Exercises to QM2, Summer Term 2018, Sheet 8

1) Lagrangian for a Klein-Gordon particle

The Lagrangian density for a complex (i.e. charged) Klein-Gordon field has the form

$$\mathcal{L} = \left(\partial_\mu \phi^*\right)\left(\partial^\mu \phi\right) - m^2 \phi^* \phi \,,$$

where the two fields ϕ and (the complex conjugated) ϕ^* are regarded as two independent fields.

(a) Use the Euler-Lagrange equations to derive the equation of motion for ϕ and ϕ^* and show that you obtain Klein-Gordon equations.

(b) Determine the generalized momenta π^* for ϕ and π for ϕ^*. (The star superscript concerning complex conjugation is the usual convention.)

(c) Show that the Hamiltonian has the form

$$H = \int d^3\vec{\mathbf{x}} \left[\pi^* \pi + (\vec{\nabla}\phi^*)(\vec{\nabla}\phi) + m^2 \phi^* \phi\right]$$

and can be rewritten as

$$H = -\int d^3\vec{\mathbf{x}}\, \phi^* \overset{\leftrightarrow}{\partial_t} \left(\frac{\partial \phi}{\partial t}\right)$$

with the help of the Klein-Gordon equation for field ϕ.

(d) Calculate the Hamiltonian (by doing the integration over $\vec{\mathbf{x}}$) for the general particle/antiparticle field

$$\phi(x) = \int d^3\vec{\mathbf{k}} \left[\, a_+(\vec{\mathbf{k}})\, \Psi^{(+)}_{\vec{\mathbf{k}}}(x) \,+\, a^*_-(\vec{\mathbf{k}})\, \Psi^{(-)}_{\vec{\mathbf{k}}}(x) \right]$$

2) Klein-Gordon "Hydrogen atom"

We calculate the bound state energies E of a relativistic spinless "electron" with electric charge $Q = -e$ and mass m trapped in the Coulomb potential $V(\vec{\mathbf{x}}) = -e\phi(\vec{\mathbf{x}}) = -\alpha/|\vec{\mathbf{x}}|$ of another much heavier particle. Because we consider the relativistic case E contains the mass of the electron, but it does not contain the mass of the much heavier central particle (which is in fact so heavy that it is nonrelativistic). Assume that the state of the bound state is stationary with total energy $E < m$. Also recall that here we can use the Klein-Gordon (KG) equation because we are having a problem where particle-antiparticle creation cannot happen. (If the latter is not clear to you, compare the problem with the potential-step problem.)

(a) Start with the usual partial wave ansatz for the stationary solution:

$$\Psi_{lm}(x) = e^{-iEt}\,\frac{u_l(\gamma r)}{r}\, Y_{lm}(\theta, \phi)$$

and show that the resulting radial KG equation can can be written in the form

$$\frac{d^2u_l(\rho)}{d\rho^2} + \left[\frac{2ZE\alpha}{\gamma\,\rho} - \frac{1}{4} - \frac{l(l+1)-(Z\alpha)^2}{\rho^2}\right] u_l(\rho) \;=\; 0$$

with $\gamma^2 = 4(m^2 - E^2)$ and $\rho = \gamma r$.

(b) Use for the radial wave function the ansatz

$$u_l(\rho) \;=\; \rho^k(1 + c_1\rho + c_2\rho^2 + c_3\rho^3 + \cdots)\, e^{-\rho/2}$$

and show that you can obtain the conditions $k = \frac{1}{2} \pm ((l+\frac{1}{2})^2 - \alpha^2)^{1/2}$ demanding a consistent behavior for $\rho \to 0$. Show from demanding that the kinetic energy is finite,

$$\int dr\; r^2 \left[\frac{d}{dr}\frac{u_l(r)}{r}\right] \;<\; \infty\,,$$

that only the $+$ solution is relevant.

(c) Determine the recursion relation for the coefficients c_i and show that, for the series to break off at some order in ρ, one obtains the energy condition

$$E \;=\; \frac{m}{\left(1 + (Z\alpha)^2 \Big/ \left[n - l - \frac{1}{2} + \sqrt{(l+\frac{1}{2})^2 - (Z\alpha)^2}\,\right]^2\right)^{1/2}}\,.$$

At this point you have to be careful to properly define the radial quantum number n consistent with definition of the corresponding nonrelativsitic hydrogen atom calculation.

(d) The nonrelativistic limit of this formula is $\alpha \to 0$. Think about why this is the correct nonrelativistic limit. Determine for $\alpha \ll 1$ the energy spectrum up to (including) $\mathcal{O}(\alpha^4)$. Compare the result to the fine structure corrections we determined for the hydrogen in QM I. Recall which types of relativistic corrections have been treated for the fine structure corrections and which of these effects can possibly be contained in the KG equation with a Coulomb potential.