Exercises "Particle Physics II" (2017)

Numerical values of physical constants and particle physics data can be found at pdg.lbl.gov (Particle Data Tables).

If not stated otherwise, we are using natural units ($\hbar = c = 1$).

1. The energy of an electron in the LEP collider (at maximum beam energy) was about 100 GeV. Compute the corresponding velocity of the electron and express your result in units of the speed of light. Hint: Expand the velocity $\vec{v} = \vec{p}/E$ in powers of m/E keeping only the leading terms.

2. The energy of a proton in the LHC (at maximum beam energy) is 6.5 TeV. Compute the corresponding velocity of the proton and express your result in units of the speed of light.

3. A particle of mass M decays into two particles with masses m_1 and m_2. Express the momenta $|\vec{p}_{1,2}|$ and the energies $E_{1,2}$ of the two decay products in the rest frame of the decaying particle in terms of M, m_1 and m_2.

4. Apply the result of the previous problem to the decay $\pi^\pm \to \ell^\pm \overset{(-)}{\nu}_\ell$, where $\ell = e, \mu$. Determine the velocity of $e^\pm$ and $\mu^\pm$, respectively.

5. Determine the range of possible values of momentum and energy of the electron in the case of the β-decay of a free neutron ($n \to p\, e^- \bar{\nu}_e$). (The neutron is assumed to be at rest.)

6. Compute the Compton length of the pion. Discuss its physical relevance.

7. $\varphi(t, \vec{x})$ is a real (classical) field fulfilling the boundary conditions $\varphi(t_{1,2}, \vec{x}) = f_{1,2}(\vec{x})$ at some times $t_1 < t_2$. Assume further that $\varphi(t, \vec{x})$ minimizes the action integral
$$\int_{t_1}^{t_2} dt \int_{\mathbb{R}^3} d^3x\, \mathcal{L}\left(\varphi(t, \vec{x}), \partial_\mu \varphi(t, \vec{x})\right) \ ,$$
whith some Lagrangian density $\mathcal{L}$. Show that $\varphi(t, \vec{x})$ has to satisfy the Euler-Lagrange equation
$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{,\mu}} = \frac{\partial \mathcal{L}}{\partial \varphi} \ .$$

8. Derive the field equation of φ^4 theory following from the Lagrangian density
$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4!} \varphi^4 \ .$$

9. What is the energy-momentum tensor $T_{\mu\nu}$ of φ^4 theory? Verify $\partial^\mu T_{\mu\nu} = 0$.

10. Defined in terms of the 4-potential $A^\mu = \left(\phi, \vec{A}\right)$, the field strength tensor of the electromagnetic field is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Derive the relations between the components of the field strength tensor and the components of the electric field $\vec{E}$ and the magnetic field $\vec{B}$, respectively. Hint: Remember $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\dot{\vec{A}} - \vec{\nabla}\phi$.

11. The action integral of the electromagnetic field (using the Heaviside system) is given by
$$S_{\text{em}}[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu\right),$$
where $j^\mu = \left(\rho, \vec{j}\right)$ is the electromagnetic 4-current density. Derive Maxwell's equations (in manifestly covariant form) by varying S with respect to A_μ. Hint: Take advantage of partial integration (no surface terms!), and the antisymmetry of $F_{\mu\nu}$.

12. Express the Lorentz invariant $F_{\mu\nu} F^{\mu\nu}$ in terms of $\vec{E}$ and $\vec{B}$.

13. The commutation relations of the creation and annihilation operators of a free hermitian scalar field (spin 0 field) with mass m are given by
$$\left[a(p), a(p')^\dagger\right] = \underbrace{(2\pi)^3 2p^0 \delta^{(3)}(\vec{p} - \vec{p}\,')}_{\delta(p,p')} \;, \quad [a(p), a(p')] = 0 \;,$$
where $p^0 = \sqrt{m^2 + \vec{p}^{\,2}}$. The one-particle momentum eigenstate $|p\rangle$ is defined by $|p\rangle = a(p)^\dagger |0\rangle$. The general form of a normalizable one-particle state $|\psi^{(1)}\rangle$ is given by
$$|\psi^{(1)}\rangle = \int \underbrace{\frac{d^3p}{(2\pi)^3 2p^0}}_{d\mu(p)} |p\rangle\, \psi^{(1)}(p) \;.$$

    (a) $\langle p | p' \rangle = ?$

    (b) Determine the normalization condition for the momentum-space wave function $\psi^{(1)}(p)$ implied by the state normalization $\langle \psi^{(1)} | \psi^{(1)} \rangle = 1$.

    (c) Show that the projection operator
    $$P^{(1)} = \int d\mu(p) |p\rangle\langle p|$$
    satisfies indeed $P^{(1)} P^{(1)} = P^{(1)}$.

14. The two-particle momentum eigenstate $|p_1, p_2\rangle$ is defined by

$$|p_1, p_2\rangle = a(p_1)^\dagger a(p_2)^\dagger |0\rangle \ .$$

(a) $\langle p_1, p_2 | p_1' p_2' \rangle =?$

(b) Determine the operator $P^{(2)}$ projecting on the two-particle subspace.

(c) Discuss the general form of a normalizable two-particle state $|\psi^{(2)}\rangle$ and the properties of the corresponding two-particle wave function in momentum space.

15. The Fourier decomposition of a free real scalar field is given by

$$\phi(x) = \int d\mu(p) \ \left[a(p) e^{-ipx} + a(p)^\dagger e^{ipx}\right] \ .$$

Show that $a(p)$ and $a(p)^\dagger$ can be obtained from $\phi(x)$ by the relations

$$a(p) = i \int d^3x \, e^{ipx} \overleftrightarrow{\partial_0} \phi(x) \ , \quad a(p)^\dagger = -i \int d^3x \, e^{-ipx} \overleftrightarrow{\partial_0} \phi(x) \ .$$

16. Use the previous formulas to show that the canonical equal-time commutation relations for ϕ and $\pi = \dot{\phi}$ imply the commutation relations for $a(p)$ and $a(p)^\dagger$ displayed in problem 13.

17. Express the Hamiltonian of the free scalar field

$$H = \frac{1}{2} \int d^3x \ : [\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2] :$$

in terms of creation and annihilation operators.

18. Express the three-momentum operator of the free scalar field

$$\vec{P} = - \int d^3x \ : \pi \, \vec{\nabla}\phi :$$

in terms of creation and annihilation operators.

19. As shown by the results of the two previous problems, the four-momentum operator P^μ is given by

$$P^\mu = \int d\mu(p) \, p^\mu \, a(p)^\dagger a(p) \ .$$

Verify the following commutation relations:

$$[P^\mu, a(p)] = -p^\mu a(p), \quad \left[P^\mu, a(p)^\dagger\right] = p^\mu a(p)^\dagger \ .$$

20. Show:
$$\exp(iPa)\phi(x)\exp(-iPa) = \phi(x+a) \ .$$
Hint: It is sufficient to check the infinitesimal version of this relation.

21. Show:
$$\langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|T\phi(x-y)\phi(0)|0\rangle \ .$$
Hint: Use the formula of the previous problem.

22. The propagator of the (free) Klein-Gordon field is defined by
$$\Delta(x) = i\langle 0|T\phi(x)\phi(0)|0\rangle \ .$$
Show: $\Delta(-x) = \Delta(x)$.

23. Show that $\Delta(x)$ (as defined in the previous problem) is a Green function of the Klein-Gordon operator, i.e.
$$(\Box + m^2)\Delta(x) = \delta^{(4)}(x) \ .$$
Discuss the behaviour of $\Delta(x)$ for positive (negative) x^0.

24. The one-dimensional harmonic oscillator is described by the Hamilton operator
$$H = \frac{P(t)^2}{2m} + \frac{m\omega^2 Q(t)^2}{2} \ .$$
The position operator $Q(t)$ and the momentum operator $P(t)$ fulfil the canonical commutation relation
$$[Q(t), P(t)] = i\hbar\mathbb{1} \ .$$
   (a) Verify that Heisenberg's equation of motion for $Q(t)$,
   $$\dot{Q}(t) = \frac{i}{\hbar}[H, Q(t)] \ ,$$
   implies the classical equation of motion $\ddot{Q}(t) + \omega^2 Q(t) = 0$.
   (b) Express $Q(t)$ in terms of the ladder operators a and $a^\dagger$, which satisfy the commutation relation $\left[a, a^\dagger\right] = \mathbb{1}$.
   (c) Express $P(t)$ in terms of a and $a^\dagger$.
   (d) Compute the two-point function $\langle 0|TQ(t_1)Q(t_2)|0\rangle$.
   (e) Compute $(\frac{d^2}{dt^2} + \omega^2)\langle 0|TQ(t)Q(0)|0\rangle$.

25. Determine the generating functional of the one-dimensional harmonic oscillator,

$$Z[f] = \langle 0|T e^{\frac{i}{\hbar}\int\limits_{-\infty}^{+\infty} dt\, f(t)Q(t)}|0\rangle \ ,$$

using the path integral representation

$$Z[f] = \frac{1}{\mathcal{N}}\int [dq] \exp\left\{\frac{i}{\hbar}\int\limits_{-\infty}^{+\infty} dt \left[\frac{m}{2}\dot{q}(t)^2 - \frac{m(\omega^2 - i\varepsilon)}{2} q(t)^2 + f(t)q(t)\right]\right\} .$$

Give a physical interpretation of the external field $f(t)$. Verify the result for the two-point function obtained with the operator method.

Hint: The path integral calculation is completely analogous to the one for a free field, discussed in detail in the lecture. The position variable $q(t)$ can be interpreted as a scalar field living in $0+1$-dimensional spacetime.

26. The generating functional of a free *non-Hermitian* scalar field $\phi(x)$,

$$Z[f] = \langle 0|T e^{i\int d^4x \left(f(x)^*\phi(x) + f(x)\phi(x)^\dagger\right)}|0\rangle \ ,$$

can be deduced from the generating functionals of two Hermitian scalar fields $\phi_{1,2}(x)$ with equal masses. Use this relation to derive the explicit form of $Z[f]$.

Hint: $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, $f = (f_1 + if_2)/\sqrt{2}$, $f_i^* = f_i$.

27. Using the result of the previous problem, discuss the pairing rule for the Green function

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(y_1)^\dagger\phi(y_2)^\dagger|0\rangle$$

of a non-Hermitian scalar field $\phi(x)$.

28. Use Noether's theorem to derive the conserved current j^μ associated with the global $U(1)$ symmetry $\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$ of the Lagrange density $\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$.

29. Compute the Gaussian mean values

$$\langle\langle \varphi(x_1)\varphi(x_2)e^{iS_{\text{int}}}\rangle\rangle \ , \qquad \langle\langle e^{iS_{\text{int}}}\rangle\rangle$$

in φ^4 theory,

$$S_{\text{int}} = -\frac{\lambda}{4!}\int d^dy\, \varphi(y)^4 \ ,$$

including the contributions of order λ. Convince yourself that the contributions of graphs with vacuum bubbles cancel when the ratio of the two terms is taken.

30. Compute the Gaussian mean values $\langle\langle\varphi(x)e^{iS_{\rm int}}\rangle\rangle$ and $\langle\langle e^{iS_{\rm int}}\rangle\rangle$ of φ^3 theory,

$$S_{\rm int} = -\frac{g}{3!}\int d^d y\, \varphi(y)^3 ,$$

including the contributions of order g^2. Draw the relevant Feynman diagrams. Express the vacuum expectation value

$$v := \langle 0|\phi(x)|0\rangle = \frac{\langle\langle\varphi(x)e^{iS_{\rm int}}\rangle\rangle}{\langle\langle e^{iS_{\rm int}}\rangle\rangle}$$

in terms of two-point functions of the free theory (to order g^2).

31. Compute the Gaussian mean value $\langle\langle\varphi(x_1)\varphi(x_2)e^{iS_{\rm int}}\rangle\rangle$ of φ^3 theory including the contributions of order g^2. Draw the relevant Feynman diagrams. Express the two-point function of the interacting theory,

$$\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)|0\rangle = \frac{\langle\langle\varphi(x_1)\varphi(x_2)e^{iS_{\rm int}}\rangle\rangle}{\langle\langle e^{iS_{\rm int}}\rangle\rangle} ,$$

in terms of two-point functions of the free theory (to order g^2).

32. Use the results of the two previous problems to determine the two-point function of the field

$$\phi'(x) = \phi(x) - v$$

to order g^2.

33. Show the following formula in dimensional regularization ($\alpha, \beta \in \mathbb{N}$):

$$\int \frac{d^d k}{(2\pi)^d}\frac{(k^2)^\beta}{(M^2 - k^2 - i\varepsilon)^\alpha} = \frac{(-1)^\beta i}{(4\pi)^{d/2}}\frac{\Gamma(\alpha-\beta-d/2)\Gamma(\beta+d/2)}{\Gamma(\alpha)\Gamma(d/2)} M^{d+2\beta-2\alpha} .$$

34. Write the finite one-loop function ($d = 4$)

$$\bar{B}(p^2, m^2) = B(p^2, m^2) - B(0, m^2)$$

in the form

$$\bar{B}(p^2, m^2) = \int_0^1 d\alpha\, f(\alpha, p^2, m^2) .$$

Hint: Use the formula for $B(p^2, m^2)$ obtained in dimensional regularization and perform the limit $d \to 4$ in the difference $B(p^2, m^2) - B(0, m^2)$.

35. The function $\bar{B}(p^2, m^2)$ exhibits a nonvanishing imaginary part in a certain range of the variable p^2. Determine this range by inspecting the properties of the function $f(\alpha, p^2, m^2)$ and compute $\mathrm{Im}\,\bar{B}(p^2, m^2)$.

Hint: The logarithm occuring in the function $f(\alpha, p^2, m^2)$ has a branch cut along the negative real axis. For the computation of $\mathrm{Im}\,\bar{B}(p^2, m^2)$, the $m^2 - i\varepsilon$ prescription must be taken into account!

36. Argue that

$$\bar{B}(s, m^2) = -\frac{1}{(4\pi)^2} \int_0^1 \ln\left[1 - \alpha(1-\alpha)\frac{s}{m^2}\right]$$

defines an analytic function in the complex s plane with a branch cut along the positive real axis for $s > 4m^2$. What is the physical reason for this behaviour of $\bar{B}(s, m^2)$? Note that $\bar{B}(p^2, m^2)$ is recovered by $s \to p^2 + i\varepsilon$.

37. Determine the asymptotic behaviour of $\bar{B}(p^2, m^2)$ for spacelike $p^2 \to -\infty$ and timelike $p^2 \to +\infty$, respectively.

38. Compute the first two terms of the Taylor expansion of $\bar{B}(s, m^2)$ around $s = 0$. Which radius of convergence do you expect for this power series?

39. Compute $\bar{B}(p^2, m^2)$ for spacelike p^2.

40. Use partial integration to show that $\bar{B}(p^2, m^2)$ can be rewritten as

$$\bar{B}(p^2, m^2) = \frac{p^2}{(4\pi)^2} \int_0^1 d\alpha \, \frac{\alpha(2\alpha - 1)}{m^2 - \alpha(1-\alpha)p^2 - i\varepsilon}.$$

41. Verify the relation

$$\frac{1}{m^2 - \alpha(1-\alpha)p^2 - i\varepsilon} = \int_{-\infty}^{+\infty} ds \, \frac{1}{s - p^2 - i\varepsilon} \, \delta(m^2 - \alpha(1-\alpha)s)$$

and insert it in the formula of the previous problem. After interchanging the order of the two integrations, perform the integration over the Feynman parameter α to arrive at the dispersion integral

$$\bar{B}(p^2, m^2) = \frac{p^2}{(4\pi)^2} \int_{4m^2}^{\infty} ds \, \frac{1}{s - p^2 - i\varepsilon} \cdot \frac{\lambda^{1/2}(s, m^2, m^2)}{s^2},$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$.

42. Using the dispersion integral for $\bar{B}(p^2, m^2)$, verify your previous result for $\mathrm{Im}\,\bar{B}(p^2, m^2)$.

    Hint: Use the formula $\frac{1}{x - i\varepsilon} = P\frac{1}{x} + i\pi\delta(x)$.

43. Show that the real part of $\bar{B}(p^2, m^2)$ can be obtained from the the imaginary part of $\bar{B}(s, m^2)$ by a dispersion relation.

44. Consider the kinematics of the scattering process $\varphi(p_1)\varphi(p_2) \to \varphi(p_3)\varphi(p_4)$ in the center of mass system. Express the Mandelstam variables $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$ in terms of $s = (p_1 + p_2)^2$ and the scattering angle θ in the center of mass system.

45. The generating functional of the free Dirac field was found to be

$$\begin{aligned} Z[\eta, \bar{\eta}] &\equiv \left\langle 0 \left| T \exp\left\{ i \int d^4x \, \left[\bar{\eta}(x)\Psi(x) + \overline{\Psi}(x)\eta(x) \right] \right\} \right| 0 \right\rangle \\ &= \exp\left\{ i \int d^4x \, d^4y \, \bar{\eta}(x) S(x-y) \eta(y) \right\} . \end{aligned}$$

Using this formula, compute

$$\langle 0 | T\{ \Psi_{a_1}(x_1) \overline{\Psi}_{b_1}(y_1) \Psi_{a_2}(x_n) \overline{\Psi}_{b_2}(y_n) \} | 0 \rangle .$$

46. Compute the two-body phase space integral

$$\int \frac{d^3k_1}{2k_1^0} \frac{d^3k_2}{2k_2^0} \delta^{(4)}(P - k_1 - k_2) f(k_1 \cdot k_2),$$

where $k_i^2 = m_i^2$. P denotes a timelike 4-vector. Give your final answer in terms of the function $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz)$.

Hint: The integral is a Lorentz scalar.

47. Compute

$$I^\mu = \int \frac{d^3k_1}{2k_1^0} \frac{d^3k_2}{2k_2^0} \, \delta^{(4)}(P - k_1 - k_2) \, k_2^\mu .$$

Hint: The integral is a Lorentz vector and can be written in the form $I^\mu = J P^\mu$, where J is a scalar function (why?).

48. Compute the tensor integral

$$I^{\mu\nu} = \int \frac{d^3k_1}{2k_1^0} \frac{d^3k_2}{2k_2^0} \, \delta^{(4)}(P - k_1 - k_2) \, k_1^\mu k_2^\nu .$$

49. Compute the invariant amplitude of electron proton scattering ($e^- p \to e^- p$) at tree level. Describe the electromagnetic interaction of the proton by simply using the appropriate covariant derivative in the Dirac Lagrangean of the proton.

Remark: In contrast to the electron, the proton is not a pointlike partice. This reflects itself by the fact that e.g. the magnetic moment of the proton is not correctly desribed by the Dirac Lagrangean alone. In a phenomenological description, an additional term $\sim \bar{\psi} \sigma_{\mu\nu} \psi F^{\mu\nu}$ is needed. In this exercise, we ignore these complications.

50. Compute the differential cross section of electron proton scattering in the rest frame of the proton, using the result of the previous problem. Determine also the total cross section of this reaction.

51. Verify the Gordon decomposition:

$$\bar{u}(p', s')\gamma^\mu u(p, s) = \bar{u}(p', s') \left[p^\mu + p'^\mu + i\sigma^{\mu\nu}(p' - p)_\nu\right] u(p, s)/(2m).$$

52. Compute $\gamma^\alpha\gamma^\mu\gamma_\alpha$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha$ and $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha$ ($d = 4$).

53. The Higgs ($M_h = 125\,\text{GeV}$) interacts with the elementary fermions by the Yukawa couplings

$$\mathcal{L} = -\sum_f g_f \bar{f}(x) f(x) h(x),$$

where $g_f = m_f/v$ ($v = 246\,\text{GeV}$). Compute the contribution of the virtual Higgs boson to the anomalous magnetic moment of the electron (at one loop).

54. The structure constants f_{abc} of a Lie algebra $\mathcal{L}$ with generators T_a are defined by

$$[T_a, T_b] = i f_{abc} T_c.$$

Show that f_{abc} is totally antisymmetric, if the T_a form an orthogonal basis of $\mathcal{L}$ ($\text{Tr}(T_a T_b) = c\,\delta_{ab}$).

55. Show that $(t_a)_{bc} = -i f_{abc}$ defines a representation of $\mathcal{L}$ (adjoint representation).

56. The transformation formula for a nonabelian gauge field $A_\mu = A_\mu^a T_a$ under a local gauge transformation is given by

$$A'_\mu = U A_\mu U^{-1} + \frac{i}{g}(\partial_\mu U) U^{-1}.$$

Show that the generalized field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

transforms as

$$F'_{\mu\nu} = U F_{\mu\nu} U^{-1}.$$

57. The positive frequency part of the massless scalar propagator is given by

$$\Delta_0^+(x) := i \int \frac{d^3p}{(2\pi)^3 2p^0}\, e^{-ip\cdot x}, \quad x^0 \to x^0 - i\varepsilon, \quad p^0 = |\vec{p}\,|.$$

Show that this function can be written in the form

$$\Delta_0^+(x) = \frac{1}{4\pi^2 i[(x^0 - i\varepsilon)^2 - \vec{x}^{\,2}]}.$$