Exercises to QM2, Summer Term 2018, Sheet 2

1) Measurement of the total cross section

In class we mostly discussed that the total cross section $\sigma_{\rm tot}$ can be determined from a measurement of the differential cross section $\mathrm{d}\sigma_{\rm tot}/\mathrm{d}\Omega$ which is then subsequently integrated over all solid angles. Another method is using a beam of particles that hits a very thin foil made of the scattering centres. For simplicity assume that the beam direction is perpendicular to the foil's surface. In this case $\sigma_{\rm tot}$ can be determined from the reduction of the particle intensity in the transmitted beam behind the foil. Show that the total cross section can be determined from the formula

$$\sigma_{\rm tot} \;=\; \frac{1}{\rho\, s}\ln\left(\frac{I(0)}{I(s)}\right), \tag{1}$$

where $I(x)$ is the particle intensity (=number of particles per unit area perpendicular to the beam direction per unit time) of the particle beam after distance x *in the foil*, s is the thickness of the foil and ρ the density (=number of scattering centres per unit volume) of scattering centres of the foil. So $I(0)$ is the beam intensity before it hits the foil. Why is the assumption of a thin foil relevant?

Hint: Determine an expression for $I(x+\delta x)$ assuming that $I(x)$ is known using the considerations discussed in class concerning the alternative interpretation of the total cross section. Derive from that relation the differential equation for $I(x)$ in x.

2) Gaussian integral

Prove the Gaussian integral formula

$$I(a,b) = \int_{-\infty}^{\infty} dx\, \exp(-ax^2+bx) \;=\; \sqrt{\frac{\pi}{a}}\,\exp\left(\frac{b^2}{4a}\right), \quad a,b\in\mathbb{C}\,, \quad \mathrm{Re}(a)>0\,, \tag{2}$$

where $\sqrt{a}\equiv\sqrt{|a|}e^{i\theta/2}$ with $a=|a|e^{i\theta}$ and $-\pi/2<\theta<\pi/2$.

To proceed show the relation first for $I(a,0)$ by using that the expression

$$[I(a,0)]^2 = \int_{-\infty}^{\infty} dx\,\exp(-ax^2)\int_{-\infty}^{\infty} dy\,\exp(-ay^2) \tag{3}$$

can be calculated rather easily using polar coordinates. Then use the Cauchy integral theorem to determine the relation between $I(a,0)$ and the integral

$$\int_{-\infty}^{\infty} dx\,\exp(-a(x-c)^2) \quad \text{for} \quad c\in\mathbb{C}\,. \tag{4}$$

From there it is straightforward to determine $I(a,b)$.

3) Integrals with $i\epsilon$ prescription I

For the following two exercises you may have to look back into your math classes. The algebra and the physics content of these relations is important for many manipulations quantum theory. In the following ϵ is an infinitesimally small positive variable, which is always much much smaller than any given quantity that arises in the same context. Show the relation

$$\frac{1}{x - x' - i\epsilon} \quad = \quad i\pi\,\delta(x - x') \; + \; \mathcal{P}\frac{1}{x - x'} \,, \tag{5}$$

where $\mathcal{P}$ is the principal value of the singular intergral. You can assume for simplicity the case $x' = 0$ (Why is this still general?) and consider an integration over a test function $f(x)$ which does not have any poles in the vicinity of the real axis. Try to use the Cauchy integral theorem. How does the RHS of the relation look for $1/(x - x' + i\epsilon)$?

4) Integrals with $i\epsilon$ prescription II

Show the relation

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dx \frac{e^{ixz}}{x - i\epsilon} \quad = \quad \Theta(z)\,, \tag{6}$$

where $\Theta(z)$ is the Heaviside step function. Determine the result for the case when the sign of the $i\epsilon$ term is reversed.