Exercises to QM2, Summer Term 2018, Sheet 9

1) Lagrangian for a Dirac particle

The Lagrangian density for a classic massive and charged Dirac particle has the form

$$\mathcal{L} \,=\, \bar{\psi}(i\partial\!\!\!/ - m)\psi\,,$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ is the adjoint Dirac field and one can consider ψ and $\bar{\psi}$ as independent.

(a) Use the Euler-Lagrange equations to derive the equation of motion for ψ and $\bar{\psi}$ and show that you correct forms of the Dirac equation.

(b) Determine the generalized momenta $\bar{\pi}$ for ψ and π for $\bar{\psi}$. (The bar superscript concerning complex conjugation is the usual convention.)

(c) Show that the Hamiltonian has the form

$$H \,=\, \int d^3\vec{\mathbf{x}}\,\psi^\dagger\Big[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\Big]\psi\,,$$

where α^i and β are the Dirac matrices discussed in class.

(d) Calculate the Hamiltonian (by doing the integration over $\vec{\mathbf{x}}$) for the general particle/antiparticle field

$$\psi(x) \,=\, \sum_s \int d^3\vec{\mathbf{k}}\,\Big[\,a(\vec{\mathbf{k}},s)\,\psi^{(+)}_{\vec{\mathbf{k}},s}(x) \,+\, b^*(\vec{\mathbf{k}},s)\,\psi^{(-)}_{\vec{\mathbf{k}},s}(x)\,\Big]\,.$$

Identify what is highly problematic about this result.

(e) Think about the promoting the Dirac particle field theory to a quantum field theory. Which problem will arise when using commutation relations assuming that $a(\vec{\mathbf{k}},s)$ and $b(\vec{\mathbf{k}},s)$ annihilate a particle and anti-particle, respectively, and $a^\dagger(\vec{\mathbf{k}},s)$ and $b^\dagger(\vec{\mathbf{k}},s)$ create a particle and anti-particle, respectively? Show how this problem is resolved when considering instead anti-commutation relations.

(f) Calculate the quantum field theory the equal-time anticommutators $\{\psi_s(t,\vec{x}),\psi^\dagger_{s'}(t,\vec{x}')\}$, $\{\psi_s(t,\vec{x}),\psi_{s'}(t,\vec{x}')\}$ and $\{\psi^\dagger_s(t,\vec{x}),\psi^\dagger_{s'}(t,\vec{x}')\}$.

2) Diracology

(a) Verify the following relations for the γ matrices. Note that for a number of cases you just need to use their representation-independent property $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$:

(1) $(\gamma^\mu)^\dagger \,=\, \gamma^0\gamma^\mu\gamma^0$,

(2) $\gamma_\alpha\gamma^\mu\gamma^\alpha \,=\, -2\gamma^\mu$,

(3) $\gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\alpha \,=\, 4g^{\mu\nu}$

(4) $\mathrm{Tr}[\gamma^\mu\gamma^\nu] \;=\; 4g^{\mu\nu}$

(5) $\not{a}\not{b} = a\cdot b - i\sigma_{\mu\nu}a^\mu b^\nu$, where $a\cdot b \equiv a_\mu b^\mu$, $\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu]$

(b) Verify the following properties of the momentum space Dirac spinors, where we have $p^\mu = (E_p, \vec{\mathbf{p}})$:

(1) $(\not{p} - m)\, u(\vec{\mathbf{p}}, s) \;=\; 0\,, \quad (\not{p} + m)\, v(\vec{\mathbf{p}}, s) \;=\; 0$

(2) $\bar{u}(\vec{\mathbf{p}}, s)\, (\not{p} - m) \;=\; 0\,, \qquad \bar{v}(\vec{\mathbf{p}}, s)\, (\not{p} + m) \;=\; 0$