Exercises for T2, Summer term 2017, Sheet 6

1) H-atom in three spatial dimensions

Calculate the expectation values of $\vec{P}$ und $\vec{P}^2$ for the ground state wave function of the H-atom. See exercise (2) on sheet 4.

2) Momentum space wave function with minimal uncertainty

The momentum space wave functions $\tilde{\psi}(p)$ (in 1 dimension) which minimize the product of position and momentum uncertainty $\Delta X \Delta P = \hbar/2$ are characterized by the equation:

$$\left(\frac{X - x_0}{\sigma} + i \, \frac{P - p_0}{\hbar/2\sigma} \right) \tilde{\psi}(p) = 0$$

where P is the momentum operator in momentum space representation and $X = i\hbar \, \partial/\partial p$ the position operator in momentum space representation. x_0 and p_0 are the expectation value of the position and momentum operator respectively. Furthermore the position uncertainty is given by $\sigma = \Delta X$. By solving the differential equation, calculate $\tilde{\psi}(p)$ which fulfills the usual normalization condition

$$\int_{-\infty}^{+\infty} dp \, |\tilde{\psi}(p)|^2 = 1$$

The result is discussed in the lecture notes, Chap. 2.9.

3) Time evolution in momentum space

Let a free particle with mass m be described at $t = 0$ by the wave function calculated in exercise (2) (see also the lecture notes, chap. 2.9.). Determine the time-dependent momentum space wave function $\tilde{\psi}(p, t)$ as well as mean and mean square deviations of X and P at time t.

4) Time evolution in configuration space

Use the result of exercise (3) for $x_0 = 0$, and determine the time-dependent configuration space wave function $\psi(x, t)$. Write down $|\psi(x, t)|^2$ as well. Use the method of quadratic completion. (See also the hand-written lecture notes, chap. 2.9. for the discussion of the Gaussian wave packet).

Hint: Let $c, d \in \mathbb{C}$ (!) where $\mathrm{Re}\, c > 0$. Then the formula

$$\int_{-\infty}^{+\infty} dx \, e^{-c(x-d)^2} = \sqrt{\frac{\pi}{c}}$$

holds as in the real case! Check this by simply performing some numerical tests (e.g. with Mathematica). (You can download the Mathematica notebook used in the lecture from the lecture webpage.)

5) Commutators

Show the following idenities for the commutator $[A, B] = AB - BA$ of two linear operators A and B. (Let C also be a linear operator and α, β two complex numbers.)

(a) $[A, B] = -[B, A]$

(b) $[\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C]$

(c) $[AB, C] = A[B, C] + [A, C]B\,, \quad [A, BC] = B[A, C] + [A, B]C$

(d) $[A, [B, C] + [B, [C, A]] + [C, [A, B]] = 0$ (Jacoby identity)

6*) Video-clip

Visualize $|\psi(x, t)|^2$ from exercise (3) as a function of t in a video-clip by using for example the `(List)Animate`-function of Mathematica. Try to think about how to choose the parameters so that one gets a nice-looking visualization.

This star exercise is not mandatory and acutally represents a small project that tries to encourage you to look beyond the scope of this lecture. Discuss your results with other students.