vec{p},\lambda))((\vec{q}+\vec{p})\vec{\epsilon}(\vec{p}',\lambda'))^*\frac{1}{E_q+p+s_0-(\vec{q}+\vec{p})^2/2\mu+i\epsilon}\right](2\pi)\,\delta(E_{q'}+|\vec{p}'|-E_q-|\vec{p}|)\,\delta^{(3)}(\vec{q}'+\vec{p}'-\vec{q}-\vec{p})$$

$$= -i\left(\frac{e^2}{4\mu^2}\right)\frac{1}{(2\pi)^2}\,\frac{1}{|\vec{p}|}\left[((\vec{q}-\vec{p}')\vec{\epsilon}(\vec{p},\lambda))(\vec{q}\cdot\vec{\epsilon}(\vec{p}',\lambda'))^*\frac{1}{(E_q-p')-(\vec{q}-\vec{p}')^2/2\mu+i\epsilon}\right.\quad ①$$

$$\left.+(\vec{q}\,\vec{\epsilon}(\vec{p},\lambda))((\vec{q}+\vec{p})\vec{\epsilon}(\vec{p}',\lambda'))^*\frac{1}{(E_q+p)-(\vec{q}+\vec{p})^2/2\mu+i\epsilon}\right](2\pi)\,\delta(E_{q'}+|\vec{p}'|-E_q-|\vec{p}|)\,\delta^{(3)}(\vec{q}'+\vec{p}'-\vec{q}-\vec{p})$$

②

① :



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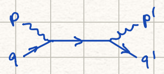

2. In the non-relativistic framework the whole calculation gets quite ugly quickly due to the time-ordering (→ s_0 integration).

In the relativistic calculation the additional antiparticle contributions lead to a much nicer result.