Exercises to QM2, Summer Term 2018, Sheet 4

1) Green's function of the time-dependent free particle Schrödinger equation

(a) Derive the (3 different) expressions for the Green's function $G(t, \mathbf{x}|t', \mathbf{x}')$ for the time-dependent Schrödinger equation for a free particle in 3 dimensions,

$$\left[i\frac{\partial}{\partial t} + \frac{\nabla_{\mathbf{x}}^2}{2\mu} \right] G(t, \mathbf{x}; |t', \mathbf{x}') \;=\; i\,\delta(t-t')\,\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,,$$

which we discussed in class. Due to causality we are only interested in the Green's function that describes forward time evolution. Use the result of exercise (2) on sheet 2 to argue which $i\epsilon$-presciption is necessary.

(b) Assume that you know the particles wave function $\Psi(t, \mathbf{x})$ on all locations $\mathbf{x}$ at a particular time $t = t'$. Show that

$$\Psi^{(+)}(t, \mathbf{x}) \;=\; \int d^3\mathbf{x}'\, G(t, \mathbf{x}|t', \mathbf{x}')\, \Psi(t', \mathbf{x}')$$

is a solution of the time-dependent Schrödinger equation for times $t > t'$ and that $\Psi^{(+)}(t', \mathbf{x}) = \Psi(t', \mathbf{x})$ is valid.

(c) Show that

$$(\Psi^{(-)}(t, \mathbf{x}))^* \;=\; \int d^3\mathbf{x}'\, \Psi^*(t', \mathbf{x}')\, G(t', \mathbf{x}'|t, \mathbf{x})$$

is the Hermitian adjoint of a solution of the time-dependent Schrödinger equation for times $t > t'$ and that $\Psi^{(-)}(t', \mathbf{x}) = \Psi(t', \mathbf{x})$ is valid.

Hint: You may use the integral

$$\int_0^\infty dk\, k^n\, e^{-r^2k^2} \;=\; \frac{1}{2r^{n+1}}\,\Gamma\left(\frac{n+1}{2}\right)\,,$$

where Γ is the Gamma function, with $\Gamma(3/2) = \sqrt{\pi}/2$. Do recall in your mind how you may derive that relation yourself.

2) Time evolution operator for an arbitrary Hamiltonian

Let $H = H(t)$ be an explicitly time-dependent Hamilton operator in the Schrödinger picture. You may *not* assume any particular form of the Hamilton operator, so you cannot split it into a free particle Hamilton operator plus some interaction potential. Construct the forward time evolution operator $G(t, t_0)$ that evolves any state given at some time t_0 to the corresponding state at time $t > t_0$ consistent with the Schrödinger equation,

$$|\psi(t)\rangle \;=\; G(t, t_0)|\psi(t_0)\rangle\,.$$

Recall that the forward time evolution operator for a time-independent Hamilton operator H has the form $G(t, t_0) = \Theta(t - t_0) \exp[-iH(t - t_0)]$. The solution represents is generalization of this expression.

Hint: Use the time-dependent Schrödinger equation to derive an operator-valued differential equation for $G(t, t_0)$ (i.e. $i\frac{\partial}{\partial t}G(t, t_0) = \ldots$). Solve the differential equation iteratively to derive a perturbative series in the number of times the Hamilton operator appears. Use the fact that $G(t_0, t_0) = \mathbb{1}$ and construct the series by integration assuming that $t - t_0$ is very small. Use the Time-ordering operator to rewrite the series into a compact form that involves the exponential function.

3) Inelastic interactions

In quantum mechanics it is possible to describe inelastic processes (decays, absorption of particles, etc.) effectively by an imaginary contribution to the potential,

$$V(\mathbf{x}) \;=\; U(\mathbf{x}) + iW(\mathbf{x})\,,$$

where U and W are real-valued functions. Assume the case of a single massive non-relativistic particle without spin, so the resulting Hamilton operator may have the form $H = \frac{\mathbf{P^2}}{2\mu} + V(\mathbf{x})$. Which fundamental property does the Hamilton operator not have any more for $W(\mathbf{x}) \neq 0$? For the case $W(\mathbf{x}) = 0$ we know from QM I that the continuity equation

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla\mathbf{j}(\mathbf{x}, t) \;=\; 0$$

is valid, where ρ is the probability density of the particle and $\mathbf{j}$ the probability current. Recall how density ρ and current $\mathbf{j}$ were defined and how the continuity equation is derived for $W(\mathbf{x}) = 0$. For the case $W(\mathbf{x}) \neq 0$ the density ρ and current $\mathbf{j}$ are defined the same way. Derive the form of the continuity equation for $W(\mathbf{x}) \neq 0$ Which different physical interpretation do the cases $W > 0$ and $W < 0$ have?

4) Unstable free particle**

Write down the effective Hamiltonian of a free particle that has an *average lifetime* τ.

5) Covariant form of the Maxwell equations I

(a) Show that Maxwell equations (i) and (iv) discussed in class can be compactly written in the form

$$\partial_\mu F^{\mu\nu} = j^\mu \quad \text{with} \quad \partial_\mu \equiv \tfrac{\partial}{\partial x^\mu}\,, \quad j^\mu = (\rho, \mathbf{j}) \quad \text{and}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix},$$

where $F^{\mu\nu}$ is called the electromagnetic *field strength tensor*. Here you have to always apply the contraction rule for relativistic 4-vectors ($a_\mu b^\mu = a_0 b_0 - \mathbf{a}.\mathbf{b}$).

(b) Show that in this form the continuity equation has the form $\partial_\mu j^\mu = 0$, and that the Maxwell equations imply this continuity equation. (You may use an important property of the matrix $F^{\mu\nu}$ to shown the continuity equation.)

6) Covariant form of the Maxwell equations II

(c) Show that you can write the field streng tensor in the form

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu , ,$$

where $A^\mu = (\phi, \mathbf{A})$ is a 4-vector constructed from the scalar potential and vector potential.

(d) Show that you can write the field streng tensor in the form

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu , ,$$

where $A^\mu = (\phi, \mathbf{A})$ is a 4-vector constructed from the scalar potential and vector potential. It is called the *4-vector potential*.

(e) How does a gauge transformation look for A^μ? Show that $F^{\mu\nu}$ is gauge-invariant.