

12. Interaction picture

In this section, we want to clarify the physical meaning of the generating functional

$$Z[f] = \langle 0 | T e^{i \int d^d x f(x) \phi(x)} | 0 \rangle$$

as vacuum-to-vacuum transition amplitude $\langle 0_{\text{out}} | 0_{\text{in}} \rangle_f$ and explain the relation between path integral formulae and operator techniques in perturbation theory.

In the Heisenberg picture (used so far), the state vectors are time-independent and the operators (representing observables) are time-dependent. The expectation value of an observable A (represented by the Heisenberg operator $A_H(t)$) at time t in the (pure) state ψ_H is given by

$$\langle \psi_H | A_H(t) | \psi_H \rangle$$

and the equation of motion reads

$$\frac{dA_H(t)}{dt} = i [H_H(t), A_H(t)] + \frac{\partial A_H(t)}{\partial t}$$

remark: the term $\partial A_H(t)/\partial t$ only occurs for operators depending explicitly on time, for instance via an external source term

in the case where the Hamiltonian does not depend on time ($H_H(t) =: H$), the solution of the equation of motion for an observable with $\partial A_H(t)/\partial t = 0$ can be written as

$$A_H(t) = e^{iHt} A_H(0) e^{-iHt}$$

and the time dependence of the expectation value is given by

$$\langle \Psi_H | e^{iHt} A_H(0) e^{-iHt} | \Psi_H \rangle$$

which is the only relevant -physical quantity

in the Schrödinger picture, the state vectors are time-dependent and the operators are time-independent (unless in the case of an explicitly-

time-dependent observable)

in the case of constant H, A , Heisenberg picture and Schrödinger picture are related by

$$\psi_s(t) = e^{-iHt} \psi_h \quad (\psi_s(0) = \psi_h)$$

and

$$A_s = A_h(0)$$

which leaves the expectation value of A unchanged:

$$\langle \psi_s(t) | A_s | \psi_s(t) \rangle$$

in the Schrödinger picture, the equation of motion is given by the Schrödinger equation

$$i \frac{d}{dt} \psi_s(t) = H \psi_s(t)$$

the form of this equation remains valid also in the case of a time-dependent Hamiltonian $H(t)$, however, the solution of the Schrödinger

equation cannot be written in exponential form but as

$$\psi_s(t) = U(t) \psi_s(0)$$

with a unitary evolution operator $U(t)$
which fulfills the differential equation

$$i \frac{d}{dt} U(t) = H(t) U(t),$$

$$U(0) = \mathbb{1} \quad (\text{initial condition})$$

we can write this also as an integral equation:

$$U(t) = \mathbb{1} - i \int_0^t d\tau H(\tau) U(\tau)$$

this integral equation can be solved by iteration;
the first step leads to

$$\begin{aligned} U(t) &= \mathbb{1} - i \int_0^t d\tau_1 H(\tau_1) \left(\mathbb{1} - i \int_0^{\tau_1} d\tau_2 H(\tau_2) U(\tau_2) \right) \\ &= \mathbb{1} - i \int_0^t d\tau_1 H(\tau_1) + (-i)^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 H(\tau_1) H(\tau_2) U(\tau_2) \end{aligned}$$

$$\Rightarrow U(t) = \mathbb{1} - i \int_0^t d\tau_1 H(\tau_1)$$

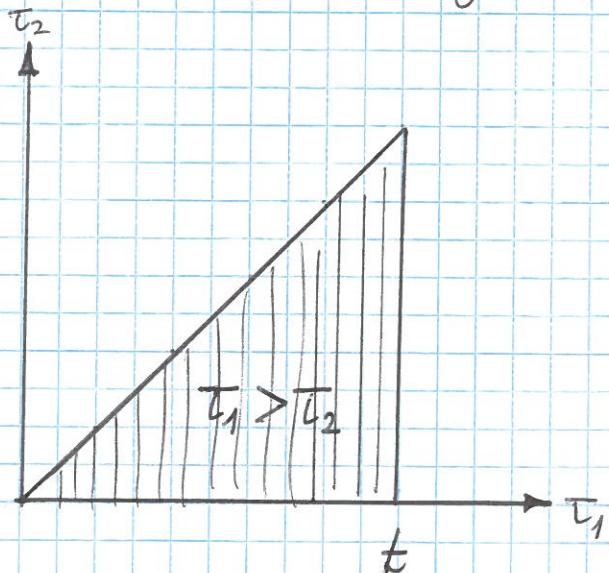
$$+ (-i)^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 H(\tau_1) H(\tau_2)$$

+ ...

$$+ (-i)^n \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{n-1}} d\tau_n H(\tau_1) \dots H(\tau_n)$$

+ ...

$$= \sum_{n=0}^{\infty} (-i)^n \int_0^t d\tau_1 \dots \int_0^{\tau_{n-1}} d\tau_n H(\tau_1) \dots H(\tau_n)$$



$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 H(\tau_1) H(\tau_2) = \frac{1}{2!} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 T(H(\tau_1) H(\tau_2))$$

general case:

$$\int_0^t d\tau_1 \dots \int_0^{t_{n-1}} d\tau_n H(\tau_1) \dots H(\tau_n) = \\ = \frac{1}{n!} \int_0^t d\tau_1 \dots \int_0^t d\tau_n T(H(\tau_1) \dots H(\tau_n))$$

$$\Rightarrow U(t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t d\tau_1 \dots \int_0^t d\tau_n T(H(\tau_1) \dots H(\tau_n)) \\ = T e^{-i \int_0^t d\tau H(\tau)}$$

interaction picture

we consider the situation where the Hamiltonian takes the form

$$H(t) = H_0 + H_1(t) \quad (\text{Schrödinger picture})$$

examples:

1) harmonic oscillator in the presence of an external time-dep.

force: $H(t) = \underbrace{P^2/2m + m\omega^2 Q^2/2}_{H_0} - \underbrace{f(t) Q}_{H_1(t)}$

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2) (self-interacting) scalar field in
the presence of an external source term

$$H = \underbrace{\int d^3x \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\vec{\nabla}\phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right]}_{H_0} - \underbrace{\int d^3x f(t, \vec{x}) \phi}_{H_1(t)}$$

the time evolution of the operators in the IP is given by

$$A_{IP}(t) = e^{iH_0 t} A_{IP}(0) e^{-iH_0 t}$$

the full time evolution of the expectation value of the observable A is given by

$$\langle \psi_H | U(t)^+ A_{IP}(t) U(t) | \psi_H \rangle$$

$$= \langle \psi_H | U(t)^+ e^{iH_0 t} A_{IP}(0) e^{-iH_0 t} U(t) | \psi_H \rangle$$

with an appropriately chosen unitary operator $U(t)$ (initial condition $U(0) = \mathbb{1}$)

the time evolution of $U(t)$ can be read off from the Schrödinger equation:

$$i \frac{d}{dt} [e^{-iH_0 t} U(t)] = [H_0 + H_1(t)] e^{-iH_0 t} U(t)$$

$$H_0 e^{-iH_0 t} U(t) + e^{-iH_0 t} i \frac{d}{dt} U(t) =$$

$$= H_0 e^{-iH_0 t} U(t) + H_1(t) e^{-iH_0 t} U(t)$$

$$\Rightarrow i \frac{d}{dt} U(t) = \underbrace{e^{iH_0 t} H_1(t) e^{-iH_0 t}}_{H_{1,IP}(t)} U(t)$$

$$\Rightarrow U(t) = T e^{-i \int_0^t H_{1,IP}(\tau) d\tau}$$

the time evolution in the Schrödinger picture and the one in the interaction picture are related by

$$|\Psi_S(t)\rangle = e^{-iH_0 t} \underbrace{U(t)}_{\substack{|\Psi_{IP}(t)\rangle \\ = |\Psi_H\rangle}} \underbrace{|\Psi_S(0)\rangle}_{= |\Psi_H\rangle}$$

$$\text{or } |\psi_{IP}(t)\rangle = e^{iH_0 t} |\psi_s(t)\rangle$$

Heisenberg picture and interaction picture are related by

$$|\psi_{IP}(t)\rangle = U(t) |\psi_H\rangle$$



$$|\psi_H\rangle = \underbrace{U(t)^{-1}}_{U(t)^\dagger} |\psi_{IP}(t)\rangle$$

We assume that the time-dependent term $H_1(t)$ vanishes for $t \rightarrow \pm\infty$ (the external force $f(t)$ tends to zero both in the remote past and the far future)

Suppose that the motion starts in the ground state $|0\rangle$ of the unperturbed system ($H_0 |0\rangle = 0$), i.e.

$$|\psi_s(-\infty)\rangle = \lim_{t \rightarrow -\infty} e^{-iH_0 t} |\psi_{IP}(t)\rangle = |0\rangle$$

being equivalent to

$$\lim_{t \rightarrow -\infty} |\Psi_{IP}(t)\rangle = |0\rangle$$

or $U(-\infty) |0, in\rangle = |0\rangle$

$\underbrace{\hspace{1cm}}$
Heisenberg state

$$\begin{aligned} |0, in\rangle &= T e^{i \int_0^{-\infty} d\tau H_{IP}(\tau)} |0\rangle \\ &= T e^{-i \int_{-\infty}^0 d\tau H_{IP}(\tau)} |0\rangle \end{aligned}$$

in the course of time, the state vector moves away from the ground state,

$$|\Psi_{IP}(t)\rangle = T e^{-i \int_{-\infty}^t d\tau H_{IP}(\tau)} |0\rangle,$$

until, for sufficiently large times, it again remains put, because $H_1(t)$ tends to zero

for $H_1 = 0$, the vector $|\psi(+\infty)\rangle$ would coincide with the ground state of H_0 , but in the presence of $H_1(t)$ it does in general not represent an eigenstate of H_0 ; the perturbation violates time-translation invariance, so that the energy fails to be conserved : the vector $|\psi(+\infty)\rangle$ does not coincide with the ground state, but may be represented as a superposition of energy eigenstates of H_0 .

The probability amplitude that the system is found in the ground state at $t \rightarrow \infty$ after the perturbation described by $H_1(t)$ is given by

$$\langle 0 | T e^{-i \int_{-\infty}^{+\infty} d\tau H_{1,P}(\tau)} | 0 \rangle$$

using

$$|0_{\text{out}}\rangle = T e^{i \int_0^\infty d\tau H_{1,\text{IP}}(\tau)} |0\rangle,$$

we may also write

$$\langle 0 | T e^{-i \int_{-\infty}^{+\infty} d\tau H_{1,\text{IP}}(\tau)} | 0 \rangle$$

$$= \langle 0_{\text{out}} | 0_{\text{in}} \rangle_{H_1(t)}$$

which is referred to as the vacuum-to-vacuum transition amplitude

in our field theoretic model, the operator $H_{1,\text{IP}}(t)$ is the space integral of $-f(x) \phi(x)$, where $\phi(x)$ evolves according to H_0 .

$$\Rightarrow \langle 0_{\text{out}} | 0_{\text{in}} \rangle_f = \langle 0 | T e^{i \int d^4x f(x) \phi(x)} | 0 \rangle$$

Perturbation series in QFT

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$$S[\varphi] = S_0[\varphi] + S_{\text{int}}[\varphi]$$

↑

bilinear in φ

Heisenberg op.

vacuum of full (interacting) theory

$$Z[f] = \langle 0 | T e^{i \int d^d x f(x) \phi_h(x)} | 0 \rangle$$

$$= \frac{\int [d\varphi] e^{i S[\varphi]} e^{i \int d^d x f(x) \varphi(x)}}{\int [d\varphi] e^{i S[\varphi]}} \quad \leftarrow \text{path integral representation}$$

$$= \frac{\int [d\varphi] e^{i S_0[\varphi]} e^{i S_{\text{int}}[\varphi]} e^{i f \cdot \varphi}}{\int [d\varphi] e^{i S_0[\varphi]}} \times \frac{\int [d\varphi] e^{i S_0[\varphi]}}{\int [d\varphi] e^{i S_0[\varphi]} e^{i S_{\text{int}}[\varphi]}}$$

$$f \cdot \varphi := \int d^d x f(x) \varphi(x)$$

$$\Rightarrow Z[f] = \frac{\langle\langle e^{i S_{\text{int}}[\varphi]} e^{i f \cdot \varphi} \rangle\rangle}{\langle\langle e^{i S_{\text{int}}[\varphi]} \rangle\rangle}$$

Gaussian mean value $\langle\langle \tilde{F}[\varphi] \rangle\rangle := \frac{\int [d\varphi] e^{i S_0[\varphi]} \tilde{F}[\varphi]}{\int [d\varphi] e^{i S_0[\varphi]}}$

perturbative expansion:

$$e^{i S_{\text{int}}[\varphi]} = \sum_{n=0}^{\infty} \frac{i^n}{n!} (S_{\text{int}}[\varphi])^n$$

translating back to operator language:

$$\begin{aligned} Z[f] &= \langle 0 | T e^{i \int d^d x f(x) \phi_H(x)} | 0 \rangle \\ &= \frac{\langle 0 | T e^{i S_{\text{int}}[\phi_{IP}]} e^{i \int d^d x f(x) \phi_{IP}(x)} | 0 \rangle}{\langle 0 | T e^{i S_{\text{int}}[\phi_{IP}]} | 0 \rangle} \\ &\quad \uparrow \\ &\quad \text{vacuum of } \underline{\text{free theory}} \\ &\quad (\text{perturbative vacuum}) \end{aligned}$$

$$T e^{i S_{\text{int}}[\phi_{IP}]} = T e^{i \int d^d x L_{\text{int}, IP}} = U(+\infty, -\infty)$$

→ n -point function

$$\begin{aligned} \langle 0 | T \phi_H(x_1) \dots \phi_H(x_n) | 0 \rangle &= \\ &= \frac{\langle 0 | T e^{i S_{\text{int}}[\phi_{IP}]} \phi_{IP}(x_1) \dots \phi_{IP}(x_n) | 0 \rangle}{\langle 0 | T e^{i S_{\text{int}}[\phi_{IP}]} | 0 \rangle} \end{aligned}$$

remark: $\langle 0 | T e^{i S_{\text{int}}[\phi_{IP}]} | 0 \rangle = e^{iL}$

is a phase factor (L is IR divergent) which
cancels all diagrams with disconnected
vacuum bubbles

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$Z[f]$ related to $Z_0[f]$ by

$$Z[f] = \frac{1}{N} e^{i S_{\text{int}} \left[\frac{1}{i} \frac{\delta}{\delta f} \right]} Z_0[f]$$

(perturbative) definition of functional integral
of interacting theory in terms of

$$Z_0[f] = e^{+\frac{i}{2} \int dx dy f(x) \Delta(x-y) f(y)}$$

remark: N determined by normalization condition

$$Z[0] = 1 \Rightarrow$$

$$N = e^{i S_{\text{int}} \left[\frac{1}{i} \frac{\delta}{\delta f} \right]} Z_0[f] \Big|_{f=0}$$