Exercises for T2, Summer term 2016, Sheet 5

1) Real valued wavefunction

Consider the **special case** of a **real valued** wavefunction $\varphi(x)$ in one dimension which vanishes for $x \to \pm\infty$. Show that in this case the expectation value of the momentum operator is zero regardless of the other properties of the wave function.

2) Consider the wave function of a particle with one degree of freedom of the form

$$\psi(x) = \varphi(x) e^{ip_0 x/\hbar}$$

with p_0 being real and a **real valued** function φ which fulfills:

$$\int_{-\infty}^{+\infty} dx\, \varphi(x)^2 = 1\,,$$

This means $\varphi(x)$ is a function similar to the one given in **1)** Calculate the expectation value of the momentum operator. What is the physical meaning of p_0?

3) Momentum operator and uncertainty relation

Use the wave function $\psi(x)$ of the Gaussian wave packet from exercise **(4)** on sheet 4 to calculate the expectation value of P and P^2. Determine the standard deviation ΔP. Check the uncertainty principle. Try to be efficient and use e.g. a suitable substitution of variables to get easier integrals or expressions known from earlier exercises and exercise sheets.

4) Same calculation in momentum space

Calculate the momentum space wave function $\tilde{\psi}(p)$ of the Gaussian wave packet from exercise **(4)** of sheet 4. Furthermore calculate the respective expectation value of X, X^2, P and P^2 using the momentum space representation wave function. Try to be efficient and use e.g. a suitable substitution of variables to get easier integrals or expressions known from earlier exercises and exercise sheets.

5) Commuting operators

Suppose A, B and C are linear operators which fulfill $[A, C] = [B, C] = 0$. Does this also mean that $[A, B] = 0$ holds?

6) Matrix elements und wave functions

Let X ($\vec{X}$) und P ($\vec{P}$) be the location and momentum operators in one dimension (three dimensionss). Let the state $|\psi\rangle$ have the configuration space wave function $\langle\vec{x}|\psi\rangle = \psi(\vec{x})$ and the momentum space wave function $\langle\vec{p}|\psi\rangle = \tilde{\psi}(\vec{p})$.

(a) Determine the following matrix elements in one dimension:
$\langle x|X|p\rangle,\ \langle p|X|x\rangle,\ \langle x|X|x'\rangle,\ \langle p|X|p'\rangle$

(b) Determine the following matrix elements in three dimensions:
$\langle\vec{x}|\vec{P}|\vec{x}'\rangle,\ \langle\vec{p}|\vec{P}|\vec{x}'\rangle$

(c) Determine the following matrix elements in three dimensions (m real and positive):
$\langle\vec{x}|\frac{1}{|\vec{X}|}e^{-m|\vec{X}|}|\vec{x}'\rangle,\ \langle\vec{x}|\frac{1}{|\vec{X}|}e^{-m|\vec{X}|}|\psi\rangle,\ \langle\vec{p}|\frac{1}{|\vec{X}|}e^{-m|\vec{X}|}|\psi\rangle$

7) Two-dimensional Hilbert space and representation of bra- and ket-vectors

Suppose a two-dimensional complex valued Hilbert space with an orthonormal basis $\{|a_1\rangle, |a_2\rangle\}$ (a-representation). Two vectors are given by:

$$|b_1\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle + i\,|a_2\rangle) \qquad |b_2\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle - i\,|a_2\rangle)\,.$$

(a) Show that $\{|b_1\rangle, |b_2\rangle\}$ also form an orthonormal basis (b-representation).

(b) Write down the coordinate representation of the ket-vectors $|a_1\rangle$, $|a_2\rangle$, $|b_1\rangle$, $|b_2\rangle$ and of the respective bra-vectors in the a-representation.

(c) Write down $|a_1\rangle$, $|a_2\rangle$ as functions of $|b_1\rangle$, $|b_2\rangle$ and calculate the corresponding coordinate representation.

(d) Calculate the entries (in terms of scalar products $\langle b_i|a_j\rangle$, $i,j = 1,2$) of the 2×2 matrix which is transforming a vector from the a- to b-representation. Use the completeness relation from the lecture.

8) Abstract linear operator

Suppose some linear operator T which acts on a complex-valued Hilbert space is defined by $T := |u\rangle\langle u|$ (with $|u\rangle \neq 0$).

(a) Is T hermitian?

(b) What feature is needed from $|u\rangle$ so that T is a projection operator?

(c) Suppose B is an arbitrary linear operator which acts on the same Hilbert space. Show that the trace of the operator TB is given by $\langle u|B|u\rangle$. Recall that the trace of an operator does not depend on the used basis.