Quantum Mechanics II – Lecture Notes

→ Aim: Going beyond the basics taught in Quantum Mechanics I (T2)

Syllabus

- Recap of QM 1 basics
- Feynman's path integral approach
- Scattering theory
  - cross section
  - phase shift, partial waves
  - optical theorem, S-matrix, unitarity
- Relativistic quantum mechanics
  - Klein-Gordon equation
  - Dirac equation
- Electro-magnetism in quantum mechanics
- Quantization of the electro-magnetic field
  - atomic transitions
  - time-dependent perturbation theory
- Second quantization, quantum field theory (basics)
- Quantum information

Literature

- Quantum mechanics 1 lecture notes
- These lecture notes

- F. Schwabl, Quantenmechanik I + II
- Griffith, Quantum mechanics
- Teilchenphysik-Gruppe web site "Studium"

Chapter 1: Principles of Non-Relativistic Quantum Mechanics (Recap)

1.1. State Vectors, Hilbert Space and Operators

→ At each instant in time t for a physical system there exists a state vector $\Phi(t)$, which contains all information on what one can possibly know about the system at this time.

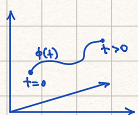
↳ Dirac-Notation (ket vector): $|\Phi\rangle$ → vector in an (in general) ∞-dimensional complex valued linear vector space $\mathcal{H}$ (→ **Hilbert space**)

Properties: $(\alpha, \beta, \gamma, \ldots \in \mathbb{C})$

- $|\phi\rangle, |\varphi\rangle \in \mathcal{H} \Rightarrow \alpha|\phi\rangle + \beta|\varphi\rangle \in \mathcal{H}$
- There exists a null vector $|0\rangle$ with $|0\rangle + |a\rangle = |a\rangle$ for all $|a\rangle \in \mathcal{H}$
- To each state $|a\rangle \in \mathcal{H}$ there is an inverse state $|b\rangle$ with $|a\rangle + |b\rangle = 0$.
- $|\phi\rangle + |\varphi\rangle = |\varphi\rangle + |\phi\rangle$ (addition commutes)
- $(|\phi\rangle + |\varphi\rangle) + |\psi\rangle = |\phi\rangle + (|\varphi\rangle + |\psi\rangle)$ (associative law for addition)
- $\alpha(\beta|\phi\rangle) = (\alpha\cdot\beta)|\phi\rangle$ (associative law for multiplication)
- $(\alpha+\beta)|\phi\rangle = \alpha|\phi\rangle + \beta|\phi\rangle$ (distributive law for multiplication)
- $\alpha(|\phi\rangle + |\varphi\rangle) = \alpha|\phi\rangle + \alpha|\varphi\rangle$ (distributive law for addition)
- For each state vector $|\phi\rangle$ the is a dual state $\langle\phi|$ that is physically equivalent and with which one can define a scalar product.
  → dual vector property: $|\varphi\rangle = \alpha|\psi\rangle + \beta|\phi\rangle \Rightarrow \langle\varphi| = \alpha^*\langle\psi| + \beta^*\langle\phi|$
- The scalar product has the properties:
  $\langle\phi|\varphi\rangle = \langle\phi||\varphi\rangle \in \mathbb{C}$
  $\langle\phi|(\alpha|\varphi\rangle + \beta|\psi\rangle) = \alpha\langle\phi|\varphi\rangle + \beta\langle\phi|\psi\rangle$
  $\langle\phi|\varphi\rangle = \langle\varphi|\phi\rangle^*$
  $\langle\phi|\phi\rangle \in \mathbb{R}$
  $\langle\phi|\phi\rangle \geq 0$ (= 0 if and only if $|\varphi\rangle = |0\rangle$)
  $|\langle\phi|\varphi\rangle|^2 \leq \langle\phi|\phi\rangle\langle\varphi|\varphi\rangle$ (Schwartz inequality)

The scalar product is used to define:
→ norm of state ϕ: $\|\phi\| \equiv \langle\phi|\phi\rangle^{1/2}$
→ orthogonality: ϕ and φ (both not null vectors) are orthogonal if $\langle\phi|\varphi\rangle = 0$
↳ ϕ and φ are then also physically inequivalent

States can be time-dependent and are therefore the basis to describe the dynamics of any system.



Note: The ground (vacuum) state of the harmonic oscillator is frequently also written as $|0\rangle$ (for $n=0$), but it is not a null state.

Orthonormal basis (ONB):

A ONB for the Hilbert space $\mathcal{H}$ is a set of state vectors $\{|n\rangle\}$ with

n: counting variable (countable or continuous)

$$\langle n|n'\rangle = \begin{cases} \delta(n-n') & , n \text{ and } n' \text{ continuous} \\ \delta_{nn'} & , \text{otherwise} \end{cases}$$

and the property $\displaystyle\sum\!\!\!\!\!\!\int_n |n\rangle\langle n| = \mathbb{1}$ (completeness relation)

$$\hookrightarrow |\phi\rangle = \sum\!\!\!\!\!\!\int_n |n\rangle\langle n|\phi\rangle = \sum\!\!\!\!\!\!\int_n \langle n|\phi\rangle \cdot |n\rangle =: \sum\!\!\!\!\!\!\int_n c_n^\phi |n\rangle \qquad |\phi\rangle \simeq \begin{pmatrix} c_1^\phi \\ c_2^\phi \\ c_3^\phi \\ \vdots \end{pmatrix}$$

$$\langle \phi|\varphi\rangle = \sum\!\!\!\!\!\!\int_n \langle\phi|n\rangle\langle n|\varphi\rangle = \sum\!\!\!\!\!\!\int_n \langle n|\phi\rangle^* \langle n|\varphi\rangle = \sum\!\!\!\!\!\!\int_n c_n^{\phi *} c_n^\varphi$$

$$\langle\phi| \simeq (c_1^{\phi *}, c_2^{\phi *}, c_3^{\phi *}, \dots)$$

1.2 Operators

→ Operators are the mathematical instrument to describe
- the dynamics of the system $\left(\frac{d}{dt}|\phi(t)\rangle = \dots\right)$
- the outcome of measurements

Linear operator: An operator $A: \mathcal{H} \to \mathcal{H}$ that assigns each vector $\psi \in \mathcal{H}$ the vector $A\psi \in \mathcal{H}$ is a linear operator, if

$$A(c_1\psi_1 + c_2\psi_2) = c_1 A\psi_1 + c_2 A\psi_2 \quad \forall\, \psi_{1,2} \in \mathcal{H} \text{ and } c_{1,2} \in \mathbb{C}$$

$\hookrightarrow$ Let $\{|n\rangle\}$ be ONB:

$$A|\phi\rangle = \sum\!\!\!\!\!\!\int_n |n\rangle\langle n|A\phi\rangle = \sum\!\!\!\!\!\!\int_n \sum\!\!\!\!\!\!\int_m |n\rangle\langle n|A|m\rangle\langle m|\phi\rangle \qquad A \simeq \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \\ a_{31} & a_{32} & a_{33} & \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

$$= \sum\!\!\!\!\!\!\int_n \sum\!\!\!\!\!\!\int_m \underbrace{\langle n|A|m\rangle}_{\text{operator matrix elements}}\langle m|\phi\rangle \cdot |n\rangle =: \sum\!\!\!\!\!\!\int_n \sum\!\!\!\!\!\!\int_m a_{nm}\, c_m^\phi \cdot |n\rangle$$

Adjoint operator: The operator A^+ is called the adjoint operator to the linear operator A, if (and only if)
aoi

$$\langle\phi|A^+\psi\rangle = \langle A\phi|\psi\rangle \quad \forall\, \phi,\psi \in \mathcal{H} \qquad A^+ \simeq \text{"A-dagger"}$$

$\hookrightarrow$ The following rules apply: $(aA+bB)^+ = a^*A^+ + b^*B^+$, $\quad A^+ = (A^T)^* = (A^*)^T$ (matrix notation)
$(AB)^+ = B^+A^+$
$(A^+)^+ = A$

Hermitian (or self-adjoint) operator: A linear operator A is called Hermitian, if aoi $A^+ = A$

Unitary operator: A linear operator is called unitary if aoi $UU^+ = U^+U = \mathbb{1}$

Eigenvalues and eigenvectors: $A\ \psi \in \mathcal{H}$ is called eigenvector to the linear operator A with the eigenvalue a if and it has the property ("Eig")

$$A\psi = a\psi \quad \text{for an } a \in \mathbb{C}$$

↳ If $A|\phi_n\rangle = a_n|\phi_n\rangle$, $A|\phi_m\rangle = a_m|\phi_m\rangle$ and $a_n = a_m$ for $n \neq m$, then the eigenvalue a_n is called degenerate.

↳ The set of eigenvalues of a linear operator A is called the spectrum of A.

↳ Eigenvalues of Hermitian operators are real.

Proof: $\langle\psi|A\psi\rangle = a\langle\psi|\psi\rangle = \langle A^+\psi|\psi\rangle = \langle A\psi|\psi\rangle = \langle\psi|A\psi\rangle^* = a^*\langle\psi|\psi\rangle \Rightarrow a = a^* \Rightarrow a \in \mathbb{R}$ ▢

↳ The eigenvectors of a Hermitian operator to two different eigenvalues $a_1 \neq a_2$ are orthogonal.

Proof: $A\psi_i = a_i\psi_i \Rightarrow \langle\psi_1|A\psi_2\rangle = \langle A\psi_1|\psi_2\rangle = a_1\langle\psi_1|\psi_2\rangle = a_2\langle\psi_1|\psi_2\rangle \Rightarrow (a_1 - a_2)\langle\psi_1|\psi_2\rangle = 0 \Rightarrow \langle\psi_1|\psi_2\rangle = 0$ ▢

Spectral Theorem: For each Hermitian operator A there is a ONB to $\mathcal{H}$ made of eigenvectors $\{|\phi_n\rangle\}$ to A, i.e. $A\phi_n = a_n\phi_n$ for some $a_n \in \mathbb{R}$.

↳ Spectral representation: $A = \sum_n a_n |\phi_n\rangle\langle\phi_n|$

↳ A, B Hermitian and $[A,B] = AB - BA = 0$:
There exists an ONB $\{|\phi_m\rangle\}$ made of eigenvectors to both A and B,
i.e. $A|\phi_m\rangle = a_m|\phi_m\rangle$, $B|\phi_m\rangle = b_m|\phi_m\rangle$

1.3. Observables

Correspondence Principle:
Each observable quantity ("observable") in quantum mechanics is represented by a Hermitian operator.

The set of eigenvalues (spectrum) of the Hermitian operator associated to an observable in quantum mechanics is the set of measurable values of the observable.

↳ Observable A: ONB $\{|\phi_n\rangle\}$ of eigenvectors: $A|\phi_n\rangle = a_n|\phi_n\rangle$

System in normalized state $|\psi\rangle$: $|\psi\rangle = \sum_n c_n|\phi_n\rangle$, $c_n = \langle\phi_n|\psi\rangle$

→ Probability to measure value a_n (in each measurement): $P(a_n) = \sum_{\{m\}_n} |c_m|^2 = \sum_{\{m\}_n} |\langle\phi_m|\psi\rangle|^2$

sum over set of states ϕ_m which have eigenvalue a_n
(can be more than one state if a_n is degenerate)

→ Average of many repeated measurements on identical systems in the state $|\psi\rangle$:

$$\sum_n a_n |c_n|^2 = \sum_n a_n \langle\phi_n|\psi\rangle\langle\psi|\phi_n\rangle = \langle\psi| \sum_n a_n |\phi_n\rangle\langle\phi_n|\psi\rangle = \langle\psi|A|\psi\rangle =: \langle A\rangle_\psi$$

Variance of these measurements: $(\Delta A)^2 = \langle (A-\langle A\rangle_\psi)^2\rangle_\psi = \langle A^2\rangle_\psi - \langle A\rangle_\psi^2$

Standard deviation: $\Delta A = (\langle A^2\rangle_\psi - \langle A\rangle_\psi^2)^{1/2}$

→ Collapse of the state: If the measurement does act minimally, so that the system is not destroyed during the measurement, the system is after measurement of value a_n in the (still unnormalized) state

$$|\psi\rangle_{a_n} = \sum_{\{m\}_n} c_m |\phi_m\rangle = \sum_{\{m\}_n} |\phi_m\rangle\langle\phi_m|\psi\rangle$$

Complete set of physical observables

≙ Sufficiently large set of commuting Hermitian operators $\{A_1, A_2, \dots, A_n\}$ $([A_n, A_m] = 0)$ such that each eigenvector of their common (and unique) ONB can be unambiguously identified by the eigenvalues w.r. to these operators.

$$A_i|\phi_m\rangle = a_i^m|\phi_m\rangle \Rightarrow |\phi_m\rangle \simeq |a_1^m, a_2^m, \dots, a_n^m\rangle$$

The set $\{a_1^m, a_2^m, \dots, a_n^m\}$ are called the quantum numbers of state $|\phi_m\rangle$.

Spin-less particle:

↳ Complete set: $\{\vec{X}\}$, $[X_i, X_j] = 0$, $\vec{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$, $\int d^3\vec{x}\, |\vec{x}\rangle\langle\vec{x}| = \mathbb{1}$, $\langle\vec{x}|\vec{y}\rangle = \delta^{(3)}(\vec{x}-\vec{y})$

$$|\psi\rangle = \int d^3\vec{x}\, |\vec{x}\rangle\overbrace{\langle\vec{x}|\psi\rangle}^{\psi(\vec{x})} = \int d^3\vec{x}\, \psi(\vec{x})|\vec{x}\rangle \qquad \psi(\vec{x}): \text{ Config. space wave function}$$

→ Interpretation of $\psi(\vec{x})$: $\frac{1}{\langle\psi|\psi\rangle}|\psi(\vec{x})|^2 d^3\vec{x}$ ≙ probability that particle is found at $\vec{x}$ within volume element $d^3\vec{x}$ in a location measurement

Complete set: $\{\vec{P}\}$, $[P_i, P_j] = 0$ $\quad \vec{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$, $\int d^3\vec{p}\, |\vec{p}\rangle\langle\vec{p}| = 1$, $\langle\vec{p}|\vec{q}\rangle = \delta^{(3)}(\vec{p}-\vec{q})$

$$|\psi\rangle = \int d^3\vec{p}\, |\vec{p}\rangle\overbrace{\langle\vec{p}|\psi\rangle}^{\tilde{\psi}(\vec{p})} = \int d^3\vec{p}\, \tilde{\psi}(\vec{p})|\vec{p}\rangle \qquad \tilde{\psi}(\vec{p}): \text{ mom. space wave function}$$

$$\langle\vec{x}|\vec{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{p}\vec{x}/\hbar},$$

$$\langle\vec{x}|\vec{P}|\vec{y}\rangle = \int d^3\vec{p}\, \vec{p}\langle\vec{x}|\vec{p}\rangle\langle\vec{p}|\vec{y}\rangle = i\hbar\frac{\partial}{\partial\vec{y}}\langle\vec{x}|\vec{y}\rangle = i\hbar\frac{\partial}{\partial\vec{y}}\delta^{(3)}(\vec{x}-\vec{y}) = -i\hbar\frac{\partial}{\partial\vec{x}}\delta^{(3)}(\vec{x}-\vec{y})$$

$$\langle\vec{p}|\vec{X}|\vec{q}\rangle = \int d^3\vec{x}\, \vec{x}\langle\vec{p}|\vec{x}\rangle\langle\vec{x}|\vec{q}\rangle = -i\hbar\frac{\partial}{\partial\vec{q}}\langle\vec{p}|\vec{q}\rangle = -i\hbar\frac{\partial}{\partial\vec{q}}\delta^{(3)}(\vec{p}-\vec{q}) = i\hbar\frac{\partial}{\partial\vec{p}}\delta^{(3)}(\vec{p}-\vec{q})$$

particle with spin-$\frac{1}{2}$:

complete set: $\{\vec{X}, \vec{n}\vec{S}\}$, $[X_i, \vec{n}\vec{S}] = 0$, $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ (Pauli matrices), $|\vec{n}|^2 = 1$, $\vec{n}$: quantization axis

$$|\vec{x}, \pm\rangle = |\vec{x}\rangle \underset{\text{direct prod.}}{\otimes} |\pm\rangle \quad , \quad \vec{X}|\vec{x}, \pm\rangle = \vec{x}|\vec{x}, \pm\rangle \quad , \quad \vec{n}\vec{S}|\vec{x}, \pm\rangle = \pm\frac{\hbar}{2}|\vec{x}, \pm\rangle$$

$$|\psi\rangle = \sum_{s=\pm} \int d^3\vec{x}\, \underbrace{\psi(\vec{x}, s)}_{\langle \vec{x}, s|\psi\rangle} |\vec{x}, s\rangle = \int d^3\vec{x} \begin{pmatrix} \psi(\vec{x},+) \\ \psi(\vec{x},-) \end{pmatrix} |\vec{x}\rangle$$

↳ Observables suitable as quantum numbers:

$\vec{X}$ (location), $\vec{P}$ (momentum), $\vec{L}$ (angular momentum), $\vec{S}^2$, $\vec{n}\vec{S}$ (spin)

but also: electric charge Q, weak isospin I, color charge, ...

Canonical Commutation Relations: $[X_i, P_j] = i\delta_{ij}\hbar$, $[X_i, X_j] = [P_i, P_j] = 0$

1.4. Temporal Dynamics

Schrödinger Picture (S)

→ mostly used for time-independent, stationary problems
bound states, exactly solvable problems
→ standard picture for construction of observable operators
using the correspondence principle

Schrödinger equation:

$$\boxed{i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle}$$

H: Hamilton operator (i.g. fct of time)
Determined from the total energy and the correspondence principle

$$E = \underbrace{T + V + \ldots}_{H = H(\vec{P}, \vec{X}, \ldots)}$$
$$E \to i\hbar\frac{\partial}{\partial t}$$

Observables: time-independent
States: time-dependent

special case: H time-independent → $|\psi(t)\rangle_s = \exp\left(-i\frac{H(t-t_0)}{\hbar}\right)|\psi(t_0)\rangle_s = U(t, t_0)|\psi(t_0)\rangle_s$

↳ There is a ONS of energy eigenstates to H.

unitary time evolution operator $U(t, t_0)$

Separation ansatz: $\psi_s(\vec{x}, t) = f(t)\psi(\vec{x})$

eigenfunction to some energy eigenvalue E

↳ $i\hbar\frac{\partial}{\partial t} f(t)\psi(\vec{x}) = H f(t)\psi(\vec{x})$

$$\Leftrightarrow \quad \underbrace{i\hbar \frac{1}{f(t)} \frac{\partial}{\partial t} f(t)}_{\text{indep. of } \vec{x}} = \underbrace{\frac{1}{\psi(\vec{x})} H \psi(\vec{x})}_{\text{indep. of } t} = E = \text{const}$$

$\hookrightarrow \quad f(t) = e^{-i\frac{Et}{\hbar}} \quad , \quad \boxed{H\psi(\vec{x}) = E\,\psi(\vec{x})}$ time-independent Schrödinger equation

$\hookrightarrow \quad |\psi_s(\vec{x},t)|^2 = |\psi(\vec{x})|^2$ stationary

see Stern-Gerlach experiment
Chap 5.3. QM1-lecture

Heisenberg Picture (H)

→ very useful to calculate time-evolution of quantum effects in "almost" classical problems
→ Does not involve determination of eigenfunctions to H.

Observables: $A_H(t) = U^\dagger(t)\, A_S\, U(t) \qquad U(t) = \exp(-i\frac{Ht}{\hbar})$ (reference time $t_0 = 0$)

Observable operator in the Schrödinger picture

Heisenberg equation:

$\neq 0$ if A_S time-dependent

$$\boxed{\frac{d}{dt} A_H(t) = \frac{i}{\hbar}[H, A_H] + \frac{\partial}{\partial t} A_H}$$

$H_H = H_S$ (always!)

States: $|\psi\rangle_H = U^\dagger(t)|\psi(t)\rangle_S = |\psi(t_0)\rangle_S$

Observables: time-dependent
States: time-independent

Interaction Picture (I)

(Dirac representation)

$H = H_0 + \delta H(t)$

exactly solvable Hamilton

time-dependent perturbation: $\delta H \ll H_0$

→ For time-dependent problems where the time-dependence cannot be solved exactly, but leads to "small" effects (suitable to perturbative treatment)

$\hookrightarrow \quad |\psi(t)\rangle_I = e^{i\frac{H_0 t}{\hbar}}|\psi(t)\rangle_S$ ← state contains only "non-trivial" time dependence from δH

$A_I(t) = e^{i\frac{H_0 t}{\hbar}} A_S(t)\, e^{-i\frac{H_0 t}{\hbar}}$ ← operator contains "trivial" time-dependence from H_0

$\hookrightarrow \quad i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle_I = e^{i\frac{H_0 t}{\hbar}}(-H_0 + H)|\psi(t)\rangle_S = \delta H\,|\psi(t)\rangle_I \qquad ([H_0, \delta H] = 0)$

$\frac{d}{dt} A_I(t) = \frac{i}{\hbar}[H_0, A_I(t)] + \frac{\partial}{\partial t} A_I(t)$

→ Later in lecture for describing time-dependent processes that are rare / slow.

1.5. Units

The fundamental constants of nature

$$\hbar = \frac{h}{2\pi} = 6.582 \cdot 10^{-16}\ eV \cdot s \qquad \text{(reduced Plank constant)}$$

$$c = 2.998 \cdot 10^{8}\ \frac{m}{s} \qquad \text{(speed of light)} \rightarrow \text{in vacuum}$$

$$k = 8.617 \cdot 10^{-5}\ \frac{eV}{K} \qquad \text{(Boltzmann constant)},$$

(K ← Kelvin)

intrinsically connect the quantities distance, time, energy, momentum, mass, temperature.

1 eV = kinetic energy a particle with one electric elementary charge e (e.g. proton p^+, electron e^-, ..) acquires / loses when going through a potential difference of 1 V (volt).

Classical mechanics: energy (E) and time (t), momentum (p) and distance (x)

↳ are unrelated quantities, can be measured using independent conventions

Quantum mechanics: quantum fluctuations and dynamics determined by products energy × time ($E \cdot t$), momentum × distance ($p \cdot x$) in units of $\hbar$

⇝ e.g. wave fct of free particle with mass μ and momentum $\vec{p}$:

$$\Psi(\vec{x},t) = e^{-i\frac{Et}{\hbar}}\, e^{i\frac{\vec{p}\vec{x}}{\hbar}}, \quad E = \frac{p^2}{2\mu} \quad \xrightarrow{\hbar = 1} \quad \Psi(\vec{x},t) = e^{-iEt}\, e^{i\vec{p}\vec{x}}$$

↳ We can adopt units such that $[E] = [t]^{-1}$, $[p] = [x]^{-1}$ ⇝ $\boxed{\hbar = 1}$

Non-relativistic physics: time (t) and distance (x), energy (E) and momentum (p)

↳ are unrelated quantities, can be measured using independent conventions

Relativistic physics: Speed of light c (in vacuum) universal, independent of reference frame

⇝ We can quantify time separations through the distance light travels.

mass is just another form of energy : $E = mc^2$

↳ We can adopt units such that $[t] = [x]^{-1}$, $[E] = [m]$ ⇝ $\boxed{c = 1}$

Recall: $[E_{kin}] = [\frac{p^2}{m}]$ ⇒ $[E] = [p]$

↳ $\boxed{\text{"Natural units"}: \hbar = c = 1}$ → All quantities can be measured e.g. in units $(eV)^{\text{some power}}$ ⇝ Simplifies notation

<u>Note 1:</u> One can always recover (unambiguously) the proper powers of $\hbar$ and c in all formulae.

<u>Note 2:</u> Classical mechanics and non-relativistic kinematics are not "wrong" physical theories but the correct physical theories in certain limits

classical mechanics $\hat{=}$ leading order in an expansion in $\hbar$ (for classical systems)

non-relativistic kinematics $\hat{=}$ leading order in an expansion in $\frac{1}{c}$ (for non-rel systems)

$\Rightarrow$ Classical mechanics, non-rel. kinematics are "effective theories" of more fundamental and more widely applicable theories

$\Rightarrow$ Basic property of an effective theory: One can (in principle) determine all higher order corrections in the expansion parameter ($\hbar$, c, ...)