22. Running coupling and running mass

p. 5/1 : $m_{ph}^2 = m^2 - \frac{i\lambda}{2}\Delta(0) + O(\lambda^2)$

loop integral $\Delta(0)$ contains a pole at $d=4$

p. 21/4 : decomposition into singular and finite parts →

$$\to \quad \Delta(0) = \frac{2im^2}{(4\pi)^2}\mu^{d-4}\Lambda_d + \Delta(0)_{ren}^{\overline{MS}}$$

bare mass m is tuned in such a way that the physical mass m_{ph} remains finite

$$\to \quad m_{ren}^2 = m^2\left[1 + \frac{\lambda}{(4\pi)^2}\mu^{d-4}\Lambda_d + O(\lambda^2)\right]$$

approaches a finite limit when the regularization is removed → m_{ren} depends on μ → running mass

remark : $[\lambda] = 4-d$ (compensated by μ^{d-4})

express $m_{ren}(\mu)$ in terms of m_{ph} :

$$m_{ren}^2(\mu) = m_{ph}^2\left[1 + \frac{\lambda_{ph}}{(4\pi)^2}\left(\ln\frac{\mu}{m_{ph}} + \frac{1}{2}\right) + O(\lambda_{ph}^2)\right]$$

scale dependence is logarithmic at one-loop order

remark : $m_{ren}(e^{-1/2}m_{ph}) = m_{ph}$

Running coupling

p. 5/22 : $\tilde{\lambda}_{phys} = \lambda - \frac{3}{2}\lambda^2 B(s_1, m_{ph}^2) + \mathcal{O}(\lambda^3)$

$$B(s_1) = -\frac{2\mu^{d-4}}{(4\pi)^2}\Lambda_d + B(s_1)_{ren}$$

$$\tilde{\lambda}_{phys} = \underbrace{\lambda + \frac{3\lambda^2}{(4\pi)^2}\Lambda_d \mu^{d-4}}_{\text{finite for } d\to 4} - \frac{3}{2}\lambda^2 B(s_1)_{ren} + \mathcal{O}(\lambda^3)$$

$$\lambda_{ren} := \lambda\mu^{d-4}\left[1 + \frac{3\lambda}{(4\pi)^2}\Lambda_d \mu^{d-4} + \mathcal{O}(\lambda^2)\right]$$

$\uparrow$

dimensionless (even for $d \neq 4$)

relation between $\lambda_{ren}(\mu)$ and $\tilde{\lambda}_{phys}$

$$\lambda_{ren}(\mu) = \mu^{d-4}\left[\tilde{\lambda}_{ph} + \frac{3}{2}\lambda^2 B(s_1)_{ren} + \mathcal{O}(\lambda^3)\right)$$

$$= \mu^{d-4}\left[\tilde{\lambda}_{ph} + \frac{3}{2}\tilde{\lambda}_{ph}^2 B(0)_{ren} + \frac{3}{2}\tilde{\lambda}_{ph}^2 \overline{B}(s_1) + \mathcal{O}(\tilde{\lambda}_{ph}^3)\right]$$

$$B(0)_{ren} \downarrow -\frac{2}{(4\pi)^2}\ln\frac{m_{ph}}{\mu}$$

$d \to 4$:

$$\lambda_{ren}(\mu) = \tilde{\lambda}_{ph}\left[1 + \frac{3\tilde{\lambda}_{ph}}{(4\pi)^2}\ln\frac{\mu}{m_{ph}} + \frac{3}{2}\tilde{\lambda}_{ph}\overline{B}(s_1) + O(\tilde{\lambda}_{ph}^2)\right]$$

$$= \tilde{\lambda}_{ph}\left[1 + \frac{3\tilde{\lambda}_{ph}}{(4\pi)^2}\ln\frac{\mu}{m_1} + O(\tilde{\lambda}_{ph}^2)\right]$$

m_1 is the value of the scale μ at which $\lambda_{ren}(m_1) = \tilde{\lambda}_{ph}$

comparison with <u>cut-off</u> <u>regularization</u>:

$|k_\mu| \leq \Lambda$

bare coupling λ_{cut}

d-dim. integral $B(s)$ replaced by $\int^\Lambda d^dk \ldots$

$\to$ diverges logarithmically

$$\lambda_{cut} = \lambda_{ph}\left[1 + \frac{3\lambda_{ph}}{(4\pi)^2}\ln\frac{\Lambda}{m_2} + O(\lambda_{ph}^2)\right]$$

same coefficient as before!

$m_2 = m_{ph} \times c \qquad c = O(1)$

$\to$ dependence of the bare coupling $\lambda_{cut} = \lambda_{cut}(\Lambda)$ on Λ is the same as the dependence of the running coupling $\lambda_{ren} = \lambda_{ren}(\mu)$ on the scale μ (apart from $m_1 \neq m_2$ → can be repaired by redefinition $\Lambda \to \Lambda \frac{m_2}{m_1}$) → at large values of the running scale, $\lambda_{ren}(\mu)$ may be interpreted as the bare coupling of cut-off regularisation with $\Lambda \simeq \mu$; for a scale of the order of the physical mass, $\lambda_{ren}(\mu) \to \lambda_{ph}$ → $\lambda_{ren}(\mu)$ interpolates between the bare and the physical coupling!

<u>remark:</u> comparison between m_{ren} and m_{cut} more complicated as 2-point-fctn quadratically divergent ($\sim \Lambda^2$)

- <u>field renormalization constant</u> $\overline{Z}$ in $\overline{MS}$ scheme:

$\overline{Z}/Z$ finite for $d \to 4$ <u>but</u> $\neq 1$

$$\overline{Z} = 1 + c_2 \frac{\lambda^2 \mu^{2(d-4)}}{(4\pi)^2} + \dots \qquad (\varphi^4 \text{ theory})$$

with an appropriate tuning of the coefficient c_2 for $d \to 4$, the n-point-functions of

$$\phi_{ren}(x) = Z^{-1/2}\phi(x)$$

are finite (at $L=2$)

- renormalization of the n-point function requires two steps

(1) the bare parameters are expressed in terms of the renormalized ones:

$$\lambda = \mu^{4-d}\lambda_{ren}\left[1 + a_1\frac{\lambda_{ren}}{(4\pi)^2} + a_2\frac{\lambda_{ren}^2}{(4\pi)^4} + \dots\right]$$

$$m^2 = m_{ren}^2\left[1 + b_1\frac{\lambda_{ren}}{(4\pi)^2} + b_2\frac{\lambda_{ren}^2}{(4\pi)^4} + \dots\right]$$

(2) the result for $\langle 0 | T\phi(x_1)\dots\phi(x_n) | 0\rangle$ is divided by the power $Z^{n/2}$

<u>remark:</u> for the determination of S-matrix elements from the (now finite) n-point function

$\langle 0 | T \, \phi_{ren}(x_1) \ldots \phi_{ren}(x_n) | 0 \rangle$, I still have to multiply the result with the finite factor $(\overline{Z}/Z)^{n/2}$!

remark: $a_1 = -3\Lambda_d$, $b_1 = -\Lambda_d$ already determined

for higher-order terms to absorb all the divergences occurring in the evaluation of the n-point functions, the coefficients must contain counter terms of the form

$$a_L = a_{L,0}\,\Lambda_d^L + a_{L,1}\,\Lambda_d^{L-1} + \ldots + a_{L,L-1}\,\Lambda_d$$

(analogously for b_L, c_L)