Exercises to QM2, Summer Term 2018, Sheet 5

1) Total momentum of the electromagnetic field (Pointing vector)

Show that the total momentum of a general photon field in a finite size box with volume V has the property

$$\vec{\mathbf{P}} = \frac{1}{4\pi c} \int_V d^3\vec{\mathbf{x}}\,(\vec{\mathbf{E}} \times \vec{\mathbf{B}}) = \frac{1}{2\pi c^2} \sum_{\vec{\mathbf{k}},\lambda} \vec{\mathbf{k}}\,\omega_{\vec{\mathbf{k}}}\,|b_{\vec{\mathbf{k}},\lambda}|\,,$$

where the photon field $\vec{\mathbf{A}}$ has the form In order to use the expression for the photon field you need to have rewrite the Pointing vector in terms of $\vec{\mathbf{A}}$.

2) Heisenberg equation for the photon field operator

The (free) photon field operator, expressed in terms of creation and annihilation operators, has the form

$$\vec{A}(t,\vec{x}) = \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar c^2}{V\omega_{\vec{\mathbf{k}}}}} \left(e^{-ik\cdot x}\vec{\varepsilon}_{\vec{\mathbf{k}},\lambda} a_{\vec{\mathbf{k}},\lambda} + h.c.\right), \quad k\cdot x = \omega_{\vec{\mathbf{k}}} t - \vec{\mathbf{k}}\cdot\vec{\mathbf{x}}.\,,$$

where "h.c." refers to hermitian conjugate. Verify that the time-dependence of the photon field operator is - as it must be, if the concept of the photon field operators makes sense - described by the Heisenberg equation

$$\frac{\partial \vec{\mathbf{A}}(t,\vec{\mathbf{x}})}{\partial t} = \frac{i}{\hbar}\left[\mathbf{H}, \vec{\mathbf{A}}(t,\vec{\mathbf{x}})\right]\,.$$

3) Spatial shifts for the photon field operators I

Show that infinitesimal spatial shifts of the photon field operator are described by the operator relation

$$\nabla_j A_k(t,\vec{x}) = -\frac{i}{\hbar}\left[\mathbf{P}_j, A_k(t,\vec{\mathbf{x}})\right]\,,$$

where $\vec{\mathbf{P}}$ is the total momentum operator (for the photon field!).

4) Space-time shifts for the photon field operators

Show that *finite* space-time shifts $x^\mu \to x^\mu + a^\mu$ of the photon field operator are described by the operator relation

$$\vec{\mathbf{A}}(x+a) = e^{iP\cdot a/\hbar}\vec{\mathbf{A}}(x)\,e^{-iP\cdot a/\hbar}\,,$$

where the totel energy-momentum operator (of the photon field) is defined as $\mathbf{P}^\mu = \left(\mathbf{H}/c, \vec{\mathbf{P}}\right)$ und $x^\mu = (ct, \vec{\mathbf{x}})$. You may set $c = 1$ throughout the calculation.