Chapter 2: Scattering Theory

2.1. Basics

Scattering experiments: One of the most important types of experiment to learn about the inner structure of matter and materials.

- e.g. crystal lattice, atomic structure, nuclear structure
- Structure too small modern high-energy particle physics to study otherwise.

- e.g. fixed-target experiments:

  \[ \text{beam of incoming particles} \to \text{scattered particles} \]

  \[ \text{accelerator} \to \text{target} \]

  \[ \text{beam axis} \]

- Two types of scattering processes:
  
  (a) Elastic scattering: particles in the incoming beam = detected particles
      \[ |E_i| = |E_f|, \quad \theta_i = \theta_f \]

  (b) Inelastic scattering: target get changed (excitation, resonance, destruction)
      incoming particle + detected particles
      \[ |E_i| \neq |E_f| \]

Big high energy collider experiments:

- Large Hadron Collider (LHC, CERN, 2008 - 2035)
  - Proton-proton (pp)

- International Linear Collider (ILC) (Japan, 2030 - ?): electron - positron (e^-e^+)


- Tevatron (Fermilab, Chicago, 1983 - 2011): proton - antiproton (p\bar{p})
Principles:
- Interaction of incoming particles very well known: \( E_i \)
- Intensity of incident beam very well known
- Measurement of scattered particles of distance \( r \) from target
- Ratio of signal strength \( S(E, x) \) with respect to beam exit: \( E_x \), particle kind, intensity

- Aim: Draw conclusions on:
  - Interactions, forces between particles
  - Inner structure of particles (translation: \( \lambda_s = \frac{2 \pi}{|\mathbf{q}|} \))

Scale hierarchies: → Basic properties of all scattering experiments

- \( R_I \): (unimportant) distance between target and detector \( (\sim 1 \text{ cm} - 1 \text{ m}) \)
- \( R_m \): Width of the beam, beam size \((\sim 10^{-4} \text{ m} - 10^{-6} \text{ m} \text{ for modern colliders})\)
- \( r_\alpha \): Size of target \( \propto \) range of interaction
  \[ (\sim 10^{-18} \text{ m} \times 10^{-16} \text{ m} \times 10^{-15} \text{ m} \ldots) \]
  \( \implies \) 
  - Potential \( V(r) \) full factor
    - (atomic radius) (nucleon radius) (interaction range)
    - then \( \frac{1}{r} \) for \( r \gg r_\alpha \)

Strong scale hierarchy: \( R_I \gg R_m \gg r_\alpha \) → approximations that are essential for the concept of scattering

- 1 GeV = \( 10^9 \text{ eV} \)
- 1 TeV = \( 10^12 \text{ eV} \)
- 1 TeV = \( 10^3 \text{ GeV} \)
- 1 TeV = \( 10^4 \text{ eV} \)
- Typical energies:
  - Mass of the electron: \( m_e = 0.51 \text{ MeV} \)
  - Universe: \( m_\text{universe} = 1 \text{ GV} \)

Higgs boson: \( m_{\text{Higgs}} = 125 \text{ GeV} \)
Center of small energy @ LHC: 13 TeV = \( 13000 \text{ GeV} \)

Strong Scale: \( 10^{13} \text{ GV} \)
- Energy where QCD becomes deconstructive
  - Seeds of quantum gravity

For most scattering experiments an actual scattering (\( \Theta = 0 \)) is quite rare.
Most of the time, incoming particles and the incoming particle emerge in the transmitted beam.
2.2. Classic Potential Scattering

**Assumption:** Potential is only \( r \)-dependent: \( V = V(r) \) ⇒ \( \Theta \)-independent (axially)

- \( b \): impact parameter
- \( \Theta \): scattering angle

\( b \) and \( \Theta \) have a **unique relation** between \( \Theta \) and \( b \)

\( \Rightarrow \) \( \Theta = \Theta(b) \)

**Example:** Scattering of a hard sphere

\[ V(r) = \begin{cases} 0, & r < R \\ \infty, & r \geq R \end{cases} \quad 0 \leq \Theta \leq \pi \]

Let here: \( b = R \cos \Theta \), \( Z \Theta = \pi \)

\[ \Rightarrow \sin \Theta = \frac{R - b}{Z} = \cos \left( \frac{\pi}{2} \right), \quad b = R \cos \left( \frac{\pi}{2} \right) \]

\[ \Rightarrow \Theta(b) = \begin{cases} \pi \cos \left( \frac{b}{R} \right), & b \leq R \\ 0, & b > R \end{cases} \]

**Differential Cross Section** \( \frac{d\sigma}{d\Omega} \)

\[ d\sigma = \frac{1}{\sin \Theta} d\Theta \]

For hard sphere: \( \frac{d\sigma}{d\Omega} = \frac{Z^2}{b^2} \sin^2 \left( \frac{\pi}{2} \right) \cos \left( \frac{\pi}{2} \right) \]

**All particles that enter through a segment \( d\sigma(\Theta, \Phi) \) are scattered into the solid angle segment \( d\Omega(\Theta, \Phi) \).** \( \Rightarrow \)

Let have: \( d\sigma = \frac{Z^2}{b^2} \sin \Phi d\Phi \)

\[ d\sigma = \frac{Z^2}{b^2} \sin \Phi d\Phi \]

**Scattering in direction \( \Theta, \Phi \)**

**Unit fragment**
Total Cross Section: \( \sigma = \int \frac{d\sigma}{d\Omega} \, d\Omega = \int_0^{2\pi} \int_0^\pi \sin\theta \, d\theta \, d\phi \). 

By definition, particles that are not scattered (i.e., scattering angle \( \theta = 0 \)) do not contribute to the total cross section.

For hard spheres: \( \sigma = \pi \cdot R^2 = \frac{\pi}{4} \cdot 4R^2 = 2\pi R^2 \) 

As expected.

Example: Rutherford Scattering

→ Scattering at a particle with charge \( q \) off a Coulomb potential of a stationary particle with charge \( q \) : \( V(r) = \frac{q^2}{4\pi \varepsilon_0 r^2} \)


\[ \sigma \text{ classical mechanics} : \sigma = \frac{\pi}{4}\left(\frac{Z^2}{4\pi \varepsilon_0} \right) \]

\[ \Rightarrow \sigma = 4\pi \left(\frac{Z^2}{4\pi \varepsilon_0} \right) \sin^2(\theta) \] 

→ Rutherford cross section 

\[ \sigma = 2\pi \left(\frac{Z^2}{4\pi \varepsilon_0} \right) \sin^2(\theta) \]

→ not finite!

→ for \( \theta \to 0 \) small angular divergence!

Divergence due to the fact that the Coulomb potential \( V(r) \to \frac{1}{r} \)

leads to “infinite range.” → Stronger possible range behavior if any potential related to a conservative force.

Note: This is not a physical problem because when \( R \to \infty \) the potential of the whole atom needs to be considered, which is electrically neutral. This is due to the fact that the whole electric charge inside a sphere with radius \( R \) is not considered.

So, Rutherford Scattering formula represents an idealization that is only valid for large scattering angles (which is nearly the correct limit to show behavior of the inner structure of the atom).
2.3. Elastic Scattering of a Spinless Particle off a Time-Independent Potential

In quantum theory, the classic notion of the cross section cannot be used because there is no direct correspondence between the area segment dS where incoming particle enters and the solid angle segment dΩ(θ, φ) in which the scattered particle is detected.

**Differential cross section:**

\[
\frac{d\sigma}{d\Omega}(\theta, \phi) = \frac{\# \text{ of particles that are scattered into the solid angle } d\Omega(\theta, \phi)}{\# \text{ of incoming particles per unit time}}
\]

\[
= \frac{\# \text{ of particles per unit time, that are scattered at } \theta, \phi}{\# \text{ of incoming particles per unit time per area } (L \to L')}
\]

**Total cross section:**

\[
\sigma_{\text{tot}} = \int d\Omega \frac{d\sigma}{d\Omega}(\theta, \phi) = \frac{\# \text{ of particles per unit time that are scattered at all}}{\# \text{ of incoming particles per unit time per area } (L \to L')}
\]

This is effective over the length representative to the incoming beam.

These definitions are more general and work in classic as well as in quantum physics.

They lead to some prescriptions used in Chap. 2.2.

**General Considerations:**

(Schrödinger picture)

\[
\hat{H} = \hat{T} + V = \frac{\hat{p}^2}{2m} + V(x) = \frac{\hat{p}^2}{2m} + V(x)
\]

\[
L \rightarrow \text{ Solution is of the form: packet kind } \quad \psi_c(x, t) = \int d^3k \psi^*(k) e^{i\vec{k} \cdot \vec{x}}
\]

\[
\psi_c(x, t) \rightarrow e^{i\hat{E}_c t} \psi_c(x)
\]

\[\psi(0) \rightarrow \text{packet} \quad \approx e^{i\hat{E}_c t} \psi_c(x)\]

We request on \(\psi_c(x, t)\) that it describes:

(a) incoming and transmitted wave (e.g. in \(z\)-direction)
(b) scattered radially outgoing wave

Let consider a stationary source flux of incoming particles \(\rightarrow \text{ one deep time-dependence } e^{iE t}\)

We found time calculation.

\[
\text{Ansatz: } \psi_c(x, t) = N \left( e^{i\gamma} + \psi_0^c(x) \right) e^{i\hat{E}_c t} \psi_c(x)
\]

Stationary Scattering Solution

\[
\text{incoming, outgoing, scattered, transmitted (at } \theta=0)\]
We can learn about the $r \to \infty$ behavior of $V_n^m(r)$ assuming that $V(r)$ has only a finite range. Near peak, $V(r \to \infty) = 0$ or $vV(r \to \infty) = 0$

For $r \to \infty$, (1.e. $r \gg a_m \gg a_s$):

$$v \left( \frac{\sigma^2}{2p} + \frac{\sigma^2}{2p} \right) c \left( \frac{s}{2p} - \frac{\sigma^2}{2p} \right) c \left( \frac{s}{2p} - \frac{\sigma^2}{2p} \right) = 0$$

$$\Rightarrow \left( \frac{\sigma^2}{2p} + \frac{\sigma^2}{2p} \right) c \left( \frac{s}{2p} - \frac{\sigma^2}{2p} \right) c = 0$$

Recall from Eq. 3.12.1:

$$\nabla^2 - \frac{1}{r^2} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \Rightarrow \text{Separation of r and } (\theta, \phi) \text{ dependence}$$

$$\nabla^2 \psi_n^m(r, \theta, \phi) = -\frac{\hbar^2}{2m_r} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \psi_n^m(r, \theta, \phi)$$

$$\Rightarrow \psi_n^m(r, \theta, \phi) = A_n \left(\frac{\epsilon}{2m_r} \right)^{1/2} e^{i k_r r}$$

$$\nabla^2 \psi_n^m(r, \theta, \phi) \Rightarrow \text{Linear wave number } k_r \text{ is analogy for } k \text{ in } \text{Eq. 3.12.1.}$$

$$\psi_n^m(0, \theta, \phi) = \left\{ \begin{array}{ll}
0 & \text{if } r < a_m \text{ (inside target)}
\end{array} \right\}$$

The scattering wave function has the general form:

$$\psi_n^m(r, \theta, \phi) \Rightarrow \left\{ \begin{array}{ll}
\frac{\epsilon}{2m_r} \frac{\partial \psi_n^m}{\partial r} & \text{if } r > a_m \text{ (outside target)}
\end{array} \right\}$$

Since detector always have a macroscopic distance to the target (i.e. $r \gg \text{good approximation}$), $f(\theta, \phi)$ contains all information that is experimentally accessible.

Derivation of the differential cross section:

We use the same boundary, $r \to \infty$ to argue that at the detector(s) only the radially outgoing wave is non-zero.

At the source, the beam can be classified such that only the incoming wave is non-zero.

Probability current of incoming wave:

$$j_i = \frac{1}{\epsilon} \left[ \psi_i^m(r^2 \epsilon)^{1/2} \frac{\partial \psi_i^m}{\partial r} - k^0 \psi_i^m (r^2 \epsilon)^{1/2} \right]$$

$$\Rightarrow j_i = \frac{1}{\epsilon} \left[ \psi_i^m (r^2 \epsilon)^{1/2} \frac{\partial \psi_i^m}{\partial r} \right]$$

At the detector, $r = 0$:

$$\text{Probability current of outgoing wave: } j_o = \frac{1}{\epsilon} \left[ \psi_o^m (r^2 \epsilon)^{1/2} \frac{\partial \psi_o^m}{\partial r} \right]$$

$$\text{Only radial direction}$$

Outputting flux entering solid angle $d\Omega (\theta, \phi)$:

$$j_o \cdot C_{\Omega} = 1 \frac{\epsilon}{\hbar} \frac{\partial \psi_o^m}{\partial r} (r^2 \epsilon)^{1/2} \frac{\partial \psi_o^m}{\partial r}$$
Incoming flux in beam direction: \( \mathcal{J}_\text{in} \cdot \mathbf{r} = \mathcal{J}_\text{in} \cdot \mathbf{r} \).

\[
\frac{d\sigma}{d\Omega}(\theta, \phi) = \left( \frac{E_0 E_\theta}{\mathcal{J}_\text{in} \cdot \mathbf{r}} \right)^2.
\]

Comments:

1. For \( \mathcal{J} = \mathcal{J}_0 \), we have \( \mathcal{J}_\text{in} \cdot \mathbf{r} \) (i.e. \( \phi \)-dependence).

2. Common unit for cross sections: \( 1 \text{ barn} = 10^{-24} \text{ m}^2 = (10^{-4} \text{ m})^2 = (0.02 \text{ GeV})^2 \)

3. Total cross section for pp collisions at the LHC: \( \sigma_{pp} = 6.8 \text{ nb} = (1.1 \times 10^{-25} \text{ m}^2) \times 10^{-24} \text{ m}^2 \) (const.)

4. The total cross section of the LHC contains many different reaction channels (elastic and inelastic). For one single channel the rule of thumb for the typical bin of the total cross section is \( \sigma_{\text{bin}} = \frac{\text{Interaction strength}}{\text{Energy of the collision}} \).

5. Alternative interpretation of \( \mathcal{J}_\text{int} \): Due to the scattering, the total cross section, that the target represents for the incoming beam can also be seen as the area for a reduction of the intensity of the incoming beam (i.e. \( \mathcal{J}_\text{out} < \mathcal{J}_\text{in} \)).

\[
\frac{\mathcal{J}_\text{out}}{\mathcal{J}_\text{in}} = \frac{I_\text{out}}{I_\text{in}} = \frac{\text{beam area}}{\text{beam area}} \times \text{fraction of beam area}
\]

6. The absolute particle density is defined as:

\( I \) of particles delivered by beam per unit time

This connection is the background of the fundamental concept of probability conservation (a particle-probability in internal quantum mechanics) in quantum theory, which is coming from the Heisenberg uncertainty.

\( \rightarrow \) see Chap. 2.4.
2.4 The Optical Theorem

For the determination of $\Delta S$ one can have neglected the interference between the incoming and the scattered waves, taking that $r \approx r_w$.

But the interference term plays an important role in the global probability flow balance of the scattering process:

\[
\Delta S = \text{Re} \left[ \Delta S \Omega \Omega^* \right] = \text{Im} \left( \Omega \right) \text{Im} \left( \Omega^* \right) + \text{Im} \left( \Omega \right) \text{Im} \left( \Omega^* \right) + \text{Re} \left( \Omega \right) \text{Re} \left( \Omega^* \right) + \text{Re} \left( \Omega \right) \text{Re} \left( \Omega^* \right)
\]

\[
= \frac{1}{\mu^2} \text{Re} \left[ \frac{\mu}{\kappa^2} \left( e^{i\kappa r} + e^{-i\kappa r} \right) \right] (\kappa \Delta \kappa + \kappa^2 \Delta \kappa + \frac{1}{\kappa^2} \Delta \kappa)
\]

\[
= \left( \frac{1}{1 + 1} \frac{1}{\mu^2} \right) \Delta S_k
\]

Due to our argument that $\Delta S \sim r^{-n}$, we conclude that the interference term is only relevant within the beam, i.e., for $\theta = 0$ (transmitted), $\theta = \pm \theta_0$ (reflected).

\[
\rightarrow \Delta S_k = 0
\]

2. Alternative definition of the total cross section:

Helmholtz: $\kappa \left( \Delta \kappa \right) = e^{i\kappa \Delta \kappa}$

\[
\rightarrow \text{continuity equation:} \quad \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right) + \nabla \cdot \nabla = 0, \quad \nabla \cdot \nabla = \left( \nabla^2 \right)
\]

We have $\int \Delta \kappa \text{exp}(i\Delta \kappa) = \text{const.} \quad \rightarrow \int \Delta \kappa \text{exp}(i\Delta \kappa) = \frac{\Delta \kappa}{\mu^2} \int \text{exp}(y) = 0$

\[
\rightarrow \int \Delta \kappa \text{exp}(i\Delta \kappa) = \lim_{\Delta \kappa \rightarrow 0} \Delta \kappa \text{exp}(i\Delta \kappa) = 0
\]

We calculate the three terms separately:

\[
\Delta \kappa = \Delta \kappa_0 + \Delta \kappa_1
\]

\[
\int \Delta \kappa_0 \text{exp}(i\Delta \kappa_0) = \int \Delta \kappa_1 \text{exp}(i\Delta \kappa_1) = 0
\]

\[
\int \Delta \kappa_0 \text{exp}(i\Delta \kappa_0) = 0
\]

\[
\int \Delta \kappa_1 \text{exp}(i\Delta \kappa_1) = 0
\]
\[ \text{Im} \left( \frac{f(E, \theta, \phi)}{E} \right) \]

\[ \text{For }\theta = \text{const}, \text{ Im} \left( \frac{f(E, \theta, \phi)}{E} \right) \text{ vanishes for } \phi \to \pm \pi. \]

**Comment:**

1. The optical theorem is also true if inelastic processes take place in the scattering ($\rightarrow$ Unitarity of Wave Equations becomes the fundamental property related to probability conservation.)

2. The optical theorem implies that $f(E, \theta, \phi)$ is an analytic function in the energy $E$.

   In physics, the analyticity of functions is related to their property that they can be unambiguously defined. Their unambiguous character is related to causality.

   For the scattering amplitude $f(E, \theta, \phi)$ causality can be expressed by the requirement that it has to describe an outgoing wave.

3. The optical theorem implies that $f(E, \theta, \phi)$ has a branch cut concerning its imaginary part along the real $E$-axis for all three energies when scattering on the plane.
2.5. The Green's Function Method

Advantages:
- Boundary conditions (e.g., causality) easy to implement
- Analytic functions
- Universal applicability:
  - Relativistic quantum field theory
  - Multi-particle systems
  - Quantum and classical systems
- Particularly suited for perturbation theory: "Hyman rules"

Schrödinger equation for the stationary scattering solution:
\[
\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) \psi(x) = E \psi(x)
\]
\[\Rightarrow D_E \psi_E(x) = \psi_E(x), \quad D_E = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - V(x), \quad f(x) = -V(x) \psi_E(x)
\]

General solution:
\[\psi(x) = \psi_E(x) + \psi_0(x) \quad a \text{ particular solution of } D_E \psi = 0
\]

Formal derivation of the particular solution:

1. We assume that there exists a fit \( G(x, x') \) ("Green's function") that satisfies
\[
D_E G_E(x, x') = \delta(x-x') \quad , \quad D_E = -\frac{\hbar^2}{2m} \left( \frac{d^2}{dx^2} + V(x) \right)
\]

Then \( \psi_E(x) = \int dx' G_E(x, x') \psi(x') \) is a particular solution.

2. Proof: \( D_E \psi_E(x) = D_E \int dx' G_E(x, x') \psi(x') = \int dx' \left( \frac{\hbar^2}{2m} \left( \frac{d^2}{dx'^2} + V(x') \right) \right) \psi(x') = f(x) \)
Formal derivation of the Green's function  

\[ G_{\nu}(x, \nu') = \begin{cases} \frac{\text{const}}{x} & \text{for } \nu = \nu' \\ \delta(x - \nu') & \text{for } \nu \neq \nu' \end{cases} \]

Method A: 
Set of all free solutions is complete: \( \phi_{\nu}(x) = \begin{cases} \frac{1}{2\pi} (\nu^2 + \nu'^2) \phi_{\nu}(x) & \text{in } (\nu^2 + \nu'^2) \text{ eigenfunctions} \\ \delta(x - \nu') & \text{otherwise} \end{cases} \)

We have 
\[ \frac{\partial}{\partial x} \phi_{\nu}(x) = -\frac{1}{2\pi} (\nu^2 + \nu'^2) \phi_{\nu}(x) \]

\[ \frac{\partial}{\partial x} G_{\nu}(x, \nu') = \begin{cases} \delta^{\nu',\nu} & \text{if } \nu = \nu' \\ \delta^{\nu',\nu} \frac{1}{x - \nu'} & \text{if } \nu \neq \nu' \end{cases} \]

\[ \text{We have } \left( \frac{\partial}{\partial x} \phi_{\nu}(x) \phi_{\nu}(x) - \frac{1}{x - \nu'} \phi_{\nu}(x) \right) \text{ orthogonal in } \text{Gilbert's completion} \]

\[ \left( \frac{\partial}{\partial x} \phi_{\nu}(x) \phi_{\nu}(x) - \frac{1}{x - \nu'} \phi_{\nu}(x) \right) \text{ complete} \]

\[ \Rightarrow \left( G_{\nu}(x, \nu') \right) = -\frac{1}{2\pi} (\nu^2 + \nu'^2) \phi_{\nu}(x) \]

\[ \Rightarrow \left( G_{\nu}(x, \nu') \right) = 2\pi \left[ \phi_{\nu}(x) \phi_{\nu}(x) - \frac{1}{x - \nu'} \phi_{\nu}(x) \right] \]

Formal operator \( G_{\nu} \):
\[ \frac{\partial}{\partial x} G_{\nu} = (x - \nu') G_{\nu} - \delta \]

Method B: 
Momentum space Green's function

Free Schrödinger equation in momentum space: \( \frac{p^2}{2\mu} - \frac{e}{\mu} \Phi(p) = 0 \)

\[ \left( \frac{p^2}{2\mu} - \frac{e}{\mu} \Phi(p) \right) G_{\nu}(p, p') = \left( E - E' \right) \tilde{\delta}(p, p') = \delta^{(0)}(p - p') \]

\[ \Rightarrow \left( \frac{p^2}{2\mu} - \frac{e}{\mu} \Phi(p) \right) G_{\nu}(p, p') = 2\pi \left[ \frac{m}{2\mu} (p - p')^{2} \right] = \langle \Phi | G_{\nu} | \Phi' \rangle \]

Comments:

- Comparing between energy and momentum Green's functions:

\[ G_{\nu}(x, x') = \langle \Phi | G_{\nu} | \Phi' \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{\delta^{(0)}(p - p')}{E - E'} = \left( \frac{2\mu}{\epsilon} \right)^{\frac{3}{2}} \exp \left( \frac{\mu}{\epsilon} |p' - p| \right) \]

- \( \text{We see that } G_{\nu}(x, x') = G_{\nu}(x', x, 0) \)

There is a serious problem in \( G_{\nu}(x, x') \) because of the singularity at \( \epsilon^2 \rightarrow 0 \).
Computation of the Green's function

So far we only made formal manipulations. The singularity at $k^2 + l^2$ in $G(x, y)$ indicates, however, that the Green's function defined that way is not ambiguity-free and certainly not an analytic function in $E_l$

Can we come up with something useful?

We need to impose some physics!

Let's resolve the issue by shifting $E_l$

infinitesimally into the upper complex half-plane. $E_l \rightarrow E_l + i\epsilon$, $\epsilon > 0$ but infinitesimally small

- Solution was ambiguity-free
- Solution is analytic (except at plus and zero)

Why not using $E_l \rightarrow E_l - i\epsilon$ or something else? - See later, let's calculate first

$$\zeta = \frac{1}{2\pi} \int \frac{\delta(\xi - x) \delta(x - y)}{p^2 - z^2} \, dp$$

$$= \frac{1}{2\pi} \int \frac{1}{(p - x)(p - y)} \, dp$$

$$= \frac{1}{2\pi i} \left[ \frac{1}{p - x} - \frac{1}{p - y} \right]$$

Integrate over 2 paths at $p = \pm (\epsilon + i\xi)$

"Closing the contour method"

The method is typically described in books as follows: "We need to close the contour in the upper complex $p$-plane, such that the singly contour does not contribute, and then pick up the path inside the contour using the residue theorem."

Consider the integral along $p(x) = p_0 e^{i\eta}, \, dp = i\eta e^{i\eta} \, dt, \, \xi_0 = const. = int. path in the upper complex plane:

$$\Rightarrow$$ A does not contribute for $p_0 = 0$

because the $e^{-i\eta}$ is opposed in the upper complex $p$-half-plane.

So we can write:

$$\int \frac{dp}{p^2 - \xi^2} = \frac{\pi}{\xi} \left( \frac{1}{\eta} \right) \left( \epsilon + i\xi - \epsilon - i\xi \right) \, dp$$

$$= \frac{\pi}{\xi} \eta$$

Let's now use the residue theorem: $\oint f(z) \, dz = 2\pi i \sum \text{(residues of } f \text{ within closed contour)}$

\begin{align*}
\end{align*}
\[ \text{Re}_\epsilon(p - \text{Re}i\epsilon) = \text{Re} \left( p - (p - \text{Re}i\epsilon) \right) / (p - \text{Re}i\epsilon) = \text{Re} \left( p - (p - \text{Re}i\epsilon) \right) / \left( p - (p - \text{Re}i\epsilon) \right) \]

\[ e^{i\epsilon} = \frac{1}{e^{i\epsilon}} \]

The prescription \( E_0 \to E_0 + i\epsilon \) is related to an outgoing wave and the Green's function.

The Green's function with the correct boundary condition and causality effects is needed for the scattering problem. It is not as simple as a straightforward calculation for the free Schrödinger equation.

Causality tells us which of the mathematically possible Green's functions is the one we need to use.

After imposing causality, the Green's function is unique and invariant under scattering.

Complete formula for the retarded Green's function:

\[ G_\text{ret}^\text{caus}(x, x') = \frac{1}{2\pi} \int \frac{d^4 \mathbf{p}}{p^0 - p^0} \frac{\mathbf{p} \cdot \mathbf{v}}{p^2} \delta(p^2 - m^2) \delta(p - (x - x')) \]

\[ G_\text{ret}^\text{caus}(x, x') = \frac{1}{2\pi} \int \frac{d^4 \mathbf{p}}{p^0 - p^0} \frac{\mathbf{p} \cdot \mathbf{v}}{p^2} \delta(p^2 - m^2) \delta(p - (x - x')) \]

\[ G_\text{ret}^\text{caus}(x, x') = \frac{1}{2\pi} \int \frac{d^4 \mathbf{p}}{p^0 - p^0} \frac{\mathbf{p} \cdot \mathbf{v}}{p^2} \delta(p^2 - m^2) \delta(p - (x - x')) \]

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\[ G_\text{ret}^\text{caus}(x, x') = \frac{1}{2\pi} \int \frac{d^4 \mathbf{p}}{p^0 - p^0} \frac{\mathbf{p} \cdot \mathbf{v}}{p^2} \delta(p^2 - m^2) \delta(p - (x - x')) \]

\[ G_\text{ret}^\text{caus}(x, x') = \frac{1}{2\pi} \int \frac{d^4 \mathbf{p}}{p^0 - p^0} \frac{\mathbf{p} \cdot \mathbf{v}}{p^2} \delta(p^2 - m^2) \delta(p - (x - x')) \]

\[ G_\text{ret}^\text{caus}(x, x') = \frac{1}{2\pi} \int \frac{d^4 \mathbf{p}}{p^0 - p^0} \frac{\mathbf{p} \cdot \mathbf{v}}{p^2} \delta(p^2 - m^2) \delta(p - (x - x')) \]

\[ G_\text{ret}^\text{caus}(x, x') = \frac{1}{2\pi} \int \frac{d^4 \mathbf{p}}{p^0 - p^0} \frac{\mathbf{p} \cdot \mathbf{v}}{p^2} \delta(p^2 - m^2) \delta(p - (x - x')) \]
General properties of the Green's function.

The dispersion of a qubit is defined as follows. We want to consider the Green's function to determine solutions of the Schrödinger equation.

\[ \mathbf{E} \in \mathbb{R}_+ \]

Only then the ambiguity in its definition arise, because the eigenenergies \( \mathbf{E} \) are only \( \mathbb{R}_+ \) in general.

But we can also consider the Green's function as an analytic function for complex energies. Then it's true that:

\[ G(E, \mathbf{r}, \mathbf{r}') = \frac{1}{2\pi i} \text{exp}(-\mathbf{r}, \mathbf{r}') (E-E') \]

\( G(E, \mathbf{r}, \mathbf{r}') \) is analytic except \( E \in \text{spec} \) spectrum of \( \mathbf{H} = \text{set of eigenergies} \)

- \( G \) has branch cuts where the spectrum is continuous
- \( G \) has poles at discrete eigenenergies

Green's function for \( \mathbf{H} = \frac{\mathbf{E}^2}{2m} \): spectrum = \( \mathbb{R}_+ \)

\[ \mathbf{E}_n = \frac{n \pi}{L} \]

\[ \text{cut} \quad \text{(See picture on page: 08/10/2016(5))} \]

Green's function for \( \mathbf{H} = \frac{\mathbf{E}^2}{2m} - \frac{1}{4m} \mathbf{E}^4 \): (Coupled problem): spectrum = \( \mathbb{R}_+ \cup \{ E_n = \frac{\pi^2}{2L^2}, n \in \mathbb{N} \} \)

\[ \mathbf{E}_n = \frac{n \pi}{L} \quad \text{cut} \]

2. The Green's function (as a function in the complex \( E \) plane) contains all information about the solutions of the system defined by the Hamilton operator \( \mathbf{H} \) and it is also the basis for perturbation theory to account for additional effects.
Determination of the Scattering Amplitude

We can now write down the physically correct solution for the stationary scattering problem:

\[ \psi_{\text{scat}}(x) = N e^{i2\pi \frac{x}{a} \cdot \mathbf{p}} \sum_{k} \int dx' \, G_{\text{scat}}(x, x') (V(x')) \psi_{\text{in}}(x') \]

\[ G_{\text{scat}}(x, x') = \frac{1}{2\pi} \frac{e^{i|x-x'|}}{|x-x'|} \quad k = (k_x, k_y) \]

\( \mathbf{2} \) incoming particle momentum

To get the expression for the scattering amplitude \( f(E, \theta, \phi) \), we shall have to expand for \( r = \left| \mathbf{r} - \mathbf{r}' \right| \to \infty \), \( \mathbf{r} = \mathbf{r}' - \mathbf{r}_0 \) because \( U \) has finite range \( r_0 \to \infty \):

\[ r = \sqrt{r^2 - 2 r \cdot \mathbf{r}_0 + r_0^2} = r (1 - \frac{2 r \cdot \mathbf{r}_0}{r^2} + \frac{r_0^2}{r^2}) = r (1 - \frac{2 r \cdot \mathbf{r}_0}{r^2}) = r - \frac{r \cdot \mathbf{r}_0}{r} = r - \frac{\mathbf{r}_0 \cdot \mathbf{r}}{|\mathbf{r}_0|} \]

\[ \psi_{\text{in}}(x) = N e^{i2\pi \frac{x}{a} \cdot \mathbf{p}} \sum_{k} \int dx' \, e^{i(x-x') \cdot \mathbf{p}} (-V(x')) \psi_{\text{in}}(x') \]

\[ f(E_n, \theta, \phi) = -\frac{\hbar}{2\pi N} \int dx \, e^{-i2\pi \mathbf{p} \cdot \mathbf{x}} V(x) \psi_{\text{in}}(x) \quad \text{not an explicit solution} \]

\[ f(E_n, \theta, \phi) = -\frac{\hbar}{2\pi N} \int dx \, e^{-i2\pi \mathbf{p} \cdot \mathbf{x}} V(x) \psi_{\text{in}}(x) \quad \text{Born approximation} \]

To get an explicit (approximate) solution (we use the assumption that the scattering overall is a very small effect):

\[ \Rightarrow \quad \psi_{\text{scat}}(x) = N e^{i2\pi \frac{x}{a} \cdot \mathbf{p}} + O(V) \]

\[ \text{'momentum transfer'} \]

\[ \text{'Born approximation'} \]

In the Born approximation, the scattering amplitude is proportional to the Fourier transform of the center spin potential \( V(x) \) to the "incoherent transfer" \( \mathbf{S} \cdot \mathbf{L} \)

\[ \text{= momentum that the scattering center is giving to the scattered particle} \]

\[ \text{The Born approximation may be already a good approximation of the actual result, however, some higher order correction is needed.} \]
2.6. Newtonian Perturbation Series

→ Approach to carry out time-independent perturbation theory for continuous states.

→ Assumption:
\[ e^{i\lambda \Phi} \approx \int d^3r \, G(x, \lambda \Phi) \psi(\Phi) \chi(x) \]
→ Must be true for \( |\Phi| \ll R_0 \) (range of potential)

→ We expand in powers of \( \Phi \).

\( C_n \approx C_n^0 \) from now on

Perturbative expansion for the time-independent Scattering solution:

We set norm \( \| \psi \|_2 = \frac{1}{2\pi} \int d\lambda \psi^\dagger \psi(\lambda) \)

0th approx.
\[ \psi_0(\Phi) = \chi(x) = \frac{e^{i\lambda \Phi}}{e^{i\lambda \Phi} + C_n^0} \langle \Phi | \]

1st approx. (Born approximation)
\[ \psi_1(\Phi) = \psi_0(\Phi) + \psi_0^\dagger(\Phi) \]
\[ = \chi(x) \left[ \int d^3r \, G(x, \lambda \Phi) \psi(\Phi) \right] \langle \Phi | \]
\[ = \langle \Phi | \left[ 1 - C_n^0 \psi(\Phi) \right] \chi(x) \langle \Phi | \]

2nd approx.
\[ \psi_2(\Phi) = \psi_1(\Phi) + \int d^3r \, G(x, \lambda \Phi) \left[ (\Phi) \right] \chi(x) \langle \Phi | \]
\[ + \langle \Phi | \left[ 1 - C_n^0 \psi(\Phi) \right] \chi(x) \langle \Phi | \]

All order solution.
\[ \psi_\infty(\Phi) = \langle \Phi | \left[ 1 - C_n^0 \psi(\Phi) \right] \chi(x) \langle \Phi | \]

(formal)
\[ = \langle \Phi | \frac{1}{1 - C_n^0 \psi(\Phi)} \langle \Phi | \]

Perturbative expansion for the Scattering amplitude:

1st approx. (Born approximation)
\[ f_{1+i}(E_\lambda, \Phi, \Phi) = \frac{1}{2\pi} \int d^3r \, e^{-i\lambda (\Phi - \Phi') \chi(x) \phi(\Phi') \Phi(x) \chi(x)} \]
\[ = \frac{1}{2\pi} \int d^3r \, \phi(\Phi') \chi(x) \phi(\Phi) \Phi(x) \chi(x) \]
\[ = \frac{1}{2\pi} \int d^3r \, \Phi(x) \]
\[ = \frac{1}{2\pi} \int d^3r \, \Phi(x) \]

→ Born transition element

→ Transition
All order solution (formal) \( \langle \kappa_{e_1} | \kappa_{e_2} \rangle \), \( T \) matrix element

\[
T_{\mu\nu}(E_{e_1}, \theta, \phi) = \frac{4\pi\hbar}{\sqrt{1 + \frac{Q^2}{m^2}}} \langle \kappa_{e_1} | \kappa_{e_2} \rangle
\]

\[
= 4\pi\hbar \left\{ -\kappa_{e_1} \kappa_{e_2} + \sum_{n=1}^{\infty} \frac{2n+1}{2n} \kappa_{e_1}^{2n} \kappa_{e_2}^{2n-1} \right. \\
\left. \times \sum_{l=0}^{\infty} \frac{2n-l}{2n-l+1} \kappa_{e_1}^{2n-l} \kappa_{e_2}^{2n-l} \right. \\
\left. \times 2n-l \sum_{k=0}^{\infty} \frac{2n-k}{2n-k+1} \kappa_{e_1}^{2n-k} \kappa_{e_2}^{2n-k} \right. \\
\left. \times \left[ -\kappa_{e_1} \kappa_{e_2} + \sum_{n=1}^{\infty} \frac{2n+1}{2n} \kappa_{e_1}^{2n} \kappa_{e_2}^{2n-1} \right] \right. \\
\left. \times \sum_{l=0}^{\infty} \frac{2n-l}{2n-l+1} \kappa_{e_1}^{2n-l} \kappa_{e_2}^{2n-l} \right. \\
\left. \times 2n-l \sum_{k=0}^{\infty} \frac{2n-k}{2n-k+1} \kappa_{e_1}^{2n-k} \kappa_{e_2}^{2n-k} \right. \\
\left. \times \left[ -\kappa_{e_1} \kappa_{e_2} + \sum_{n=1}^{\infty} \frac{2n+1}{2n} \kappa_{e_1}^{2n} \kappa_{e_2}^{2n-1} \right] \right. \\
\left. \times \sum_{l=0}^{\infty} \frac{2n-l}{2n-l+1} \kappa_{e_1}^{2n-l} \kappa_{e_2}^{2n-l} \right. \\
\left. \times 2n-l \sum_{k=0}^{\infty} \frac{2n-k}{2n-k+1} \kappa_{e_1}^{2n-k} \kappa_{e_2}^{2n-k} \right. 
\]

Feynman Rules for the T-matrix element in Momentum Space

- Graphical notation for the analytic expression which -conceptually- intuitively illustrates the actual physical process that takes place.

- Comment: How “Feynman rules” come historically from relativistic quantum field theory, whose importance is enhanced byopped up due to complexity. Hence, the advantage of Feynman rules is not so obvious.

Born approximation

\[
\begin{array}{c}
\text{Incoming state: } \langle \kappa_{e_1} \rangle \\
\text{Interaction factor: } -\bar{\nu}(E_{e_1}, \kappa_{e_1}) \\
\text{for transition } \langle \kappa_{e_1} \rangle \rightarrow \langle \kappa_{e_2} \rangle
\end{array}
\]

\[
\Rightarrow \quad -\bar{\nu}(E_{e_1}, \kappa_{e_1}) \\
\text{related to classical limit}
\]

2nd order

\[
\begin{array}{c}
\text{Intermediate propagation factor of particle without interaction: } \\
\frac{2\nu}{(\kappa_{e_1}^2 C_{e_1})} \quad \text{for all } \kappa_{e_1}
\end{array}
\]

\[
\Rightarrow \quad \int d\Sigma \frac{2\nu}{(\kappa_{e_1}^2 C_{e_1})} \frac{2\nu}{(\kappa_{e_2}^2 C_{e_2})} - \bar{\nu}(E_{e_1}, \kappa_{e_1}) - \bar{\nu}(E_{e_2}, \kappa_{e_2})
\]

3rd order

\[
\begin{array}{c}
\text{Virtual particle IC.}
\end{array}
\]

\[
\text{Virtual particle IC.}
\]

\[
\text{Virtual particle IC.}
\]

\[
\text{Virtual particle IC.}
\]
Figurative Rules for the $T$-matrix element in Configuration Space

Born approximation

\[ R_{\text{Born}} \]

2nd order

\[ R_{\text{2nd order}} \]

When is the Neumann series a good expansion? We carry out a number of qualitative considerations.

1. We already see from formulae: Series appears divergent alternating for $V(x) > 0$.
   2. Convergence better for isospin-potentiate.

2. Consider $k = 0$ (at centre of potential) - we have $\frac{1}{k^2} - 1 \implies$ we must - $|k|^2 \phi(k) = 1$

\[ \left( \phi^2(x) \right)^{\text{total}} = \sum_{k} \left| \frac{k}{2\pi} \int V(x) e^{ikx} dx \right| = \frac{1}{2\pi} \int V(x) \sin(kx) \sin(kx) dx \]

Case 1: Low-energy scattering: $k' \ll k, \nu \ll 1$ (that's only within range of potential!)

\[ |\phi^2(x)| \ll \frac{k}{2\pi} \int_{x'} \phi^2(x') \frac{k}{2\pi} e^{ikx} V(x) e^{ikx'} dx' \ll 1 \]

1. $V_0 = \frac{\mu e^2 V_0}{\nu}$

What does the inequality mean? Consider ground state (lowest energy bound state) of an attractive potential

\[ \text{average momentum} \approx \frac{1}{\mu} \left( \text{average free} \right) \]

\[ \text{average kinetic energy} \approx \left( \frac{1}{2} \right) \frac{1}{\mu} \left( \frac{1}{\mu} \right) \]

\[ V_0 \ll \frac{1}{\mu} \] means that $V$ is too shallow to allow for a bound state

\[ \implies \text{Converge better if the potential does not have bound states.} \]

because otherwise they states can to additional non-thrid order effects not properly treated by the Neumann series.
Case 2: High-energy scattering (but still non-relativistic)

\[ 2 \cdot (k \cdot r) \gg 1 \quad \text{and} \quad r \ll k^{-1} (e^{ikr} - e^{-ikr}) \]

\[ |\psi(r)|^2 = \frac{1}{L} \left( \int dB (e^{2ikr} - 1) \psi(r) \right) = \frac{1}{L} \left( \int dB \psi(r) \right) \sim \frac{1}{L} \psi(r) \quad \text{as } L \rightarrow \infty \]

Where \( L \) can be always readily satisfied if only the energy \( E \sim \frac{1}{k^2} \) is high enough.

2. Higher energy \( \rightarrow \) interaction time with potential smaller.

Note: The Heaviside step is in general not convergent — even if the conditions are satisfied and the first terms in the step are convergent. \( \rightarrow \) "Asymptotic expansion." 

2.7 Generalizations

Differential cross section for spinless particles at stationary scattering centre, typically.

\[ \frac{d\sigma}{d\Omega} = \left| \frac{\langle E_1 | \hat{v} \cdot \hat{n} | E_2 \rangle}{E_2 - E_1} \right|^2 \]
Final state phase space. We want to understand the counting of states using statistical mechanics.

Density of free particle states: Consider a cube with side length $L$.

$$ V = L^3 ightarrow \infty $$

Only integer multiples of $\frac{2\pi}{L}$ fit into the cube.

Possible momenta (four vectors):

$$ k = \frac{2\pi}{L} (n_x, n_y, n_z), \quad n_i = 0, 1, 2, 3, ... $$

Each vector $V(n_x, n_y, n_z)$ corresponds to one state.

Number of states for momentum in interval $[k_0, k_0 + \Delta k_0]$: 

$$ \frac{1}{k_0} \frac{dV}{(\Delta k_0)^2} = \frac{V}{(2\pi)^3} \frac{d^3 \mathbf{k}}{\Delta k_0^2} $$

Number of states associated to the momentum interval around $\mathbf{k}^0$: 

$$ \frac{V}{(2\pi)^3} d^3 \mathbf{k} $$

Norm of free particle states (in cube with $V = L^3$): 

$$ \langle \mathbf{k} | L^3 \rangle = \frac{(2\pi)^3}{6} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} = \frac{V}{6} $$

We see that 

$$ \frac{V}{(2\pi)^3} \frac{d^3 \mathbf{k}^0}{\Delta k_0^2} = \frac{V}{(2\pi)^3} \text{sum over all possible final state states} [\mathbf{k}] = \text{constant} $$

and also divides out their norm.

So using 1D states as dual states and $\Delta k_0$ for the final state integration correctly sums over all possible states.

We now can generalize to multi-particle final states:

$$ | \langle \mathbf{k}_1 \ldots \mathbf{k}_n | L^3 \rangle |^2 d^3 \mathbf{k}_1 d^3 \mathbf{k}_2 \ldots d^3 \mathbf{k}_n $$

for

$$ | \langle \mathbf{k}_1 \ldots \mathbf{k}_n | L^3 \rangle |^2 = \frac{1}{(2\pi)^3} \langle \mathbf{k}_1 \ldots \mathbf{k}_n \rangle $$

Notes:

1. It appears that only positive integers $k > 0$ fit inside the box and that the integral is only defined by excluding the origin from the interval $k > 0$.

2. When we add the solutions of the problem to particles inside an incompressible box (on axis, not off axis), which can represent waves moving in positive ($k > 0$) and negative directions ($k < 0$).

The way this works mathematically is that each solution can be written in terms of free particle eigenstates $\mathbf{k}$, $\mathbf{k}(x, y, z) = \frac{1}{2} ((e^{ix} - e^{-ix})$)

Alternative arguments for the counting of states that fit in the box:

We guess boundary conditions:

$$ \mathbf{\psi}(x, y, z) = \mathbf{\psi}(x, y, L) $$

Possible momenta (four vectors):

$$ \mathbf{k} = \frac{2\pi}{L} (n_x, n_y, n_z), \quad n_i = 0, 1, 2, 3, ... $$

Number of states for momentum in interval $[k, k + \Delta k]$: 

$$ \frac{V}{(2\pi)^3} \frac{d^3 \mathbf{k}}{(2\pi)^3} = \frac{V}{(2\pi)^3} \frac{d^3 \mathbf{k}}{\Delta k^2} $$

2. Same result.
Scattering of a bound particle

- Scattering center is a particle A (bound by the potential $W$)
- Beam particles scatter off the potential $V$ generated by the
  particle A and are not subject to potential $W$ (which
  is related to a different type of interaction)

→ example: electron-proton scattering ("deep inelastic scattering")
- protons are bound inside nucleons due to the strong nuclear force
- electrons do not feel the strong force and only interact via the
  electromagnetic force due to the electric charge of the quarks.

→ We treat the potential $V$ perturbatively

For $V = 0$: 2 decoupled systems

\[ V = \left\{ \begin{array}{c} \text{free particles (beam)}: \quad \left( \frac{-i}{2\hbar} \cdot \nabla - \frac{e^2}{2R} \right) \Phi(x) = 0 \\ \text{bound particle A (target)}: \quad \left( \frac{-i}{2\hbar} \cdot \nabla - u(x) - E_o \right) \Psi(x) = 0 \end{array} \right\} \]

Hilbert space: direct product of the free and bound particle Hilbert spaces

→ We assume that the bound particle A is initially in its ground state \( |\psi_0\rangle \), and we can track (perturbatively) the scattering formula for this case.

Born level: Time-matrix element

incoming state: \( |\psi_{in}\rangle \otimes |\psi'_0\rangle = |\psi_{in}, \psi'_0\rangle \)

outgoing state: \( \langle \psi'_n | \otimes |\psi_{out}\rangle = \langle \psi_{in}, \psi_{out}\rangle \quad \Rightarrow \quad n = 0, 1, 2, \ldots \) (bound particle may be excited due to scattering)

\[ \langle \psi_{in}, \psi_{out} | - V | \psi_{in}, \psi'_n \rangle \]

\[ = - \frac{i}{2\hbar} \int d^3x_1 \int d^3x_2 \left( \frac{e^2}{2R} \right) \psi(x_1) \Phi(x_2) \left( \frac{-i}{2\hbar} \cdot \nabla - u(x_2) - E_o \right) \psi(x_2) \]

\[ = - \frac{i}{2\hbar} \int d^3x_1 \int d^3x_2 \left( \frac{e^2}{2R} \right) \left( \frac{-i}{2\hbar} \cdot \nabla \right) \left( \psi(x_1) \Phi(x_2) \right) \left( \frac{-i}{2\hbar} \cdot \nabla \right) \psi(x_2) \]

\[ n = 0: \] particle A not excited

\[ \psi_{in} (x_1) = \frac{1}{\sqrt{\omega_{in}}} \int d^3x_2 \left( \frac{e^2}{2R} \right) \psi(x_2) \Phi(x_2) \]

\[ \Rightarrow \quad d\sigma = \frac{1}{4\pi^2} \left| \psi_{in} (x_1) \right|^2 \left| \psi_{out} (x_2) \right|^2 \| \Phi(x_2) \| \]

\[ d\sigma = \frac{1}{4\pi^2} \left| \psi_{in} (k_1, \omega_1) \right|^2 \left| \psi_{out} (k_2, \omega_2) \right|^2 \| \Phi(k_2) \|^2 \]

\[
= \frac{1}{4\pi^2} \left| \psi_{in} (k_1, \omega_1) \right|^2 \left| \psi_{out} (k_2, \omega_2) \right|^2 \| \Phi(k_2) \|^2
\]
\( u = 0 \) : Inelastic scattering with excitation of scattering center
\[ E_{eD} - E_{\gamma} = E_{\gamma'} - E_{\gamma} \quad \text{(energy conservation)} \]

\[ \mathcal{L} \left\{ \left[ \frac{E_{\gamma} - E_{\gamma'}}{E_{\gamma}} \right] - \left[ \frac{E_{\gamma'} - E_{\gamma}}{E_{\gamma}} \right] \right\} \quad \text{"momentum transfer"} \]

\[ = \frac{1}{(2\pi)^3} \int d^3k \ e^{-iE_{\gamma}k} \frac{1}{(2\pi)^3} \int d^3k' \ e^{-iE_{\gamma'}k'} \psi^*_\gamma(k) \psi_{\gamma'}(k') \]

\[ = \frac{1}{(2\pi)^3} \int d^3k \ e^{-iE_{\gamma}k} \psi^*_\gamma(k) \psi_{\gamma}(k) \]

"structure function" or "form factor"

Contains information on the bound state properties of the target

\[ \mathcal{L} \left\{ \left( \frac{2\pi}{E_{\gamma}} \right)^3 \right\} \int d^3k \ e^{-iE_{\gamma}k} \psi^*_\gamma(k) \psi_{\gamma}(k) \]

\[ \implies \text{one can learn a lot about the bound state structure if } \langle \psi^2/2 \rangle \text{ were to be known} \]

\[ \text{Target dissociation: In case of inelastic scattering it may also be possible that the bound particle is broken out of the bound state and is a free particle after the scattering.} \]

\[ \mathcal{L} \left\{ \left[ \frac{E_{\gamma} - E_{\gamma'}}{E_{\gamma}} \right] - \left[ \frac{E_{\gamma'} - E_{\gamma}}{E_{\gamma}} \right] \right\} \]

\[ = \frac{1}{(2\pi)^3} \int d^3k \ e^{-iE_{\gamma}k} \frac{1}{(2\pi)^3} \int d^3k' \ e^{-iE_{\gamma'}k'} \psi^*_\gamma(k) \psi_{\gamma'}(k') \]

\[ = \left( \frac{2\pi}{E_{\gamma}} \right)^3 \int d^3k \ e^{-iE_{\gamma}k} \psi^*_\gamma(k) \psi_{\gamma}(k) \]

(5) add final state integration \( d^3k_{\gamma} \)

\[ \mathcal{L} \left\{ \left[ \frac{E_{\gamma} - E_{\gamma'}}{E_{\gamma}} \right] - \left[ \frac{E_{\gamma'} - E_{\gamma}}{E_{\gamma}} \right] \right\} \]

\[ \text{Finally, we can also use the form factor parameterization for the elastic case} \]

\[ \mathcal{L} \left\{ \left[ \frac{E_{\gamma} - E_{\gamma'}}{E_{\gamma}} \right] - \left[ \frac{E_{\gamma'} - E_{\gamma}}{E_{\gamma}} \right] \right\} \]

\[ \text{For small scattering angles } \theta, \quad \frac{1}{E_{\gamma}} \rightarrow 0 \]

\[ \mathcal{L} \left\{ \left[ \frac{E_{\gamma} - E_{\gamma'}}{E_{\gamma}} \right] - \left[ \frac{E_{\gamma'} - E_{\gamma}}{E_{\gamma}} \right] \right\} \]

\[ \text{primarily distribution } g_{\gamma'}(E_{\gamma}) \text{ (renormalized)} \]

\[ \mathcal{L} \left\{ \left[ \frac{E_{\gamma} - E_{\gamma'}}{E_{\gamma}} \right] - \left[ \frac{E_{\gamma'} - E_{\gamma}}{E_{\gamma}} \right] \right\} \]

\[ \text{primarily distribution } g_{\gamma'}(E_{\gamma}) \text{ (renormalized)} \]

\[ \begin{align*}
\text{dipole moment} & \quad \text{linear dipoles squared} \quad \text{quadrapole moment} \\
\text{(also for form factor above)} & \quad \text{of distribution } g_{\gamma'}(E_{\gamma}) \quad \text{of distribution } g_{\gamma'}(E_{\gamma})
\end{align*} \]
2 Particle Scattering

Head-on head collision is the major collision type in modern collider experiments.

Example: LHC: p+p collision

Fermi-Dirac Collider: e^+e^- collisions

LEP: e^+e^- collisions

HERA: e^-p collisions

We consider elastic scattering

\[ A(C_1) \cdot \mathcal{E}(C_2) \rightarrow A(C_3) \cdot \mathcal{E}(C_4) \]

We again prove intuitively without proof.

Born level T matrix element

Incoming state: \( |L_C > \otimes |E_C > = |L_{C_E} > \)

Outgoing state: \( |C_A > \otimes |C_B > = |C_{A_B} > \)

\[ Z \cdot |L_{C_E} > \cdot V \cdot |L_{C_E} > \]

\[ = - \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_q} \cdot e^{i(q \cdot x)} \cdot e^{-i(q \cdot x)} \cdot V(q \cdot \bar{q}) \cdot e^{i(q \cdot x)} \cdot e^{-i(q \cdot x)} \]

\[ = - \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_q} \cdot e^{-i(q \cdot \bar{q})} \cdot V(q) \cdot e^{i(q \cdot \bar{q})} \cdot \bar{q} \cdot \bar{q} \]

\[ = - \delta(q^2 - \bar{q}^2) \cdot \delta(\bar{q} - q) \cdot V(q) \cdot \delta(q^2 - \bar{q}^2) \]

\[ = - \delta(q^2 - \bar{q}^2) \cdot \delta(q) \cdot V(q) \]

\[ = 2 \cdot \delta(q^2 - \bar{q}^2) \cdot V(q) \cdot \delta(q^2 - \bar{q}^2) \]

In addition to energy conservation

Problems:
(a) How to deal with \( |L_{C_E} > \cdot V \cdot |L_{C_E} > \) ?
(b) Both initial states are plane waves. \( \rightarrow \) how to select "target" \( (A \text{ or } B) = 1 \)
Which is the incoming probability current ?

We can handle both problems by considering initial state wave packets:

\[ \langle \phi(t_1), C_1, \rangle = \int d^3x \cdot \phi_\mathcal{E}(x) \langle C_1 | \mathcal{E}(x) \rangle \]

We impose \( \langle \phi(t_1), C_1, \rangle \cdot \mathcal{E}(t_1) \rangle = 1 \)

\[ \int d^3x \cdot \phi_\mathcal{E}(x) \langle C_1 | \mathcal{E}(x) \rangle \cdot \mathcal{E}(t_1) \rangle = 1 \]

\[ \Rightarrow \langle \phi(t_1), C_1, \rangle \cdot \mathcal{E}(t_1) \rangle = \frac{1}{\mathcal{E}(t_1)} \]

\[ \Rightarrow |L_{C_E} > \cdot V \cdot |L_{C_E} > \]

\[ = \text{probability that the final state packet on scattered into incoming initial state packets} \]

\[ d^3p_\mathcal{E} \cdot \delta(p_\mathcal{E} - p) \cdot |E_\mathcal{E} > \]

\[ = \text{probability that the final state packet on scattered into incoming initial state packets} \]

\[ d^3p_\mathcal{E} \cdot \delta(p_\mathcal{E} - p) \cdot |E_\mathcal{E} > \]
Because in an experiment, the scattering of individual particles, we cannot easily use the probability current to properly normalize the cross section.

We have to go back to the basic definition of the cross section:

\[ \frac{\sigma}{A} = \frac{N_{\text{events}}}{N_{\text{beam}}} \]

- \( N_{\text{events}} \): \# of events where scattering happened (within time \( T \))
- \( N_{\text{beam}} \): \# of particles of type \( 1/2 \) in beam

\( A \): overlap area of the beams \( L \) to beam direction

\( A \): size of cross product

We have simplified the situation of a target particle (2) and a beam of particles (8): \( \rightarrow A = 1 \)

It now must account for the fact that the beam particles can have any impact parameter \( t \) to the beam axis. We cannot distinguish the different situations and must sum over them.

A solid square for particle A beam packet

\[ N_{\text{beam}} \propto \int d^2q \left( \phi_{\text{beam}} \phi_{\text{target}} \right)^2 \]

The cross section \( \sigma \) is now determined from:

\[ \sigma = \frac{N_{\text{events}}}{n_{\text{beam}}} \]

- \( N_{\text{events}} \): \# of events where scattering happened (within time \( T \))
- \( n_{\text{beam}} \): are density of beam (within beam)

\[ \Rightarrow N_{\text{events}} \propto \int d^2q \left( \phi_{\text{beam}} \phi_{\text{target}} \right)^2 \]

- \( \phi_{\text{beam}} \): beam function

We use the following facts:

\[ \sigma \propto \sum_{i,j} \left( \phi_{\text{beam}} \phi_{\text{target}} \right)^2 \]

- \( \phi_{\text{beam}} \): beam function

We use two \( E \)-invariant \( S \)-folds, because the matrix elements have different indices.

This agrees at low \( E \) but otherwise we cannot implement energy conservation for the same particle. It is also meaningless for dimensional reasons (\( \sigma \in \text{a}^2 \text{L}^2 \)).

\[ \sigma \propto \text{will be justified rigorously when we discuss time-dependent perturbation theory} \]
\[ d^2 \sigma = \frac{1}{2 S_{\text{inel}}} |S(C_{i+}, C_{i-})|^2 \frac{d^3 S}{dE_{\text{kin}} d^2p_i} \frac{d^2 \sigma_{\text{inel}}}{dE_{\text{kin}} d^2p_i} \]

Final result for 2-particle scattering cross section (in any frame)
Comments:

Stationary target limit:

For $p_2 \to 0$ we have $\vec{v}_{2,a} \to 0$, $E_{k,1} \to E_{k,1} - \vec{v}_{2,a}$ then $E_{k,2} \to 0$. Particle 2 does not move at all.

\[
\int d\omega = \frac{1}{16\pi} \left| \vec{v}(\vec{v}_{1,a}, \vec{v}_{2,a}) \right|^2 \delta^3(\vec{v}_{2,a} - \vec{v}_{2,a} - \vec{v}_{2,b} - \vec{v}_{2,c}) \delta(E_{k,1} - E_{k,1} - E_{k,2} - E_{k,2})
\]

Particle 2 final state momentum can have any size $\vec{v}_{2,a}$ and does not affect the cross section.

We can integrate over it.

\[
\int d\omega = \frac{1}{16\pi} \left| \vec{v}(\vec{v}_{1,a}, \vec{v}_{2,a}) \right|^2 \delta(E_{k,1} - E_{k,1}) d^3v_{2,a}
\]

We denote exactly the cross section of elastic scattering of particle 1 off the stationary potential caused by particle 2 (which is infinitely heavy).

Our all disc manipulations are fully consistent (and in fact completely correct).

Differential cross sections

From Eq. (6) we have plenty of options to define differential cross sections. We can pick any variable that we want as long as it can be computed from the final state momenta $\vec{v}_{2,a}$ and $\vec{v}_{2,b}$:

\[
\frac{d\sigma}{d\omega} = \left( \frac{d^3v_{2,a}}{d^3v_{2,b}} \right) \int d\omega \left| \vec{v}(\vec{v}_{1,a}, \vec{v}_{2,a}) \right|^2 \delta^3(\vec{v}_{2,a} - \vec{v}_{2,a} - \vec{v}_{2,b} - \vec{v}_{2,c})
\]

\[
= \left( \int d\omega \right) \delta(E_{k,1} - E_{k,1} - E_{k,2} - E_{k,2}) \delta(X - x(\vec{v}_{2,a}, \vec{v}_{2,b}))
\]

The multi-differential cross sections are possible:

\[
\frac{d\sigma}{d\omega dX} = \left( \frac{d^3v_{2,a}}{d^3v_{2,b}} \right) \int d\omega \left| \vec{v}(\vec{v}_{1,a}, \vec{v}_{2,a}) \right|^2 \delta^3(\vec{v}_{2,a} - \vec{v}_{2,a} - \vec{v}_{2,b} - \vec{v}_{2,c})
\]

\[
= \left( \int d\omega \right) \delta(E_{k,1} - E_{k,1} - E_{k,2} - E_{k,2}) \delta(X - x(\vec{v}_{2,a}, \vec{v}_{2,b})) \delta(X - x(\vec{v}_{2,a}, \vec{v}_{2,b})) 
\]

Fixed target frame

One particle (let's say \(2\)) has $E_{k,1} = 0$

Before scattering: $E_{k,2} = 0$.

Due to energy and momentum conservation $E_{k,2} + E_{k,3} = 0$ if $\Theta = 0$.
Center of mass frame:

Initial state wavevectors vanish: \( \vec{k}_1 + \vec{k}_2 = 0 \)
\[ \vec{k}_1 + \vec{k}_2 = 0 \]

Used e.g. at the LHC.

Cross section in different reference frames:

\[ \text{Total cross section } \sigma \text{ in frame - independent (Area on frame-independent.)} \]
\[ \sigma = \frac{(I_i \cdot I_f)}{I_i} \]

Intensities can change, but only globally by an overall factor.

That \( \sigma \) is frame-independent is also true in the relativistic case (although areas are not frame-independent.)

Differential cross sections are in general frame-dependent since they depend on kinematic quantities that are frame-dependent.

Example: Angular differential cross section

Solid angle frame 1: \( \theta_1, \phi_1 \)
Solid angle frame 2: \( \theta_2, \phi_2 \)
\[ \frac{d^2\sigma}{d\theta_1 d\phi_1} \text{ frame 1} = \frac{d\theta_2 d\phi_2}{d\theta_1 d\phi_1} \text{ frame 2} \]

Consider change of fixed target to center of mass frame (for particle 1)

\( \vec{F}_1 \) frame \( \rightarrow \) \( \vec{F}_1^{CM} \)
\( \vec{C}_1 \) frame \( \rightarrow \) \( \vec{C}_1^{CM} \)

\( \vec{F}_1 \) frame moves with velocity \( (\vec{v}_1)^{CM} \), w.r.t. the \( \vec{C}_1 \) frame.
\[ \text{vector velocities in the } \mathbf{F} \text{ frame: } V_L = V_0 - \left( V_c \right)_L; \]

\[ \text{Convection transformation: } \mu_L \left( \mathbf{V} \right)_L = - \mu_0 \left( \mathbf{V} \right)_0; \quad \Rightarrow \quad \left( \mathbf{V} \right)_L = - \frac{\mu_0}{\mu_L} \left( \mathbf{V} \right)_0; \]

\[ z_0 = \frac{\theta_L}{\theta_0} = \frac{\theta_c}{\theta_0} \]

\[ \tan \theta_L = \frac{\left( V\theta \right)_L}{\left( V\theta \right)_0} = \frac{\left( V\theta \right)_L}{\left( V\theta \right)_0 + \frac{1}{\mu_0} \left( \mathbf{V} \right)_0}; \quad \Rightarrow \quad \left( \mathbf{V} \right)_L = \frac{\sin \theta_L}{\cos \theta_L} \left( \mathbf{V} \right)_0 \]

\[ z \Rightarrow \cos \theta_L = \frac{\mu_0 + \mu_c \cos \theta_c}{\sqrt{\mu_0^2 + \mu_c^2 \cos^2 \theta_c}} \]

\[ \text{Special case: } \mu_L = \mu_0 = \mu, \quad \Rightarrow \quad \tan \theta_L = \frac{\sin \theta_L}{\cos \theta_L + \frac{\mu_c}{\mu}} = \frac{\sin \theta_L}{\frac{\mu_c}{\mu}} \]

\[ z \Rightarrow \theta_L = \frac{\theta_c}{\nu \theta_L}; \quad \theta_L = \frac{\theta_c}{\mu \theta_0} \]

\[ d \left( \theta_0 \cos \theta_0 \right) = \frac{d \cos \theta_0}{d \cos \theta_0} = d \cos \theta_0 \frac{d \theta_0}{d \theta_0} = \frac{2 \sin \left( \theta_0 \right) \cos \theta_0}{\sin \theta_0} \]

\[ d \left( 4 \cos \theta_c \phi_0 \right) = \frac{4 \cos \theta_c \phi_0}{d \cos \theta_c \phi_0} = 4 \cos \theta_c \phi_0 \]

\[ \Rightarrow \frac{d \cos \theta_c \phi_0}{d \cos \theta_c \phi_0} = -\frac{1}{4 \cos \theta_c \phi_0} \]
2.8. Time - Dependent Perturbation Theory

Propagator (Green's function method)

\[ i \frac{\partial}{\partial t} \Psi(x,t) + 0 , \quad H = \frac{p^2}{2m} + V(x,t) = H_0 + V(x,t) \] → Exercise

Retarded Green's function for the free Schrödinger equation (propagator):

\[ \Psi_0(x,x') = \begin{cases} \frac{\hbar}{2 \pi \hbar} & \text{if} \quad x > x' \\ 0 & \text{if} \quad x < x' \end{cases} \]

\[ = \Theta(x-x') \left( \frac{\hbar}{2 \pi \hbar} \right) \left( \frac{x}{x'} \right) \left( \frac{t}{t'} \right) \]

Momentum space:

\[ \tilde{\Psi}_0(q,q') = i \left( 2 \pi \right)^3 \delta(q-q') \]

Comments:

- We use the 4-vector notation: \( \rightarrow \) Essential when we consider relativistic physics.

\[ x^0 = (t, \vec{x}) , \quad q^0 = (q, \vec{0}) \quad (\text{contravariant 4-vector}) \]

\[ x^1 = (t, \vec{x}) , \quad q_i = (q_i, \vec{0}) \quad (\text{covariant 4-vector}) \]

\[ x^0 = x' q^0 = \frac{q^0}{c} \]

- Alternative solution: \( \Psi(x) = \psi(x,t) , \quad \tilde{\Psi}(q) = \tilde{\psi}(q,t) , \quad \delta^0(q-x) = \delta(q^0-c^{-1} q^0) \)

\[ \tilde{\Psi}(q,q') = \left( \frac{\hbar}{2 \pi \hbar} \right) \left( \frac{t}{t'} \right) \left( \frac{x}{x'} \right) \]

- \( \tilde{\Psi}(q,q') \) is a "spaced time" solution operator: \( \rightarrow \) causality implemented!

\[ \tilde{\Psi}(q,q') = \left( \frac{\hbar}{2 \pi \hbar} \right) \delta(q-x') \delta(t-t') \]

- Given \( \tilde{\Psi}(q,q') \) we can use \( \tilde{\Psi}(q,x') \) to determine \( \tilde{\Psi}(q,t-x) = \int d^3x' \tilde{\Psi}(q,x') \chi(t-x) \)

Check:

\[ \left( \frac{\hbar}{2 \pi \hbar} \right) \chi(t-x) = \int d^3x' \left[ \frac{\hbar}{2 \pi \hbar} \right] \delta(q-x') \chi(t-x') \]

\[ = \left[ \frac{\hbar}{2 \pi \hbar} \right] \chi(t-x) \]

\[ = 0 \]

\[ (t-t_x) \]
\[ S(x, y) \] "propagates" a particle located at \( x' = (x', \xi) \) dynamically (i.e. due to \( \nabla \))

to the location \( x'' = (x'', \xi') \), \[ \xi' = \xi + (x'' - x')/2 \]

orIGIN of wave propagator

\[ S(x, y) \sim \delta^{(3)}(x - r) \delta^{(3)}(y - r) \]

2. Implementation of energy-momentum conservation.

\[ S_{\text{tot}}(x, y, t, \xi) = \int d^{3}x' \, S(x', y, t, \xi) \, S(x, x', t, \xi') \]

\[ \text{for } t < t' < t'_{2} \]

**Time-dependent scattering problem**

\[ \rightarrow \text{ Physical setup for time dependent scattering ("Asymptotic States") } \]

\[ \rightarrow \text{ We consider again elastic} \]

\[ \rightarrow \text{ Atomic scattering for} \]

\[ t_{1} \rightarrow -\infty : \text{ Wave starts as incoming free wave with fixed momenta } \xi_{1}. \]

\[ \text{ (It has no effect on the potential which has finite range.)} \]

\[ E_{1} \rightarrow -\infty : \text{ Wave is detected as outgoing free wave with fixed momenta } \xi_{2}. \]

\[ \text{ (It has no effect on the potential which has finite range.)} \]

\[ \approx \text{ Much longer as the time span in which the interaction takes place.} \]

\[ \text{ (Typically one can really calculate only finite at } t_{0} \text{, but one has to be}\]

\[ \text{ aware of the approximation.)} \]

2. When problem arise i.e. (a) localized wave packets & finite range potential.

\[ S(\xi_{1} \rightarrow \xi_{2}, t_{1}, t_{2}) \text{ then finite } T \rightarrow \infty \]

\[ \rightarrow \text{ S-matrix element } \]

\[ \text{ amplitude that a free incoming particle at } t_{1} \rightarrow -\infty \]

\[ \text{ with momentum } \xi_{1} \text{ is later at } t_{2} \rightarrow +\infty \text{ detected as a free} \]

\[ \text{ outgoing particle with momentum } \xi_{2} \text{.} \]

\[ 2 \rightarrow \text{ } \text{In the case } t_{1} \rightarrow -\infty \text{, then} \]

\[ \text{ the potential is negligible.} \]

\[ \begin{align*}
S_{\text{inc}}(t_{1}, x) &= \frac{1}{2m_{0}} e^{i \frac{-E_{1}t_{1}}{2m_{0}}} \\
S_{\text{out}}(t_{2}, x) &= \frac{1}{2m_{0}} e^{i \frac{-E_{2}t_{2}}{2m_{0}}} \\
E_{1} &= \frac{\xi_{1}^{2}}{2m_{0}} \\
E_{2} &= \frac{\xi_{2}^{2}}{2m_{0}} \quad \text{(initial state)}
\end{align*} \]

\[ \begin{align*}
S_{\text{tot}}(t_{1}, x, t_{2}, x) &= \frac{1}{2m_{0}} e^{i \frac{-E_{1}t_{1}}{2m_{0}}} \\
S_{\text{out}}(t_{2}, x) &= \frac{1}{2m_{0}} e^{i \frac{-E_{2}t_{2}}{2m_{0}}} \\
E_{1} &= \frac{\xi_{1}^{2}}{2m_{0}} \\
E_{2} &= \frac{\xi_{2}^{2}}{2m_{0}} \quad \text{(inertial final state)}
\end{align*} \]

2. We want the time-evolved state at \( t = t_{2} \) that starts as \( S_{\text{inc}}(t_{1}, x) \) : \( S_{\text{inc}}^{(t_{1}, x)} \)

\[ \begin{align*}
(\frac{\partial}{\partial t} - H_{0} - V(t, x)) \phi_{\text{inc}}^{(t_{1}, x)} &= 0 \quad \text{for } t_{1} \text{ and } \phi_{\text{inc}}^{(t_{1}, x)} = \phi_{\text{inc}}(t_{1}, x)
\end{align*} \]

\[ \text{so} \quad (\frac{\partial}{\partial t} - H_{0}) \phi_{\text{inc}}^{(t_{1}, x)} = V(t, x) \phi_{\text{inc}}^{(t_{1}, x)} \]
Configuration space Feynman rules:

\[ G(x, y, t, \xi) \quad \text{propagator} \]
\[ -i V(x) \quad \text{vertex factor} \]
\[ \Psi_i(x, t) \quad \text{incoming wave} \]
\[ \Psi_{ou}(x, t) \quad \text{outgoing wave} \]

Additional rule: \( \times \) with sum over all intermediate states \( t \) and \( \xi \)

Hamiltonian space Feynman rules:

\[ G(x, y, t, \xi) = \frac{1}{i} \frac{\delta}{\delta \xi(x)} \quad \text{propagator} \]
\[ \left( -i \right) \frac{\delta}{\delta \xi(x)} \left( \sum_{-\infty}^{\infty} \right) \quad \text{vertex factor} \]
\[ = \frac{1}{i \omega_0} \int \text{d}x \text{d}x' \exp \left( -\frac{i}{\hbar} \int_{x(t)}^{x'(t')} \right) V(x) \quad q(x) = \frac{\text{d}x}{\text{d}t} \]

Additional rule: \( \times \) with sum over all intermediate states \( t \) and \( \xi \)

Dirac application: time-independent potential for 2-2 scattering

\[ V(t) = V(x) \]
\[ \Rightarrow \Psi(t) = \frac{1}{\sqrt{(2\pi \hbar)^2}} \int \text{d}x' e^{it\frac{x'^2}{2\hbar}} V(x') = (2\pi) \delta(p_f - p_i) \] \[ \text{energy conservation!} \]

This justifies precisely our treatment for 2-particle scattering in Sec 2.2.

Everything is derived in analogy to 1-particle scattering:

\[ \Psi_\text{in}(t_0, \xi) \sim \frac{1}{\sqrt{(2\pi \hbar)^2}} e^{-it_0 \frac{p_0^2}{2\hbar}} \]
\[ \Psi_{\text{fin}}(t_\infty, \xi) \sim \frac{1}{\sqrt{(2\pi \hbar)^2}} e^{it_\infty \frac{p_\infty^2}{2\hbar}} \] \[ E_{\text{fin}} = \frac{E_0 \pm \hbar k}{2\mu} \] (initial condition)

\[ \Psi_{\text{fin}}(t_0, \xi) \sim \frac{1}{\sqrt{(2\pi \hbar)^2}} e^{it_0 \frac{p_0^2}{2\hbar}} \]
\[ \Psi_{\text{fin}}(t_\infty, \xi) \sim \frac{1}{\sqrt{(2\pi \hbar)^2}} e^{it_\infty \frac{p_\infty^2}{2\hbar}} \] \[ E_{\text{fin}} = \frac{E_0 - \hbar k}{2\mu} \] (measurement final state)
2.4. The Interaction Picture

**So far we have used the Schrödinger picture to solve the scattering problem.**

The interaction picture is a more efficient way to formulate perturbation theory for time-dependent potentials.

**What is the canonical formulation of quantum field theory?**

**2.** The final results of course equivalent to what we have obtained previously.

Recall: Time-evolution operator \( \mathcal{U}(t_1, t_2) = e^{-iH(t_2 - t_1)} \) for time-independent \( H \)

- \( t_2 > t_1 \): \( \mathcal{U}(t_2, t_1) \) propagated forward in time
- \( t_1 < t_2 \): backward

Properties:
- \( \mathcal{U}(t_2, t_1) = \mathcal{U}(t_1, t_2) \)
- \( \mathcal{U}(t_2, t_2) = 1 \)
- \( \mathcal{U}(t_2, t_1)\mathcal{U}(t_3, t_2) = \mathcal{U}(t_3, t_1) \)
- \( \mathcal{U}(t_1, t_1) = 1 \)
Consider \( H(t) = H_0 + \delta H(t) \)

\[
\uparrow \quad \uparrow
\]

"Small" time-dependent contribution

Exactly solve & time-independent (eq. free particle, harmonic oscillator)

'Umberturbed system'

**Interaction picture:**

\[
U_0(t_1,t) = e^{-\frac{i}{\hbar} H_0 (t_1 - t)} \quad \Rightarrow \text{Time evolution operator of the unperturbed system.}
\]

\[
\text{States:} \quad |\psi(t)\rangle = U_0^+(t_1,t) |\psi(t_1)\rangle \quad \text{Schrödinger picture state}
\]

\[
= U_0(t_0,t) |\psi(t)\rangle \quad \rightarrow |\psi(t)\rangle = |\psi(t_0)\rangle + t \text{ arbitrary reference time.}
\]

1. Time evolution of the interaction picture state \( |\psi(t)\rangle \) due to \( H_0 \) has been eliminated.
   They contain only the time evolution coming from \( \delta H(t) \)

\[
\text{Operator:} \quad A(t) = U_0^+(t_1,t) A(t) U_0(t_1,t)
\]

\[
= U_0(t_0,t) A(t) U_0(t_0,t) \quad \rightarrow A(t) = A_0(t)
\]

2. Interaction picture operator contains time evolution of the unperturbed \( H_0 \).

Recall: Scattering element for particle scattering in the Schrödinger picture

\[
S_G = \left\{ T \right\} \quad \text{with coupled } H(t) \text{ time-evolved in state.}
\]

\[
= \langle \psi(t), \psi(t) | \psi(t), \psi(t) \rangle
\]

\[
= \langle \psi(t), \psi(t) | U_0^{\dagger}(t_1,t) | \psi(t), \psi(t) \rangle + \langle \psi(t), \psi(t) | U_0^{\dagger}(t_1,t) \delta H(t) U_0(t_1,t) | \psi(t), \psi(t) \rangle + \ldots
\]

\[
= \langle \psi(t), \psi(t) | U_0^{\dagger}(t_1,t) | \psi(t), \psi(t) \rangle + \langle \psi(t), \psi(t) | U_0^{\dagger}(t_1,t) \delta H(t) U_0(t_1,t) | \psi(t), \psi(t) \rangle + \ldots
\]

\[
\Rightarrow \quad |\psi(t)\rangle = U_0(t_0,t) |\psi(t)\rangle
\]
\[ S_{\text{free}} = \langle \phi_{\text{free}} | T \exp \left\{ -\frac{\mu}{\hbar} \int_{t_0}^{t} \mathcal{H}_2(t) dt \right\} | \phi_{\text{free}} \rangle \]
Application: System with a small periodic perturbation

\[ H = H_0 + H_\gamma e^{i\omega t} \]

We will reconsider this case when we treat radiation transitions of the 1-atom.

Rooz approximation for S-matrix element: (only transition term) \( t_{<0} \)

\[
S_{E_0}^{1<0} = -i \left[ \frac{d}{dt} \langle f(\omega) \mid U(\omega) H_\gamma e^{i\omega t} U(\omega) \mid i(\omega) \rangle \right] \\
= -i \langle f(\omega) \mid H_\gamma e^{i\omega t} \rangle \frac{d}{dt} \langle i(\omega) \mid e^{-i\omega t} \rangle \\
= -i \langle f(\omega) \mid H_\gamma \rangle \langle i(\omega) \mid 2\pi \delta(E_F - E_i - \omega) \rangle
\]

We see:
- Periodic oscillation \( e^{i\omega t} \) corresponds to interaction that adds energy \( \omega \) into the system.
- Periodic oscillation \( e^{-i\omega t} \) corresponds to interaction that removes energy \( \omega \) out of the system.

In a more later stage we deal with interaction of the 1-atom with real photons.