Exercises to QM2, Summer Term 2018, Sheet 7

1) Tensorial solid angle integration I

In class we have shown by symmetry arguments the relation

$$\int \mathrm{d}\Omega \left(\delta^{ij} - n^i n^j\right) \;=\; \frac{8\pi}{3}\,\delta^{ij}$$

in 3 (Cartesian) dimensions, where $\vec{\mathbf{n}}$ is an arbitrary unit vector. Show the relation by an explicit solid angle integration.

2) Tensorial solid angle integration II

Use rotational symmetry in 3 (Cartesian) dimensions to compute the integrals

$$\int \mathrm{d}\Omega \, (\vec{\mathbf{p}}\vec{\mathbf{n}})^{\alpha}\, n^i n^j \quad \text{and} \quad \int \mathrm{d}\Omega \, \frac{n^i n^j}{(1+\vec{\mathbf{p}}\vec{\mathbf{n}})^2}$$

where α is a positive real number, $\vec{\mathbf{n}}$ is an arbitrary unit vector and $\vec{\mathbf{p}}$ is a vector with some finite length. Of coure you can proceed as in exercise (1), but using symmetry argument makes you life substantially easier.

3) Single photon state wave function

Show using the photon quantization formalism that the scalar product of a single photon state with a location operator eigenstate has the form

$$\langle \vec{\mathbf{x}} | \vec{\mathbf{p}}, \lambda \rangle \;=\; \frac{1}{(2\pi)^{3/2}}\, e^{i\vec{\mathbf{p}}\vec{\mathbf{x}}}\,,$$

where $|\vec{\mathbf{p}}, \lambda\rangle = a^{\dagger}(\vec{\mathbf{p}}, \lambda)|0\rangle$.

4) Two-particle Fock space

Show that the two-particle Fock space operator

$$P^{(2)} \;=\; \frac{1}{2}\sum_{\lambda_1\lambda_2}\int d^3\vec{\mathbf{p_1}}\, d^3\vec{\mathbf{p_2}}\, |\vec{\mathbf{p_1}}, \lambda_1; \vec{\mathbf{p_2}}, \lambda_2\rangle\langle \vec{\mathbf{p_1}}, \lambda_1; \vec{\mathbf{p_2}}, \lambda_2|$$

is indeed a projection operator, as stated in class.

5) Partial derivative as a 4-vector

Show that the partial derivative $\partial^{\mu} = \partial/\partial x_{\mu}$ is a 4-vector. To prove this you can make the assumption that x^{μ} is a 4-vector satisfying the correct Lorentz transformation property. So starting from there you need to show that $\partial^{\mu} \to \Lambda^{\mu}{}_{\nu}\partial^{\nu}$ for a Lorentz transformation.

6) Fixed-target experiment kinematics

Consider a fixed-target experiment, where particle A is accelerated and hits particle B at rest. In the reaction the particles C_1, C_2, $\dots C_n$ are being produced. Determine the minimal energy E_{min} particle A needs to have, such that the reaction can just take place. The particle masses are m_A, m_B, m_{C_1}, m_{C_2}, etc.

7) Particle decay kinematics

Particle A at rest decays into the massive particles B and C, $A \to B + C$. The particle masses are m_A, m_B and m_C, respectively. Use energy-momentum conservation to address the following questions.

(a) When can the decay happen in the first place? Determine the energy of B and C and their momentum (modulus) as a function of their masses. For the analytic formula for the momenta use the function $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$.

(b) How far can a muon that is produced in the decay of a pion at rest travel in average before it decays? You need to look up the average lifetime τ_μ of the muon from the web. Note that τ_μ is defined in the muon rest frame.

8) General form for a Lorentz transformation

The orthochonous and proper Lorentz transformations (i.e. boosts and rotations) can be written down in a very compact and useful form as the expression

$$\Lambda^\mu{}_\nu = \exp\left[\frac{1}{2}\omega_{\alpha\beta}(\tilde{J}^{\alpha\beta})^\mu{}_\nu\right]$$

where $(\tilde{J}^{\alpha\beta})_{\mu\nu} = \delta^\alpha{}_\mu\delta^\beta{}_\nu - \delta^\alpha{}_\nu\delta^\beta{}_\mu$. Note the location of the index μ and that you need the metric tensor to lower or raise indices. Also note that the notation of the μ-ν indices inside the exponential means that the exponentiation happens on these indices, so e.g. $(A^\mu{}_\nu)^2$ actually means $A^\mu{}_\sigma A^\sigma{}_\nu$. The choice for $\omega_{\alpha\beta}$ determines which kind of transformation you have. Here $\omega_{\alpha\beta}$ has three parameters for spatial rotations and three for boosts in the form $(i, j = 1, 2, 3)$

$$\omega_{\alpha\beta} = \begin{pmatrix} 0 & \omega_{0j} \\ \omega_{i0} & \omega_{ij} \end{pmatrix}.$$

(Why is $\omega_{00} = 0$?) A spatial rotation has $\omega_{ij} = \theta_k\epsilon_{kij}$ with the convention $\epsilon_{123} = 1$ and all other entries of $\omega_{\alpha\beta}$ being zero. Determine $\Lambda^\mu{}_\nu$ for a rotation by the angle θ around the z-axis, i.e. $\vec{\theta} = (0, 0, \theta)$. For this it is useful to first derive a matrix form for $(\tilde{J}^{12})^\mu{}_\nu$ and determine the matrix that has to be exponentiated.