9. Dirac field

Lagrangean of (free) Dirac field:

$$\mathcal{L} = \overline{\psi}\,(i\not{\partial} - m)\,\psi = \overline{\psi}_a\, i(\gamma^\mu)_{ab}\partial_\mu \psi_b - m\,\overline{\psi}_a\,\psi_a$$

$\uparrow$ Dirac index $(a = 1, 2, 3, 4)$

$\Rightarrow$ field equation $\quad (i\not{\partial} - m)\,\psi(x) = 0$

general solution of free Dirac equation:

$$\psi(x) = \sum_s \int d\mu(p)\, [\, b(p,s)\, u(p,s)\, e^{-ipx}$$

$$+\, d(p,s)^*\, v(p,s)\, e^{+ipx}\,]$$

$$\not{p}\, u(p,s) = m\, u(p,s)$$

$$\not{p}\, v(p,s) = -m\, v(p,s)$$

Fourier coefficients $b, d^* \xrightarrow{\text{quantization}}$ operators $b, d^\dagger$

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} = i\psi_a^\dagger = \pi_a \;, \quad \frac{\partial \mathcal{L}}{\partial \dot{\overline{\psi}}_a} = 0$$

$$\{ \psi_a(t,\vec{x}), \underbrace{i\,\psi_b^\dagger(t,\vec{y})}_{\pi_b(t,\vec{y})} \} = i\,\delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y})$$

$\{A, B\} = AB + BA$ <u>anti-commutator</u>

$$\{\psi_a(x), \psi_b(y)\}\Big|_{x^0=y^0} = 0$$

$$\{\psi_a^\dagger(x), \psi_b^\dagger(y)\}\Big|_{x^0=y^0} = 0$$

equivalent to anti-commutation rules for b, d:

$$\{b(p,s), b(p',s')^\dagger\} = \{d(p,s), d(p',s')^\dagger\} =$$

$$= \delta(p,p')\,\delta_{ss'}$$

$$\{b, b\} = \{d, d\} = \{b, d\} = \{b, d^\dagger\} = 0$$

<u>energy-momentum vector of the Dirac field</u>

energy-momentum density: $T^{0\mu} = \sum_a \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a}\partial^\mu \psi_a - g^{0\mu}\mathcal{L}$

$$= i\,\psi_a^\dagger\,\partial^\mu\psi_a - g^{0\mu}\mathcal{L} = i\,\bar{\psi}\gamma^0\partial^\mu\psi - g^{0\mu}\bar{\psi}(i\not{\partial}-m)\psi$$

second term with $\mathcal{L} = \overline{\psi}\,(i\not{\partial} - m)\,\psi$ does not contribute because of field equation $(i\not{\partial}-m)\,\psi = 0$

$$\Rightarrow \quad P^\mu = \int d^3x \; i\,\overline{\psi}\,\gamma^0\,\partial^\mu\psi = \int d^3x \; \psi^\dagger\, i\,\partial^\mu \psi$$

insert Fourier decomposition of ψ:

$$\overset{ex.}{\Rightarrow} \quad P^\mu = \sum_s \int d\mu(p)\; p^\mu \left[b(p,s)^\dagger b(p,s) - d(p,s)\, d(p,s)^\dagger \right]$$

$$= \sum_s \int d\mu(p)\; p^\mu \left[b(p,s)^\dagger b(p,s) + d(p,s)^\dagger d(p,s) \right]$$

$$- \sum_s \underbrace{\int d\mu(p)\; p^\mu\; \delta(p,p)\,\delta_{ss}}_{\int d^3p\; p^\mu\, \underbrace{\delta^{(3)}(\vec{0})}_{\frac{V}{(2\pi)^3}}}$$

minus sign for fermions (in favour of super symmetry? → cancellation of bosonic contributions)

normal ordering of fermionic operators

b, d to the right of all $b^\dagger, d^\dagger$; factor (-1) if two fermionic operators ar interchanged

$$\Rightarrow \quad P^\mu = \int d^3x \; :\psi^\dagger i \partial^\mu \psi: =$$

$$= \sum_s \int d\mu(p) \, p^\mu \left[b(p,s)^\dagger b(p,s) + d(p,s)^\dagger d(p,s) \right]$$

$$= \sum_s \int \left[dn(p,s) + d\bar{n}(p,s) \right] p^\mu$$

$$dn(p,s) = d\mu(p) \, b(p,s)^\dagger b(p,s)$$

$$d\bar{n}(p,s) = d\mu(p) \, d(p,s)^\dagger d(p,s)$$

$$[P^\mu, b(p,s)] = -p^\mu b(p,s)$$

$$[P^\mu, b(p,s)^\dagger] = p^\mu b(p,s)^\dagger$$

$$[P^\mu, d(p,s)] = -p^\mu d(p,s)$$

$$[P^\mu, d(p,s)^\dagger] = p^\mu d(p,s)^\dagger$$

$b(p,s)$ / $d(p,s)$ <u>annihilates</u> a particle / antiparticle with momentum p^μ and spin s

$b(p,s)^\dagger$ / $d(p,s)^\dagger$ <u>creates</u> a particle / antiparticle with momentum p^μ and spin s

why <u>anti</u>particle ?

$\bar{\psi}\gamma^\mu\psi$ is a conserved current

$$j^\mu = q\,\bar{\psi}\gamma^\mu\psi$$

$$Q = \int d^3x \, :j^0(x): \, = q\int d^3x \, :\psi^\dagger(x)\,\psi(x):$$

$$\overset{\text{ex.}}{=} q\sum_s \int \left[dn(p,s) - d\bar{n}(p,s)\right]$$

$$\Rightarrow \quad [Q, b(p,s)] = -q\, b(p,s)$$

$$[Q, b(p,s)^\dagger] = q\, b(p,s)^\dagger$$

$$[Q, d(p,s)] = q\, d(p,s)$$

$$[Q, d(p,s)^\dagger] = -q\, d(p,s)^\dagger$$

i.e. $b(p,s)^\dagger$ creates a state with charge $+q$

$d(p,s)^\dagger$ —ıı— —ıı— —ıı— —ıı— —ıı— $-q$

construction of <u>Fock space</u>:

vacuum state defined by

$$b(p,s)\,|0\rangle = d(p,s)\,|0\rangle = 0 \quad \forall\, \vec{p}, s$$

further states obtained by applying products of $b^\dagger$ and $d^\dagger$ on $|0\rangle$

Pauli exclusion principle guaranteed by anti-commutation relations

e.g. $$b(p,s)^\dagger b(p,s)^\dagger\,|0\rangle =$$

$$= - b(p,s)^\dagger b(p,s)^\dagger\,|0\rangle$$

$$\Rightarrow \quad b(p,s)^\dagger b(p,s)^\dagger\,|0\rangle = 0$$

two identical fermions cannot sit in the same one-particle state

Feynman propagator of the Dirac field

$$\langle 0 | \psi_a(x) \; \psi_b(y) | 0 \rangle = 0$$

$$\left(\langle 0 | (b + d^\dagger)(b + d^\dagger) | 0 \rangle = \langle 0 | b \, d^\dagger | 0 \rangle = \right.$$

$$\left. = - \langle 0 | d^\dagger b | 0 \rangle = 0 \right)$$

$$\langle 0 | \psi_a(x) \; \overline{\psi}_b(y) | 0 \rangle \qquad \left(\langle 0 | (b + \cancel{d^\dagger})(b^\dagger + \cancel{d}) | 0 \rangle \right)$$

$$= \sum_{s,s'} \int d\mu(p) \, d\mu(p') \; u_a(p,s) \, e^{-ipx} \; \overline{u}_b(p',s') \, e^{+ip'y}$$

$$\underbrace{\langle 0 | b(p,s) \; b(p',s')^\dagger | 0 \rangle}_{\delta_{ss'} \, \delta(p,p')}$$

$$= \int d\mu(p) \underbrace{\sum_s u_a(p,s) \, \overline{u}_b(p,s)}_{(\not{p} + m)_{ab}} e^{-ip(x-y)}$$

$$= (i \not{\partial}_x + m)_{ab} \int d\mu(p) \, e^{-ip(x-y)}$$

$$\langle 0 | \overline{\psi}_b(y)\, \psi_a(x) | 0 \rangle \qquad \left(\langle 0 | (\cancel{b^\dagger} + d)(\cancel{b} + d^\dagger) | 0 \rangle \right)$$

$$= \sum_{s,s'} \int d\mu(p)\, d\mu(p')\; \overline{v}_b(p',s')\, e^{-ip'y}\; v_a(p,s)\, e^{+ipx}$$

$$\underbrace{\langle 0 | d(p',s')\, d(p,s)^\dagger | 0 \rangle}_{\delta_{ss'}\, \delta(p,p')}$$

$$= \int d\mu(p)\, \underbrace{\sum_s v_a(p,s)\, \overline{v}_b(p,s)}_{(\not p - m)_{ab}}\; e^{ip(x-y)}$$

$$= -\,(i \not\partial_x + m)_{ab} \int d\mu(p)\, e^{ip(x-y)}$$

definition of <u>time ordering</u> for fermionic fields:

$$T\, \psi_a(x)\, \overline{\psi}_b(y) := \begin{cases} \psi_a(x)\, \overline{\psi}_b(y) & \text{for } x^0 > y^0 \\ -\overline{\psi}_b(y)\, \psi_a(x) & \text{for } x^0 < y^0 \end{cases}$$

$$\Rightarrow \langle 0 | T\, \psi_a(x)\, \overline{\psi}_b(y) | 0 \rangle =$$

$$= \Theta(x^0 - y^0)\, \langle 0 | \psi_a(x)\, \overline{\psi}_b(y) | 0 \rangle$$

$$- \Theta(y^0 - x^0)\, \langle 0 | \overline{\psi}_b(y)\, \psi_a(x) | 0 \rangle$$

$$= \Theta(x^0-y^0)\,(i\partial\!\!\!/_x + m)_{ab} \int d\mu(p)\, e^{-ip(x-y)}$$

$$+\Theta(y^0-x^0)\,(i\partial\!\!\!/_x + m)_{ab} \int d\mu(p)\, e^{+ip(x-y)}$$

$$= (i\partial\!\!\!/_x + m)_{ab} \underbrace{\left[\Theta(x^0-y^0)\int d\mu(p)\, e^{-ip(x-y)} + \Theta(y^0-x^0)\int d\mu(p)\, e^{+ip(x-y)}\right]}_{\frac{1}{i}\Delta(x-y)}$$

explonation of last step:

$$\left[\frac{\partial}{\partial x^0}\Theta(x^0-y^0)\right]\int d\mu(p)\, e^{-ipx} + \left[\frac{\partial}{\partial x^0}\Theta(y^0-x^0)\right]\int d\mu(p)\, e^{+ip(x-y)}$$

$$= \delta(x^0-y^0)\left[\int d\mu(p)\, e^{-ip(x-y)} - \int d\mu(p)\, e^{+ip(x-y)}\right] = 0$$

$$\Rightarrow \langle 0|T\,\psi_a(x)\,\overline{\psi}_b(y)|0\rangle = \frac{1}{i}\,(i\partial\!\!\!/_x + m)_{ab}\,\Delta(x-y)$$

$$=: \frac{1}{i}\,S(x-y)$$

$$\Rightarrow (m - i\partial\!\!\!/_x)\,S(x-y) =$$

$$= (m - i\partial\!\!\!/_x)(m + i\partial\!\!\!/_x)\,\Delta(x-y)$$

$$= (m^2 + \Box_x)\,\Delta(x-y) = \delta^{(4)}(x-y)$$

$\Rightarrow$ $S(x)$ is a <u>Green's function</u> of $m - i\not\partial$ with <u>Feynman boundary conditions</u>

Fourier representation of $S(x-y)$ can be obtained from the Fourier representation of the scalar Green's function $\Delta(x-y)$:

$$S(x-y) = (i\not\partial_x + m)\, \Delta(x-y)$$

$$= (i\not\partial_x + m) \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{m^2 - k^2 - i\varepsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{\not k + m}{m^2 - k^2 - i\varepsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{m - \not k - i\varepsilon}$$

$$\langle 0 | T\, \psi_a(x)\, \psi_b(y) | 0 \rangle = 0$$

$$\langle 0 | T\, \overline{\psi}_a(x)\, \overline{\psi}_b(y) | 0 \rangle = 0$$

vacuum expectation value of time-ordered product of arbitrary number of Dirac field operators: <u>nonvanishing</u> result only if # of ψs = # of $\overline{\psi}$s:

$$\langle 0 | T\, \psi_{a_1}(x_1)\, \overline{\psi}_{b_1}(y_1) \ldots \psi_{a_n}(x_n)\, \overline{\psi}_{b_n}(y_n) | 0 \rangle$$

$$= \sum_{\sigma \in S_n} (-1)^\sigma \frac{1}{i} S_{a_1 b_{\sigma(1)}}(x_1 - y_{\sigma(1)}) \ldots \frac{1}{i} S_{a_n b_{\sigma(n)}}(x_n - y_{\sigma(n)})$$

$$(-1)^\sigma = \begin{cases} +1 & \text{if } \sigma \text{ even permutation of } 1, \ldots, n \\ -1 & \text{—''— —''— odd —''— —''— —''—} \end{cases}$$

<u>Discrete transformations of the Dirac field</u>

<u>Parity</u>

$x' = Px \quad , \quad P^\mu{}_\nu = \mathrm{diag}(1, -1, -1, -1)$

$(x^0, \vec{x}) \to (x^0, -\vec{x})$

$\det P = -1 \quad , \quad P \in \mathcal{L}^\uparrow_-$

$$\psi'(x') = S_P \psi(x) = S_P \psi(P^{-1}x)$$

$$\Rightarrow \quad \psi(x^0, \vec{x}) \xrightarrow{P} S_P \psi(x^0, -\vec{x})$$

$$\Rightarrow \quad \overline{\psi}(x^0, \vec{x}) \xrightarrow{P} \overline{\psi}(x^0, -\vec{x}) \gamma^0 S_P \gamma^0$$

action integral $S = \int d^4x \, \overline{\psi}(x) \, (i\gamma^\mu \partial_\mu - m) \psi(x)$

$$\xrightarrow{P} \int d^4x \, \overline{\psi}(x') \gamma^0 S_P^\dagger \gamma^0 (i\gamma^\mu \partial_\mu - m) S_P \psi(x')$$

$$= \int d^4x \, \overline{\psi}(x) \gamma^0 S_P^\dagger \gamma^0 (i\gamma^\mu \varepsilon(\mu) \partial_\mu - m) S_P \psi(x),$$

where $\varepsilon(0) = 1, \quad \varepsilon(1) = \varepsilon(2) = \varepsilon(3) = -1$

action S <u>invariant</u>, if

1) $\gamma^0 S_P^\dagger \gamma^0 S_P = \mathbb{1}$

2) $\gamma^0 S_P^\dagger S_P = \gamma^0 \quad \Rightarrow \quad S_P^\dagger S_P = \mathbb{1}$

3) $- \underbrace{\gamma^0 S_P^\dagger \gamma^0}_{\overset{1)}{=} S_P^{-1} \overset{2)}{=} S_P^\dagger} \gamma^i S_P = \gamma^i \qquad (i = 1,2,3)$

$$\Rightarrow \quad - S_P^\dagger \gamma^i S_P = \gamma^i \quad \overset{2)}{\Rightarrow} \quad - \gamma^i S_P = S_P \gamma^i$$

$$\Rightarrow \quad \{\gamma^i, S_P\} = 0$$

$\Rightarrow$ the following conditions must be fulfilled by S_P:

$$S_P^\dagger = S_P^{-1}\,,\quad [\gamma^0, S_P] = 0\,,\quad \{\gamma^i, S_P\} = 0 \quad (i=1,2,3)$$

$\Rightarrow$ solution $S_P = e^{i\varphi}\gamma^0$

$P^2 = \mathbb{1} \quad \Rightarrow \quad e^{i\varphi} = (\pm)\, 1$

$$\Rightarrow \quad \psi(x^0, \vec{x}) \xrightarrow{P} \gamma^0 \psi(x^0, -\vec{x}) = \mathcal{P}\, \psi(x^0, -\vec{x})\, \mathcal{P}^{-1}$$

$\uparrow$

operator acting on Fock space

$\mathcal{P}|0\rangle = |0\rangle$

$$\gamma^0 u(\vec{p}, s) = u(-\vec{p}, s)\,,\quad \gamma^0 v(\vec{p}, s) = -v(-\vec{p}, s) \quad (\text{ex.})$$

$$\Rightarrow \quad \mathcal{P}\, b(\vec{p}, s)\, \mathcal{P}^{-1} = b(-\vec{p}, s)$$

$$\mathcal{P}\, d^\dagger(\vec{p}, s)\, \mathcal{P}^{-1} = -d^\dagger(-\vec{p}, s) \quad (\text{ex.})$$

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fermion field operator $\psi(x)$ does <u>not</u> correspond to an observable

$\rightarrow$ <u>but</u> expressions of the form $\bar{\psi}\underbrace{\Gamma}_{\text{complex } 4\times 4 \text{ matrix}}\psi$

are candidates for observable quantities

$\exists$ 16 linear independent complex 4×4 matrices Γ_n ($n = 1, \dots 16$) $\rightarrow$ a general 4×4 matrix Γ can be uniquely represented in the form

$$\Gamma = \sum_{n=1}^{16} c_n \Gamma_n \qquad (c_n \in \mathbb{C})$$

standard basis:

	#
$\Gamma_S = \mathbb{1}$	1
$\Gamma_V^\mu = \gamma^\mu$	4
$\Gamma_T^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] =: \sigma^{\mu\nu}$	6
$\Gamma_P = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\,\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma =: \gamma_5$ $(\varepsilon_{0123} = 1)$	1
$\Gamma_A^\mu = i\gamma^\mu\gamma_5$	4

remarks: $\mathrm{Tr}\,\Gamma_n = 0$ (except $\Gamma_S = \mathbb{1}$)

$$\mathrm{Tr}(\Gamma_m \Gamma_n) = 4\,\delta_{mn}$$

completeness relation:

$$\frac{1}{4}\sum_n (\Gamma_n)_{ab}(\Gamma_n)_{cd} = \delta_{ad}\,\delta_{bc}$$

properties of γ_5:

$$\gamma_5^2 = \mathbb{1}$$

$$\{\gamma_5, \gamma^\mu\} = 0 \Rightarrow [\gamma_5, \sigma^{\mu\nu}] = 0 \Rightarrow [\gamma_5, S_L] = 0$$

$$S_L = e^{-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}} \qquad L \in \mathcal{L}_+^\uparrow$$

$$\gamma_5^\dagger = \gamma_5$$

$$P_L := \frac{1-\gamma_5}{2}, \quad P_R := \frac{1+\gamma_5}{2}$$

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = 0$$

$$\psi_L := P_L \psi, \quad \psi_R := P_R \psi$$

"lefthanded" "righthanded"

component of the field

physical observables of the form $\overline{\psi}(x)\Gamma\psi(x)$:

scalar: $\overline{\psi}(x)\psi(x)$

$x' = Lx, L \in \mathcal{L}$ $\quad \overline{\psi}'(x')\psi'(x') = \overline{\psi}(x)\psi(x)$ (also for P transf.)

pseudoscalar: $\overline{\psi}(x)\gamma_5\psi(x)$

$$\overline{\psi}'(x')\gamma_5\psi'(x') = \det L \; \overline{\psi}(x)$$

vector: $\overline{\psi}(x)\,\gamma^\mu\,\psi(x)$

$$\overline{\psi}'(x')\,\gamma^\mu\,\psi'(x') = L^\mu{}_\nu\,\overline{\psi}(x)\,\gamma^\nu\,\psi(x)$$

pseudovector: $\overline{\psi}(x)\,\gamma^\mu\gamma_5\,\psi(x)$

$$\overline{\psi}'(x')\,\gamma^\mu\gamma_5\,\psi'(x') = \det L \;\; L^\mu{}_\nu\,\overline{\psi}(x)\gamma^\nu\psi(x)$$

tensor: $\overline{\psi}(x)\,\sigma^{\mu\nu}\,\psi(x)$

$$\overline{\psi}'(x')\,\sigma^{\mu\nu}\,\psi'(x') = L^\mu{}_\rho\,L^\nu{}_\sigma\,\overline{\psi}(x)\,\sigma^{\rho\sigma}\,\psi(x)$$

charge conjugation

$\psi^c = C\overline{\psi}^T$, $C = -i\gamma^2\gamma^0$

properties of the charge conjugation matrix C:

$C\gamma_\mu^T C^{-1} = -\gamma_\mu$

$C^T = -C$

$C^\dagger = C^{-1}$

behaviour of the current density under charge conjugation:

$$\overline{\psi}\gamma^\mu\psi \xrightarrow{C} \overline{\psi^c}\gamma^\mu\psi^c = \overline{C\overline{\psi}^T}\gamma^\mu C\,\overline{\psi}^T =$$

$$= \overline{\psi}^* \underbrace{C^\dagger}_{C^{-1}} \gamma^0\gamma^\mu C\,\overline{\psi}^T = \overline{\psi}^*\gamma^{0T}\gamma^{\mu T}\overline{\psi}^T = \psi^T\underbrace{\gamma^{0*}\gamma^{0T}}_{\mathbb{1}}\gamma^{\mu T}\overline{\psi}^T$$

$$= -\,\overline{\psi}\gamma^\mu\psi$$