23. Renormalization group

we had: $\lambda = \mu^{4-d} F(\lambda_{ren}(\mu), d)$

with $F(x,d) = x - 3\Lambda_d \frac{x^2}{(4\pi)^2} + \dots$ (indep. of μ)

bare coupling λ does not depend on μ $\Rightarrow \frac{d\lambda}{d\mu} = 0$

$$\Rightarrow \quad \mu \frac{d}{d\mu} \lambda_{ren}(\mu) = (d-4) \frac{F(\lambda_{ren}(\mu), d)}{F'(\lambda_{ren}(\mu), d)}$$

$d \to 4 \quad \Rightarrow \quad \mu \frac{d}{d\mu} \lambda_{ren}(\mu) = \beta(\lambda_{ren}(\mu))$ renormalization group equation (RGE)

with $\beta = \lim_{d \to 4} (d-4) \frac{F}{F'}$

perturbative expansion $\rightarrow \beta(\lambda) = \beta_0 \frac{\lambda^2}{(4\pi)^2} + \beta_1 \frac{\lambda^3}{(4\pi)^4} + \dots$

coefficients $\beta_0, \beta_1, \dots$ are pure numbers

β_0 may be obtained from $\frac{d}{d\mu} \lambda_{ren}(\mu)$ (see ch. 22)

$\rightarrow \beta_0 = 3 \quad \rightarrow$ for weak coupling: $\beta(\lambda) \simeq \frac{3\lambda^2}{(4\pi)^2} > 0$

$\rightarrow \lambda_{ren}(\mu)$ grows if μ is increased

analogously: $m^2 = m^2_{ren}(\mu)\, \tilde{F}(\lambda_{ren}(\mu), d)$

$\rightarrow$ differential equation for running mass: $\mu \frac{d}{d\mu} m^2_{ren}(\mu) = \gamma(\lambda_{ren}(\mu))\, m^2_{ren}(\mu)$

ch. 22 $\Rightarrow$ $\beta(\lambda) = \frac{\lambda}{(4\pi)^2} + \mathcal{O}(\lambda^2)$

<u>note:</u> expressed in terms of the bare parameters λ, m, the d-dimensional n-point fctn of the bare field ϕ are <u>scale-independent</u>; <u>but:</u> renormalized n-point fctns are μ-dependent, because of fixed renormalisation constant

$$\Rightarrow \langle 0 | T \phi_{ren}(x_1) \ldots \phi_{ren}(x_n) | 0 \rangle \Big|_{\lambda_{ren}, m_{ren}, \mu}$$

$$= Z^{n/2} \langle 0 | T \phi_{ren}(x_1) \ldots \phi_{ren}(x_n) | 0 \rangle \Big|_{\lambda'_{ren}, m'_{ren}, \mu'}$$

$$Z = \lim_{d \to 4} \frac{Z'}{Z}$$ renormalisation group invariance

scale changes may be performed successively

$\rightarrow$ <u>renormalisation group</u> parametrized by the factor μ'/μ

$\rightarrow$ μ is a matter of choice, which does <u>not</u> affect physical quantities if $\beta(\lambda)$ is known, the RGE fully determines the scale dependence of $\lambda_{ren}(\mu)$; even if only the leading term β_0 is known (the evaluation of β_n requires a calculation to $n+1$ loops), the RG implies that the coefficients $\overline{a}_L$ in the relation between the running and the physical couplings,

$$\lambda_{ren} = \lambda_{ph}\left[1 + \overline{a}_1 \frac{\lambda_{ph}}{(4\pi)^2} + \overline{a}_2 \frac{\lambda_{ph}^2}{(4\pi)^4} + \dots\right]$$

are polynomials of rank L in the variable $t = \ln(\mu/m_{ph})$ (ex.: show this!):

$$\overline{a}_L = \overline{a}_{L,0}\, t^L + \overline{a}_{L,1}\, t^{L-1} + \dots + \overline{a}_{L,L} .$$

last term $\overline{a}_{L,L}$ may be eliminated by replacing the scale of the logarithm with the scale m_1, at which $\lambda_{ph} = \lambda_{ren}(m_1)$, $t = \ln(\mu/m_1)$;
term with highest power of $\ln\mu$ $\equiv$ <u>leading logarithmic contribution</u> is <u>fully determined</u> by the first coefficient of the β-fctn: $\overline{a}_{L,0} = (\beta_0)^L$

$\Rightarrow$ leading logarithms $\rightarrow$ geometric series

$$\lambda_{ren} = \frac{\lambda_{ph}}{1 - \lambda_{ph} \beta_0 t/(4\pi)^2} + \dots \qquad (*)$$

$\hat{=}$ reordering of the perturbation series

running coupling depends on the scale μ only through t $\rightarrow$ straightforward perturbative expansion yields a double series in powers of λ_{ph} and t $\rightarrow$ since the power t^L occurs only at order λ_{ph}^{L+1} or higher, the variable t may be replaced by $\bar{t} = \lambda_{ph} t$ and expand in powers of λ_{ph} at fixed $\bar{t}$ $\rightarrow$ in this reordered expansion, $(*)$ is valid to first order in λ_{ph} but to all orders in $\bar{t}$ $\rightarrow$ one loop result

$$\lambda_{ren} = \lambda_{ph} \left[1 + \beta_0 \frac{\bar{t}}{(4\pi)^2} \right]$$

is less accurate, as it only contains λ_{ph} and $\lambda_{ph} \bar{t}$

$\rightarrow$ significant improvement by reordering if $\ln(\mu/m_1)$ is large!

remark: (*) is an approximate solution of the RGE for $\beta(\lambda) \simeq \beta_0 \frac{\lambda^2}{(4\pi)^2}$

$$\rightarrow \quad \mu \frac{d}{d\mu} \frac{1}{\lambda_{ren}} = -\beta_0/(4\pi)^2$$

$$\rightarrow \quad \frac{1}{\lambda_{ren}} = -\frac{\beta_0}{(4\pi)^2} \ln \frac{\mu}{\Lambda_\phi} \qquad \Lambda_\phi = \text{constant of int.}$$

$$\Rightarrow \quad \frac{\lambda_{ren}}{(4\pi)^2} = \frac{1}{\beta_0 \ln \frac{\Lambda_\phi}{\mu}}$$

take $\mu = m_1 \rightarrow \Lambda_\phi = m_1 \exp\left[(4\pi)^2/\beta_0 \lambda_{ph}\right]$

$\rightarrow$ (*) ✓

Λ_ϕ is independent of μ (by def.) $\rightarrow$ <u>renormalisation group invariant scale</u> of the model

<u>see</u>: J. Gasser, H. Leutwyler: Skript zur Vorlesung Quantenfeldtheorie I, Sommersemester 1997, Univ. Bern