Exercises for T2, Summer term 2017, Sheet 8

1) Spatial translation

The operator T_a acts on a wave function $\psi(x)$ (in one spatial dimension) as

$$(T_a\psi)(x) = \tilde{\psi}_a(x) = \psi(x-a), \quad a \in \mathbb{R}\,. \tag{1}$$

Show that T_a has the explicit form $T_a = \exp(-a\frac{d}{dx})$. Also write this expression as a function of the momentum operator P and formulate Eq. (1) in abstract form for Ket states.

2) Particle scattering on the potential barrier I

Let there be a one-dimensional system of a particle with mass m in the potential $V(x) = V_0\Theta(x)\Theta(a-x)$. Take now the eigenfunctions $\phi_k(x)$ of the eigenvalue $E(k) = \hbar^2k^2/2m > V_0$ that correspond to a particle entering from the left, which have been discussed in the lecture (english lecture notes, chapter 3.4).

(a) Determine the amplitudes A, B und T by using continuity of the wave function at the points $x = 0$ and $x = a$.

(b) Determine the probability current in the three regions $x < 0$, $0 < x < a$ and $x > a$ as a function of A, B and T. Interpret the individual contributions.

(c) Show that conservation of particle number is valid in all regions.

3) Particle scattering on the potential barrier II

Work through exercise (2) for the case $E(k) = \hbar^2k^2/2m < V_0$.

4) Distributions

Calculate the first and second derivative of the following functions in the distributional sense ($\theta(x)$ is the Heaviside step function).

- $\theta(x)$
- $\theta(-x)$
- $|x| = -x\,\theta(-x) + x\,\theta(x)$
- $e^{-a|x|} = e^{ax}\,\theta(-x) + e^{-ax}\,\theta(x),\ a > 0.$

5) Particle in the delta-potential

Let the wave function of a particle with one degree of freedom be given by

$$\psi(x) = \mathcal{N}\exp(-a|x|), \quad a > 0.$$

(a) Convince yourself with the help of results from exercise (4) that the wave function $\psi(x)$ is, for a suitable choice of the parameter a, an energy eigenfunction of the Hamilton operator

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \lambda\,\delta(x), \quad (\lambda > 0)$$

What is the necessary choice for a and what is the result for the energy eigenvalue E?

(b) Determine the probability current for $|x| > 0$ and argue that the wave function has to be interpreted as a bound state. What is the interpretation of the corresponding energy eigenvalue?

(c) Argue why there cannot be any further bound states.