Running coupling and running mass

\[ m_{\text{ph}}^2 = m^2 - \frac{\alpha}{2} \Delta(\alpha) + O(\alpha^2) \]

Loop integral \( \Delta(\alpha) \) contains a pole at \( d=4 \)

\[ \Delta(\alpha) = \frac{2i\alpha}{(4\pi)^2} \mu^{d-4} \Lambda_d + \Delta(\alpha)_{\overline{\text{MS}}} \]

Bare mass \( m \) is tuned in such a way that the physical mass \( m_{\text{ph}} \) remains finite

\[ m_{\text{ren}}^2 = m^2 \left[ 1 + \frac{\alpha}{(4\pi)^2} \mu^{d-4} \Lambda_d + O(\alpha^2) \right] \]

approaches a finite limit when the regularization is removed \( \rightarrow m_{\text{ren}} \) depends on \( \mu \) \( \rightarrow \) running mass

Remark: \( [\alpha] = 4 - d \) (compensated by \( \mu^{d-4} \))

Express \( m_{\text{ren}}(\mu) \) in terms of \( m_{\text{ph}} \):

\[ m_{\text{ren}}^2(\mu) = m_{\text{ph}}^2 \left[ 1 + \frac{\alpha_{\text{ph}}}{(4\pi)^2} \left( \ln \frac{\mu^4}{m_{\text{ph}}^2} + \frac{1}{2} \right) + O(\alpha^2_{\text{ph}}) \right] \]

Scale dependence is logarithmic at one-loop order

Remark: \( m_{\text{ren}}(e^{-\frac{1}{2}m_{\text{ph}}}) = m_{\text{ph}} \)
Running coupling

\[ \tilde{\lambda}_{\text{phys}} = \lambda - \frac{3}{2} \lambda^2 B(\Lambda_1, m_{\text{ph}}^2) + O(\lambda^3) \]

\[ B(\Lambda_1) = -\frac{2\mu^{d-4}}{(4\pi)^2} \Lambda d + B(\Lambda_1)_{\text{ren}} \]

\[ \tilde{\lambda}_{\text{phys}} = \lambda + \frac{3\lambda^2}{(4\pi)^2} \Lambda d \mu^{d-4} - \frac{3}{2} \lambda^2 B(\Lambda_1)_{\text{ren}} + O(\lambda^3) \]

finite for \( d \to 4 \)

\[ \lambda_{\text{ren}} = \lambda \mu^{d-4} \left[ 1 + \frac{3\lambda}{(4\pi)^2} \Lambda d \mu^{d-4} + O(\lambda^2) \right] \]

\[ \lambda_{\text{ren}} \text{ is dimensionless (even for } d \neq 4) \]

relation between \( \lambda_{\text{ren}}(\mu) \) and \( \tilde{\lambda}_{\text{phys}} \)

\[ \lambda_{\text{ren}}(\mu) = \mu^{d-4} \left[ \tilde{\lambda}_{\text{phys}} + \frac{3}{2} \lambda^2 B(\Lambda_1)_{\text{ren}} + O(\lambda^3) \right] \]

\[ = \mu^{d-4} \left[ \tilde{\lambda}_{\text{phys}} + \frac{3}{2} \tilde{\lambda}_{\text{phys}}^2 B(\Lambda_1)_{\text{ren}} + \frac{3}{2} \tilde{\lambda}_{\text{phys}}^2 \tilde{B}(\Lambda_1) + O(\lambda^3) \right] \]

\[ \downarrow \]

\[ -\frac{2}{(4\pi)^2} \ln \frac{m_{\text{ph}}}{\mu} \]
\[
\alpha_{\text{ren}}(\mu) = \widetilde{\alpha}_\text{ph} \left[ 1 + \frac{3\widetilde{\alpha}_\text{ph}}{(4\pi)^2} \ln \frac{\mu}{m_\text{ph}} + \frac{3}{2} \widetilde{\alpha}_\text{ph} B(\lambda_1) + O(\alpha^3) \right]
\]

\[
= \widetilde{\alpha}_\text{ph} \left[ 1 + \frac{3\widetilde{\alpha}_\text{ph}}{(4\pi)^2} \ln \frac{\mu}{m_1} + O(\alpha^2) \right]
\]

\(m_1\) is the value of the scale \(\mu\) at which \(\alpha_{\text{ren}}(m_1) = \widetilde{\alpha}_\text{ph}\).

Comparison with cut-off regularization:

\[|k_\mu| \leq \Lambda\]

bare coupling \(\alpha_{\text{cut}}\)

d-dim. integral \(B(0)\) replaced by \(\int d^d k\) ...

\(\to\) diverges logarithmically

\[
\alpha_{\text{cut}} = \widetilde{\alpha}_\text{ph} \left[ 1 + \frac{3\widetilde{\alpha}_\text{ph}}{(4\pi)^2} \ln \frac{\Lambda}{m_2} + O(\alpha^2) \right]
\]

some coefficient as before.

\[
M_2 = M_{\text{ph}} \times c \quad c = O(1)
\]

\(\to\) dependence of the bare coupling \(\alpha_{\text{cut}} = \alpha_{\text{cut}}(\Lambda)\) on \(\Lambda\) is the same as the dependence of the
running coupling $\lambda_{\mu} = \lambda_{\mu} (\mu)$ on the scale $\mu$
(apart from $m_{1} \neq m_{2} \rightarrow$ can be repaired by redefinition
$\lambda \rightarrow \lambda \frac{m_{2}}{m_{1}}$ ) \rightarrow at large values of the
running scale, $\lambda_{\mu} (\mu)$ may be interpreted as
the bare coupling of cut-off regularisation
with $\lambda = \mu$; for a scale of the order of the
physical mass, $\lambda_{\mu} (\mu) \rightarrow \lambda_{0} \rightarrow \lambda_{\mu} (\mu)$
interpolates between the bare and the physical
coupling !

remark: comparison between $m_{\mu}$ and more
complicated as 2-point-fan quadratically
divergent ($\sim \Lambda^{2}$)

field renormalization constant $\overline{Z}$ in $\overline{MS}$ scheme:
$\overline{Z} / Z$ finite for $d \rightarrow 4$ but $\neq 1$

$$\overline{Z} = 1 + c_{2} \frac{\alpha^{2} \mu^{2(d-4)}}{(4\pi)^{2}} + \ldots \quad (\phi^{4} \text{theory})$$
with an appropriate tuning of the coefficient $c_\zeta$ for $d \to 4$, the $n$-point functions of

$$\Phi_{\text{ren}}(x) = \Xi^{-\frac{\zeta}{2}} \Phi(x)$$

are finite (at $L=2$).

Renormalization of the $n$-point function requires two steps:

1. The bare parameters are expressed in terms of the renormalized ones:

$$\lambda = \mu^{4-d} \lambda_{\text{ren}} \left[ 1 + \alpha_1 \frac{\lambda_{\text{ren}}}{(4\pi)^2} + \alpha_2 \frac{\lambda_{\text{ren}}^2}{(4\pi)^4} + \ldots \right]$$

$$m^2 = m_{\text{ren}}^2 \left[ 1 + \beta_1 \frac{\lambda_{\text{ren}}}{(4\pi)^2} + \beta_2 \frac{\lambda_{\text{ren}}^2}{(4\pi)^4} + \ldots \right]$$

2. The result for $\langle 0 | T \Phi(x_1) \ldots \Phi(x_n) | 0 \rangle$ is divided by the power $\Xi^{n/2}$.

Remark: for the determination of $S$-matrix elements from the (now finite) $n$-point function
\[ \langle 0 | T \phi_{\alpha_1}(x_1) \ldots \phi_{\alpha_n}(x_n) | 0 \rangle, \] I still have to multiply the result with the finite factor \((Z/Z)^{n/2}\)!

**Remark:** \(a_i = -3 \Lambda d, \; b_i = -\Lambda d\) already determined.

For higher-order terms to absorb all the divergences occurring in the evaluation of the n-point functions, the coefficients must contain counter terms of the form

\[ a_L = a_{L,0} \Lambda^L + a_{L,1} \Lambda^{L-1} + \ldots + a_{L,L-1} \Lambda \]

(analogously for \(b_L, c_L\))