Exercises for T2, Summer term 2017, Sheet 11

1) Harmonic oscillator in the Heisenberg picture

Let there be a one-dimensional harmonic oscillator

$$H = P^2/2m + m\omega^2 X^2/2.$$

Determine the time evolution of the position and momentum operators using the Heisenberg equations of motions. Write down the relation between the position and momentum uncertainties at the time t and those at the time $t = 0$ for an arbitrary state. What do you get if the state has the minimal product of uncertainties at the time $t = 0$. What if, additionally to that, the relation $\Delta X(0) = \Delta P(0)/m\omega$ is fulfilled as well?

2)Spin precession in a time-independent magnetic field

Let there be a spin-1/2 system with the Hamiltonian

$$H = -\vec{\mu}\,\vec{B}\,, \qquad \vec{\mu} = \gamma\vec{S}\,, \qquad \vec{B} = B\,\vec{e}_z\,.$$

(a) Determine the Heisenberg equations for the Heisenberg spin operators $\vec{S}_H(t)$ and solve them with the initial condition that the Heisenberg spin operators at $t = 0$ coincide with the corresponding Schrödinger spin operators, i.e $\vec{S}_H(0) = \vec{S}_S$.

(b) Solve the problem in the Schrödinger picture for the two-component spin wave function

$$\begin{pmatrix} a_+(t) \\ a_-(t) \end{pmatrix}$$

(c) Show the equivalence of the two solutions by calculating the time evolution of the expectation values for a spin state that initially points in x-direction at $t = 0$, in both the Heisenberg and the Schrödinger picture.

3) Orbital angular momentum

Show that the components of the orbital angular momentum operator

$$L_k = \varepsilon_{klm} X_l P_m$$

fulfill the following commutation relations:

$$[L_k, L_l] = i\hbar\varepsilon_{klm} L_m, \quad [L_k, X_l] = i\hbar\varepsilon_{klm} X_m, \quad [L_k, P_l] = i\hbar\varepsilon_{klm} P_m.$$

Hint: Use the commutation relations $[X_k, P_l] = i\hbar\delta_{kl}$ and $[X_k, X_l] = [P_k, P_l] = 0$, without specifying any particular representation.

4) Spin-1

A spin-1 system is prepared in the eigenstate of L_3 with eigenvalue 0. What are the probabilities to get the results $+\hbar$, 0 and $-\hbar$ for a measurement of L_1?

5) Spin operator

Take the spin operator $\vec{S}$ in the fundamental (spin-1/2) representation of SU(2). This is a so-called vector operator, since it has three components for the three Cartesian coordinates. Determine the vector operator $\vec{Q} \equiv \vec{S} \times \vec{S}$, und the commutator $[S_k, Q_l]$.

6) Spherical harmonics

Find all four other spherical harmonics for angular momentum $\ell = 2$, by acting with L_- sufficiently many times on $Y_{2,2}(\theta, \phi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$.