Exercises for T2, Summer term 2017, Sheet 10

1) Spin measurements

Initially, let there be a spin-1/2 system in a pure spin state $\chi_{z,+}$, where the spin is pointing in the z-direction.

(a) For which spin operator is $\chi_{z,+}$ an eigenstate and what is the corresponding eigenvalue?

(b) Write down the density matrix for the spin state $\chi_{z,+}$.

(c) What are the probabilities to get the values $\pm\hbar/2$ if a measurement of the spin in x-direction is taken? What are the corresponding probabilities for a measurement of the spin in y-direction? What are the respective expectation values?

(d) Let's suppose you get the value $-\hbar/2$ for a measurement of the spin in x-direction. Assume that the spin is still there and not destroyed by the measurement. In which state is the system after the measurement?

(e) Now take another measurement of the spin in the $\vec{n}$ direction on this spin. The vector $\vec{n}$ is a unit vector which is rotated with respect to the x direction by the angle ϕ around the z axis. What is the probability to get the values $\hbar/2$ for this measurement?

(f) Determine the expectation value for that spin measurement in the $\vec{n}$ direction.

2) Magnetic moment im thermodynamic equilibrium

Let there be a spin-1/2 system in an external magnetic field $\vec{B} = B\vec{e}_3$. The operator of the magnetic moment is $\vec{\mu} = \gamma\vec{S}$ and the Hamilton operator is $H = -\vec{\mu}\cdot\vec{B}$. If the spin is in contact with a heat bath of temperature T, then the corresponding equilibrium state is described by the density matrix

$$\rho = \mathcal{N}\exp(-\beta H), \quad \beta = 1/kT$$

Determine the normalization factor $\mathcal{N}$. Calculate the expectation value and the mean square deviations of μ_i $(i = 1, 2, 3)$ and H. Sketch the expectation value of μ_3 as function of temperature T.

3) Computation with an observable in a Hilbert space with finite dimensions

Let $\mathcal{H} = \mathbb{C}^3$ be the Hilbert space of complex 3-dimensional vectors with the usual scalar product $\langle\chi|\psi\rangle = \sum_{k=1}^{3} \chi_k^* \psi_k$. An observable is given by the Matrix A which has the form

$$A = \begin{pmatrix} 1 & \sqrt{2}i & \sqrt{2}i \\ -\sqrt{2}i & 2 & -1 \\ -\sqrt{2}i & -1 & 2 \end{pmatrix} \qquad \left[|\phi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right].$$

(a) What are the possible values $a_1, \ldots$ for a measurement of this observable?

(b) Determine the corresponding orthonormalized eigenvectors - which are of course only unique up to a complex phase. You should use the Schmidt's orthogonlization method.

(c) Determine the projector onto the eigenspace that corresponds to the positive eigenvalue.

(d) Determine for each possible measurement value the probability to obtain it in a measurement when the system is in the state $|\phi\rangle$.

(e) Determine for each of the possible measurement values the probability to obtain it in a measurement, when the system is in the state given by the density matrix

$$\rho = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(f) Determine the expectation value in the measuremnt of the observable A you obtain in taking many measurements on systems that are in the state given by the density matrix ρ.