

21. Minimal subtraction

back to φ^4 theory!

form of the relations

$$\lambda = \lambda(m_{ph}, \lambda_{ph}), \quad m = m(m_{ph}, \lambda_{ph})$$

requires a computation of the 2- and 4-point function to the relevant order in perturbation theory and depends also on the convention used for the definition of $\lambda_{ph} \rightarrow$ rather cumbersome!

different approach: minimal subtraction method ('t Hooft)

MS minimal subtraction scheme

$\overline{\text{MS}}$ modified minimal subtraction scheme

basic observation: in dimensional regularization,

divergences appear as singularities in the variable d

$\rightarrow$ finite part of any divergent loop integral

may be defined in a uniform manner by

removing these singularities

example:

$$B(0, m^2) = \frac{1}{i} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i\varepsilon)^2} = \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} m^{d-4}$$

↑
defined on p. 5/14

↑
see p. 5/18

$$\times \Gamma(x) = \Gamma'(x+1) \Rightarrow B(0, m^2) = - \frac{2 \Gamma(3 - \frac{d}{2})}{(d-4) (4\pi)^{d/2}} m^{d-4}$$

→ pole at $d=4$ with residue $-\frac{2}{(4\pi)^2}$

$$\rightarrow \text{define } B(0, m^2)_{\text{ren}} \Big|_{\text{MS}} := B(0, m^2) + \frac{2 \mu^{d-4}}{(4\pi)^2} \frac{1}{d-4} \text{ - finite!}$$

↑
MS scheme

remark: factor μ^{d-4} (μ has dimension of a mass, but otherwise arbitrary) was introduced as $[B] = d-4$

renormalization scale or running (mass-) scale μ

explicit form of $B(0, m^2)_{\text{ren}} \Big|_{\text{MS}}$:

$$\begin{aligned} B(0, m^2)_{\text{ren}} \Big|_{\text{MS}} &= -2 \frac{\Gamma(3 - \frac{d}{2})}{(d-4) (4\pi)^{d/2}} m^{d-4} + \frac{2 \mu^{d-4}}{(4\pi)^2 (d-4)} \\ &= -\frac{2}{(4\pi)^2 (d-4)} \left[\frac{\Gamma(3 - \frac{d}{2})}{(4\pi)^{\frac{d-4}{2}}} m^{d-4} - \mu^{d-4} \right] \end{aligned}$$

expand around
 $d=4$

$$\begin{aligned}
&= - \frac{2}{(4\pi)^2 (d-4)} \left\{ \left[\Gamma(1) - \frac{1}{2} \Gamma'(1) (d-4) + \dots \right] \right. \\
&\quad \times \left[1 + (d-4) \ln m - \frac{1}{2} (d-4) \ln 4\pi + \dots \right] \\
&\quad \left. - \left[1 + (d-4) \ln \mu + \dots \right] \right\} \\
&= - \frac{2}{(4\pi)^2} \left[- \frac{1}{2} \Gamma'(1) - \frac{1}{2} \ln(4\pi) + \ln \frac{m}{\mu} \right]
\end{aligned}$$

ugly terms $\Gamma'(1) = -\gamma_E$ and $\ln(4\pi)$ can be avoided in modified minimal subtraction scheme ($\overline{MS}$):

$$B_{ren}^{\overline{MS}}(0) = B(0) + \frac{2\mu^{d-4}}{(4\pi)^2} \underbrace{\frac{\Gamma(3-\frac{d}{2})}{(4\pi)^{\frac{d-4}{2}} (d-4)}}_{\Lambda_d}$$

$$\Lambda_d := \frac{\Gamma(3-\frac{d}{2})}{(d-4) (4\pi)^{\frac{d-4}{2}}} = - \frac{1}{2} \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d-4}{2}}}$$

$$= \frac{1}{d-4} - \frac{1}{2} [\Gamma'(1) + \ln(4\pi)] + \mathcal{O}(d-4)$$

new: $B_{ren}^{\overline{MS}}(0) = - \frac{2}{(4\pi)^2} \ln \frac{m}{\mu} + \mathcal{O}(d-4)$

this procedure may be applied to any function X that contains a simple pole at $d=4$:

→ determine the residue $R = \lim_{d \rightarrow 4} [(d-4)X]$

→ choose the number p such that $R\mu^p$ has the same dimension as X

→ in $\overline{\text{MS}}$, the finite part of X is given by

$$X_{\text{ren}} = X - R\mu^p \Lambda_d$$

example: $\Delta(0) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{m^2 - k^2 - i\varepsilon} = i \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{d/2}} m^{d-2}$

→ residue $\frac{2im^2}{(4\pi)^2}$

→ dimension of $\Delta(0)$: $d-2$

→ $\Delta(0)_{\text{ren}}^{\overline{\text{MS}}} = \Delta(0) - \frac{2im^2}{(4\pi)^2} \mu^{d-4} \Lambda_d$

→ $\Delta(0)_{\text{ren}}^{\overline{\text{MS}}} = i \frac{m^2}{(4\pi)^2} \left(\ln \frac{m^2}{\mu^2} - 1 \right)$

extension to graphs with L loops:

- contains in general divergences $\sim \frac{1}{(d-4)^R}$ $1 \leq R \leq L$
- corresponding divergent part may be defined by

$$[R_0 \Lambda_d^L + R_1 \Lambda_d^{L-1} + \dots + R_{L-1} \Lambda_d] \mu^p$$

R_0 is the residue of the pole of highest order,

R_1 is the residue of the pole of order $L-1$ that remains when the term $R_0 \Lambda_d^L \mu^p$ is removed, etc.