Exercises for T2, Summer term 2017, Sheet 9

1) General uncertainty principle

The variance $(\Delta_\omega A)^2$ of some observable A with respect to a state ω is defined by

$$(\Delta_\omega A)^2 = \omega((A - \omega(A))^2)$$

Given two hermitian operators $A, B \in \mathrm{L}(\mathcal{H})$, show that for an arbitrary state ω the inequality

$$\Delta_\omega A \Delta_\omega B \geq |\omega(\frac{i}{2}[A, B])|.$$

holds. For this one can use the (non-hermitian) operator

$$C = \frac{A - \omega(A)}{\Delta_\omega A} + i\frac{B - \omega(B)}{\Delta_\omega B}$$

and the functional properties (a)-(c) of a general state ω as discussed in Chapter 4.2 of the lecture notes.

2) Mixed state

(a) Show that a state which is given by the density matrix ρ is a mixed state if $\rho^2 \neq \rho$ holds.

(b) Show that a state which is given by the density matrix ρ is a mixed (pure) state if $\mathrm{Tr}[\rho^2] < 1$ ($\mathrm{Tr}[\rho^2] = 1$) holds.

3) Harmonic oscillator in thermal equilibrium

Given a harmonic osciallator with angular frequency ω which is in thermal equilibrium with an external heatbath of absolute temperature T. The density matrix then has the form:

$$\rho = \frac{\exp(-\mathbf{H}/kT)}{\mathrm{Tr}[\exp(-\mathbf{H}/kT)]},$$

where $\mathbf{H}$ is the Hamilton operator and k the Boltzmann constant.

(a) Calculate the spectral representation of the mixed state ρ in Bra-Ket notation as a function of the temperature, where $|\phi_n\rangle$ is the normalized eigenstate with occupation number n. Note that the sum of the geometric series is very helpful for this calculation.

(b) Calculate the average occupation number $\langle N \rangle$ and the average energy $\langle H \rangle$ as a function of temperature T. ($\langle N \rangle = (\exp(\hbar\omega/kT) - 1)^{-1}$)

(c) Calcuate the average occupation number for visible light ($\lambda = 550\mathrm{nm}$) at room temperature ($T = 295\mathrm{K}$) and at the surface of the sun ($T = 5500\mathrm{K}$).

1) Pauli matrices

The Pauli matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show by explicit calculation that the following relations hold:

a) $\sigma_1\sigma_2\sigma_3 = i\mathbb{1}_{2x2}$

b) $[\sigma_k, \sigma_l] = 2i\varepsilon_{klm}\sigma_m$ (note: sum convention)

c) $\sigma_k\sigma_l + \sigma_l\sigma_k = 2\delta_{kl}\mathbb{1}_{2x2}$

d) Show from b) and c) that: $\sigma_k\sigma_l = \delta_{kl}\mathbb{1}_2 + i\varepsilon_{klm}\sigma_m$

e) Use d) to show that $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\mathbb{1}_2 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}, \quad \vec{a},\vec{b} \in \mathbb{R}^3$