Exercises to QM2, Summer Term 2018, Sheet 6

You may take $c = 1$ on this exercise sheet.

1) Lagrange density for the electromagnetic field

The action S for the electromagnetic field coupled to some conserved electric 4-current in Heaviside-Lorentz units has the form

$$S = \int \mathrm{d}^4x\, \mathcal{L}\Big(A^\mu(x), \partial^\mu A^\nu(x)\Big)\,, \quad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu\,, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,.$$

(a) Derive from the least action principle form of the general Euler-Lagrange equations:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0\,.$$

(b) Derive from the Euler-Lagrange equations the Maxwell equations $\partial_\mu F^{\mu\nu} = j^\mu$.

(c) Show that the action is indeed gauge invariant under transformations $A^\mu(x) \to A^\mu(x) + \partial^\mu\Lambda(x)$, where $\Lambda(x)$ is an arbitrary (well behaved) scalar function. Which assumption (!) in addition to using periodic boundary conditions in the finite box picture do you have to make so that you can show gauge-invariance? Think of the following question: Is the fact that the current is conserved a consequence of or a condition for gauge-invariance?

2) Lagrange density for the electromagnetic field in Coulomb gauge I

Start from the Lagrangian from exercise (1) accounting for the effects of the electric current j^μ.

(a) The generalized momentum conjugates of the vector fields $A^\mu(x)$ are defined as

$$\pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^\mu)},\,.$$

Calculate the generalized momentum conjugates for all four comments of A^μ. What do the results mean physically?

(b) Since A^0 is not a physically independent degree of freedom, it can be expressed in terms of other quantities. Use the Maxwell equations $\partial_\mu F^{\mu\nu} = j^\mu$ to derive the equation of motion for A^0 and show that, applying the Coulomb gauge condition $\nabla\mathbf{A} = 0$, the solution for A^0 is

$$A^0(t,\mathbf{x}) = \frac{1}{4\pi}\int \mathrm{d}^3\mathbf{x}\,\frac{\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,,$$

where the 4-current has the form $j^\mu(t,\mathbf{x}) = (\rho(t,\mathbf{x}), \mathbf{j}(t,\mathbf{x}))$.

(c) Derive the equation of motion for $\mathbf{A}(t,\mathbf{x})$.

3) Lagrange density for the electromagnetic field in Coulomb gauge II

Start from the Lagrangian from exercise (1) accounting for the effects of the electric current j^μ.

Calculate the explicit form of the Lagrangian $L = \int \mathrm{d}^3\mathbf{x}\,\mathcal{L}(t,\mathbf{x})$ in Coulomb gauge and show that the result can be written in the form

$$L = \int \mathrm{d}^3\mathbf{x}\left[\frac{1}{2}\,\mathbf{E}^2(t,\mathbf{x}) - \frac{1}{2}\,\mathbf{B}^2(t,\mathbf{x}) + \mathbf{j}_\perp(t,\mathbf{x})\cdot\mathbf{A}(t,\mathbf{x})\right] - \frac{1}{8\pi}\int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}'\,\frac{\rho(t,\mathbf{x})\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,,$$

where the $\mathbf{E}$, $\mathbf{B}$ fields are defined exactly as in the vacuum case discussed in class,

$$\mathbf{j}_\perp(t,\mathbf{x}) \equiv \mathbf{j}(t,\mathbf{x}) - \nabla\partial_t A^0(t,\mathbf{x})\,,$$

and you have to use the explicit form of A^0 derived in exercise (2). Note that you may have to use the integration-by-parts trick several times. Explain the reasons why you can use it. Argue why one can drop the second term in the definition of $\mathbf{j}_\perp$, so that $\mathbf{j}_\perp = \mathbf{j}$.

4) Hamilton operator for an electron, proton and the electromagnetic field

Assume now that you have an electron and a proton located at the generalized coordinates $\mathbf{q}_e(t)$ and $\mathbf{q}_p(t)$, so that the electric current adopts the form

$$j^\mu(t,\mathbf{x}) = -\,e\,\delta^{(3)}(\mathbf{x}-\mathbf{q}_e(t))\,(1,\dot{\mathbf{q}}_e(t)) + e\,\delta^{(3)}(\mathbf{x}-\mathbf{q}_p(t))\,(1,\dot{\mathbf{q}}_p(t))\,,$$

and add the to the Lagrangian of exercise (3) the (non-relativistic) Lagrangian for a free electron and a free proton. expressed as a function of the $\mathbf{q}_i$ and their generalized velocities. Use canonical Lagrange formalism to show that the Hamiltonian for this system can be written in the form

$$\begin{aligned} H \;=\; & \int \mathrm{d}^3\mathbf{x}\left[\frac{1}{2}\,\mathbf{E}^2(t,\mathbf{x}) + \frac{1}{2}\,\mathbf{B}^2(t,\mathbf{x})\right] - \frac{1}{4\pi}\frac{e^2}{|\mathbf{q_e}(\mathbf{t})-\mathbf{q_p}(\mathbf{t})|} \\ & + \frac{1}{2m_e}\Big(\mathbf{p_e}(t) + e\mathbf{A}(t,\mathbf{q_e}(t))\Big)^2 + \frac{1}{2m_p}\Big(\mathbf{p_p}(t) - e\mathbf{A}(t,\mathbf{q_p}(t))\Big)^2\,, \end{aligned}$$

where $\mathbf{p}_e$ and $\mathbf{p}_p$ are the generalized momenta of the electron and positron, respectively. To get to this result one has to discard the so-called self-energy contributions. What is the physical interpretation of these self-energies. How does the Hamiltonian look for a infinitely heavy proton at the origin?