Exercises for T2, Summer term 2017, Sheet 4

1) Parametric integral

Calculate the integral that depends on the parameter $u > 0$

$$I(u) = \int_0^\infty dr\, e^{-ur}.$$

Then calculate

$$\int_0^\infty dr\, r^n\, e^{-ur}$$

($n \in \mathbb{N}$) from $I(u)$ without doing any additional integration.

2) H-Atom

The wave function of the ground state of an electron in an hydrogen atom has the form

$$\psi(\vec{x}) = \mathcal{N} \exp(-r/a).$$

Here $r = |\vec{x}|$ is the distance to the nucleus, $a = \hbar/m_e \alpha c$ the Bohr radius, m_e the mass of the electron and $\alpha = e^2/\hbar c \simeq 1/137$ the fine structure constant.

(a) What numerical value do you find for the Bohr radius?

(b) How do you have to choose $\mathcal{N}$, such that the wave function is normalized correctly?

Hint: In this problem you have to consider a wave function in 3 spatial dimensions, but the state $|\psi\rangle$ is still defined in an infinite dimensional Hilbert space. Use pherical coordinates to do the computation and use the result from exercise 1.

3) Gaussian integral

(a) Show that $\int\limits_{-\infty}^{+\infty} dx \exp(-ax^2) = \sqrt{\pi/a}$, with $a > 0$.

(b) Calculate $\int\limits_{-\infty}^{+\infty} dx\, x \exp(-ax^2)$.

(c) Calculate $\int\limits_{-\infty}^{+\infty} dx\, x^2 \exp(-ax^2)$.

(d) Calculate $\int\limits_{-\infty}^{+\infty} dx\, x^n \exp(-ax^2)$, for an arbitrary natural number n.

4) Gaussian wave packet

Consider a wave function in one spatial dimension, given by

$$\psi(x) = \mathcal{N} \exp(-x^2/4\sigma^2)\,, \qquad (\sigma \in \mathbb{R}^+)$$

(a) What is the normalization constant $\mathcal{N}$?

(b) What is the expectation value for a measurement of the position?

(c) Is the square of the position operator X^2 a hermitian operator? What is the expectation value for a measurement of X^2?

(d) Calculate the expected standard deviation Δx, that one will get in the limit of infinitely many position measuremnts (on identical copies, each of them in the state $\psi(x)$).

5) Computation with an observable in a Hilbert space with finite dimensions

Let $\mathcal{H} = \mathbb{C}^3$ be the Hilbert space of complex 3-dimensional vectors with the usual scalar product $\langle \chi | \psi \rangle = \sum_{k=1}^{3} \chi_k^* \psi_k$. Let a particular observable be given by the matrix A

$$A = \begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad \left[|\phi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$$

(a) What are the possible values a_1, a_2, a_3 that can occur in a measuerment of this observable?

(b) Determine the orthonormalized eigen vectors that correspond to each eigen value. These eigen vectors are unique up to a complex phase which you can choose as you wish.

(c) Determine that probability for each measuerment value $a_1, \ldots$ that is occurs in the system is in the state $|\phi\rangle$.

6) Spatial translation

The operator T_a acts on a wave function $\psi(x)$ (in one spatial dimension) as

$$(T_a \psi)(x) = \psi(x - a), \quad a \in \mathbb{R}$$

Give an intuitive interpretation of the action of T_a. What is the product $T_a T_b$? What is $T_a^\dagger$, $T_a T_a^\dagger$ and $T_a^\dagger T_a$? Classify the operator T_a with respect to the properties discussed in the lecture (linearity, unitariy, hermiticity).

7) Group properties

Show that $\{T_a | a \in \mathbb{R}\}$ is an abelian group with respect to the product $T_a T_b$.