Quantum Mechanics (T2) Lecture Exam Question Sheet

The following problems show you the type of problems you can expect for the lecture exam. In the exam you have to work on 5 different problems.

For the lecture exam you are NOT allowed to use any helping equipments.

Axioms of quantum theory I

Consider the case of a *finite-dimensional* space of states $\mathcal{H}$. Here the space of states $\mathcal{H}$ is a unitary vector space, i.e. a vector space over the field of complex numbers, equipped with a scalar product. Use bra and ket notation when suitable.

1. What are the defining properties of a scalar product?
2. How can pure states and linear operators A be represented assuming that $\mathcal{H}$ is n-dimensional? What is the explicit form of the scalar product of two states ? What is the definition of the adjoint operator $A^\dagger$ to a given linear operator A?
3. How are (i) Hermitian operators, (ii) projection operators, (iii) unitary operators defined?
4. How is an observable (observable quantity) represented in the mathematical formalism of quantum theory? What is the correspondence principle?
5. What are the mathematical equivalents of the possible measurement values of an observable?
6. What are the eigenvalues of a projection operator?
7. Write down the *general* mathematical definition of a state in quantum theory.

Axioms of quantum theory II

Consider the case of a *finite-dimensional* space of states $\mathcal{H}$. Here the space of states $\mathcal{H}$ is a unitary vector space, i.e. a vector space over the field of complex numbers, equipped with a scalar product. Use bra and ket notation when suitable.

1. Write down the *general* mathematical definition of a state in quantum theory.
2. What is a pure state? What is a mixed state? Which type of state is more general?
3. What is a complete orthonormal system?
4. What is the "spectral representation" pf an observable A.
5. Explain the meaning of the "completeness relation".
6. The system shall be in a pure normalized state $|\phi\rangle$ and one makes a measurement of observable A. Assume that all eigenvalues of A are non-degenerate. What kind of outcome can you expect for a single measurement?

7. In which state is the system after the measurement?
8. What is the mean value for many measurements on identical copies of the system in the state $|\phi\rangle$? What is the standard deviation for these measurements?
9. What is the Hamilton operator? How does the state $|\phi(t = 0)\rangle\rangle$ given at time $t = 0$ evolve in time? What is the time evolution operator? Assume that the Hamiltonoperator is time-independent.

Axioms of quantum theory III

Consider the case of a *finite-dimensional* space of states $\mathcal{H}$. Here the space of states $\mathcal{H}$ is a unitary vector space, i.e. a vector space over the field of complex numbers, equipped with a scalar product. Use bra and ket notation when suitable.

1. What are the defining properties of a density operator? What is its physical meaning?
2. What additional property does the density operator have in case of a pure state?
3. Write down the form of the density operators for a state with maximal mixing.
4. Write down the form of the density operator for the state if the system is in thermodynamical equilibrium with an external heat bath with absolute temperature T.
5. Explain the content of the Neumann's Projection Theorem. How does the density operator of the system change after the measurement of a general observable A?
6. Explain "Schrödinger Picture" and "Heisenberg Picture" for a system where the Hamilton operator is time-independent.
7. How does the von-Neumann equation for the time-evolution $\rho(t)$ of a density operator in the Schr"odinger picture look like? ($\dot{\rho}(t) = \ldots ?$) Also write down the relation between $\rho(t)$ and $\rho(0)$ in case of a time-independent Hamilton operator.
8. In the Heisenberg picture an observable is represented by a time-dependent operator $A(t)$. How does the Heisenberg equation look like? Also write down the relation between $A(t)$ and $A(0)$ in case of a time-independent Hamilton operator.

Uncertainty relations

1. Write down the *general* mathematical definition of a state ω in quantum theory.
2. Let there be a Hermitian operator A and a state ω. What is the definition of the mean square deviation of A in the state ω?
3. What is the general definition of the uncertainty relation for two hermitian operators A, B, if the system is in the state ω?
4. Write down the uncertainty relation for the special case of position and momentum operators in one spatial dimension.

5. Verify the result of part 4 by evaluating the formula from part 3 concretely.
6. The Hamilton operator of the one-dimensional harmonic oscillator is:

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2 \,,$$

where ω is the frequency. Determine the ground state energy of the harmonic oscillator by means of the uncertainty relation for X and P. You can assume that the ground state is a state with minimal uncertainty.

Calculations with observables

Let $\mathcal{H} = \mathbb{C}^3$ with the usual scalar product $\langle \chi | \psi \rangle = \sum_{k=1}^{3} \chi_k^* \psi_k$. Furthermore let a certain observable be represented by the matrix

$$A = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad \text{[example for illustration !]}$$

1. What are the possible values a_1, a_2, a_3 for a measurement of this observable?
2. Determine the corresponding orthonormalized eigenvectors e_1, e_2, e_3.
3. Determine the projectors P_1, P_2, P_3 onto the eigenspaces that correspond to the different eigenvalues.
4. Determine the spectral representation of A.
5. For every possible measurement value a_1, a_2, a_3 determine the probability to get this value, if the system is in a state described by the density matrix $\rho = \mathbb{1}_3/3$. Determine the expectation value for measurement of A in this state.
6. Calculate the matrix $\exp(-i\alpha A)$ with α being a real number.

Particle in a Delta function potential

Let the wave function of a particle in one dimension be given by

$$\psi(x) = \mathcal{N} \exp(-a|x|) = \mathcal{N}(\, e^{ax}\theta(-x) + e^{-ax}\theta(x)\,), \quad a > 0.$$

(The function $\theta(x)$ is the Heaviside step function.)

1. Calculate the norm $\mathcal{N}$ of the wave function.
2. Calculate the first and second derivative of $\psi(x)$.
3. Show that the wave function $\psi(x)$ is, for a suitable choice of the parameter a, an energy eigenfunction of the Hamilton operator

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \lambda\,\delta(x), \quad (\lambda > 0)$$

What is the necessary choice for a and what is the result for the energy eigenvalue E?

4. How is the probability current for a given wave function $\Phi(x,t)$ defined? Determine the probability current for the energy eigenfunction $\psi(x,t)$ from part 3 for $|x| > 0$ and argue that the wave function has to be interpreted as a bound state.

5. Assume the particle experiencing the delta-potential is in the state with the wave function

$$\phi(x) = \mathcal{N}' \begin{cases} 1+x\,, & -1 < x \leq 0 \\ 1-x\,, & 0 < x \leq 1 \\ 0\,, & \text{elsewhere} \end{cases} .$$

   Determine the average energy obtained in many repeated energy measurements on the system in this state.

6. What is the probability to receive the energy value E from part 3 in an energy measurement? Hint: The relation $\pm\, x\, e^{\pm ax} = \frac{d}{da} e^{\pm ax}$ is quite useful for the calculation of the integrals.

Harmonic oscillator

The Hamilton operator of the one-dimensional harmonic oscillator is:

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2.$$

The usual ladder operators have the form $a = \alpha X + i\beta P$ and $a^\dagger = \alpha X - i\beta P$, where α, β are real numbers. The number operator is given by $N = a^\dagger a$.

1. Write down the possible energy eigenvalues of H.

2. What are the eigenvalues of the number operator N?

3. What is the commutation relation between a and $a^\dagger$? What condition for α and β follows?

4. It is possible to rewrite the Hamilton operator as $H = \hbar\omega(a^\dagger a + 1/2)$. Determine α and β from this form.

5. Calculate the commutator $[N, a^\dagger]$.

6. Let the state $|\phi\rangle$ be an energy eigenstate with eigenvalue E. Calculate the energy eigenvalue of $a^\dagger|\phi\rangle$.

Electrically charged harmonic oscillator

Consider a particle in one dimension (x-direction) with mass m and electric charge q. This particle is trapped inside a potential of the form $V_{\text{ho}}(x) = \frac{m\omega^2}{2} x^2$. In addition there is a homogeneous time-independent electric field E in the x-direction, so that there is an additional potential $V_{\text{E}}(x) = -qxE + c$, where c is a constant with the dimension of energy.

1. Write down the total potential energy $V_{\text{tot}}(x)$ of the particle assuming that $V_{\text{tot}}(0) = 0$. Write down the Hamilton operator $H(X, P)$ for the system and the Schrödinger equation for the particles wave function.

2. Show that you can define a shifted position operator of the form $X' = X + b$, where b is a constant with dimenion of distance, such that the Hamilton operator $H(X', P)$ takes the form of a regular harmonic oscillator with an additional energy shift but without an electric field. Determine the constant b.

3. Which important relation do X and P satisfy? Does X' satisfy this relation as well? (Only then you can consider also X' being a location operator.

4. Give the general form of the normalized energy eigenfunctions $\tilde{\phi}_n(x')$ and the corresponding energy eigenvalues $\tilde{E}_n$.

5. Now consider the electric field to be very small such that you can use time-independent perturbation theory to determine the effects of the electric field. Determine the eigenenergies $\tilde{E}_n$ and the wave functions $\tilde{\phi}_n$ to first order in the electric field. Show that the result is consistent with the exact solution of part 4.

6. Determine the eigenenergies $\tilde{E}_n$ to second order in the electric field. Show that the result is compatible with the result given in part 4.

Hints: The normalized eigenfunctions of the regular harmonic oscillator with potential $V(x) = \frac{m\omega^2}{2}x^2$ without electric field are

$$\phi_n(x) = \langle x|\phi_n\rangle = (2^n n! \sqrt{\pi}\tilde{x})^{-1/2} \exp(-\tfrac{1}{2}(\tfrac{x}{\tilde{x}})^2) H_n(\tfrac{x}{\tilde{x}}) \, ,$$

where the H_n are the Hermite-Polynomials and $\tilde{x} = (\frac{\hbar}{\omega m})^{1/2}$. The ladder operators $a = \frac{1}{\sqrt{2}}(\frac{X}{\tilde{x}} + i\frac{\tilde{x}}{\hbar}P)$ and $a^\dagger = \frac{1}{\sqrt{2}}(\frac{X}{\tilde{x}} - i\frac{\tilde{x}}{\hbar}P)$ satisfy the relations $a|\phi_n\rangle = \sqrt{n}|\phi_{n-1}\rangle$ and $a^\dagger|\phi_n\rangle = \sqrt{n+1}|\phi_{n+1}\rangle$. Useful relations among the Hermite polynomials are $H_n'(x) = 2n\, H_{n-1}(x)$ and $xH_n(x) = nH_{n-1}(x) + \frac{1}{2}H_{n+1}(x)$. In the lectures we derived for time-independent perturbation theory at first order the expressions $E_n^1 = \langle n^0|H_1|n^0\rangle$ and $|n^1\rangle = \sum_{m\neq n} \frac{|m^0\rangle\langle m^0|}{E_n^0 - E_m^0} H_1|n^0\rangle$, and at second order $E_n^2 = \langle n^0|H_1|n^1\rangle$.

Particle in an impenetrable box

Consider a particle in one dimension that is confined inside an impenetrable box, but can otherwise move freely inside the box in the interval $[-L, L]$. The box potential has the form

$$V(x) = \begin{cases} 0\,, & -L \leq x \leq L \\ \infty\,, & |x| > L \end{cases} \, .$$

1. Write down the Hamilton operator H for the system.

2. Write down the Schrödinger equation for the particles wave function and the boundary conditions that the wave function has to satisfy.

3. Determine the energy eigenfunctions $\phi_n(x)$ and the corresponding energy eigenvalues E_n, by solving the Schrödinger equation and accounting for the boundary conditions.

4. Normalize the wave functions $\phi_n(x)$ such that the condition

$$\int_{-L}^{L} dx\, |\phi_n(x)|^2 = 1 \qquad \text{is satisfied.}$$

5. Let $\psi(x)$ be a non-normalized wave function of the particle. Write down the expression for the probability that the lowest possible value is obtained in an energy measurement.

6. Calculate that probability for $\psi(x) = \Theta(L^2 - x^2)$.

Spin measurements

Initially, let there be a spin-1/2 system in a pure spin state $\chi_{z,+}$, where the spin is pointing in the z-direction.

1. For which spin operators is $\chi_{z,+}$ an eigenstate and what are the corresponding eigenvalues?

2. Write down the density matrix for the spin state $\chi_{z,+}$.

3. What are the probabilities to get the values $\pm\hbar/2$ if a measurement of the spin in x-direction is taken? What are the corresponding probabilities for a measurement of the spin in y-direction? What are the respective expectation values?

4. Let's suppose you get the value $-\hbar/2$ for a measurement of the spin in x-direction. Assume that the spin is still there and not destroyed by the measurement. In which state is the system after the measurement?

5. Now take another measurement of the spin in the $\vec{n}$ direction on this spin. The vector $\vec{n}$ is a unit vector which is rotated with respect to the x direction by the angle ϕ around the z axis. What is the probability to get the values $\hbar/2$ for this measurement?

6. Determine the expectation value for that spin measurement in the $\vec{n}$ direction.

7. How does the density matrix of a general state look like? Under which condition does the density matrix describe a pure state?

Hints: The spin operators have the form

$$\vec{\mathbf{S}} = \frac{\hbar}{2}\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Angular momentum and representations I

Consider the abstract and general angular momentum operators J_k ($k = 1, 2, 3$) and the orbital angular momentum operator L_k ($k = 1, 2, 3$).

1. Write down: $[J_k, J_l]$.

2. The components of the orbital angular momentum operator L_k can be written as functions of the position operators X_l and the momentum operators P_m, $L_k = \epsilon_{klm} X_l P_m$. Calculate the commutator $[L_k, P_m]$ by means of the fundamental commutation relations for X_l, P_n without using any special representation.

3. Calculate $[L_k, L_l]$.

4. What eigenvalues are in principle possible for the operator $\vec{J}^2$? What eigenvalues are possible for J_3 for a given fixed eigenvalue of $\vec{J}^2$?

5. In position space representation the components of the orbital angular momentum operator have the form $L_k = \epsilon_{klm} x_l(-i\hbar\nabla_m)$. Complete the statements:

   (1) The spherical harmonics $Y_{lm}(\theta,\varphi)$ are simultaneous eigenfunctions of the operators .... and .... with the corresponding eigenvalues .... and ....

   (2) $L_+ Y_{lm}(\theta,\varphi) =$ .... $(L_+ = L_1 + iL_2)$

6. Consider two irreducible angular momentum representations with quantum numbers j_1 and j_2 with respect to the total angular momentum operator $\vec{J}^2$. What irreducible representations follow from the direct product of these two representations?

Angular momentum and representations II

Consider the abstract and general angular momentum operators J_k $(k = 1, 2, 3)$.

1. Write down: $[J_k, J_l]$

2. The components of the orbital angular momentum operator L_k can be written as functions of the position operators X_l and the momentum operators P_m, $L_k = \epsilon_{klm} X_l P_m$. Calculate the commutator $[P_n, \vec{L}^2]$ by means of the fundamental commutation relations for X_l, P_n without using any special representation.

3. What eigenvalues are in principle possible for the operator $\vec{J}^2$? What eigenvalues are possible for J_3 for a given fixed eigenvalue of $\vec{J}^2$?

4. In classic physics one can write the spatial angular momentum $\vec{L}$ either as $\vec{x}\times\vec{p}$ or as $-\vec{p}\times\vec{x}$. Write down the resulting two possibilities to define quantum mechanical angular momentum operators using the correspondence principle. Explain why this could in principle lead to an ambiguity in the definition of the angular momentum operators and why - in the case of the angular momentum operator - both expressions turn out to be equivalent.

5. Consider a system of two particles with spin. Particle 1 has spin $s_1 = \frac{1}{2}$ and particle 2 has spin $s_2 = 2$, where $\hbar^2 s_i(s_i+1)$ stands for the eigenvalue of the spin operator $\vec{S}_i^2$ $(i = 1, 2)$. Assume now that the combined system has spin $s = \frac{5}{2}$, where $\hbar^2 s(s+1)$ is the eivenvalue of the spin operators $\vec{S}^2$ where $\vec{S} = \vec{S}_1 + \vec{S}_2$. The spin in the z-direction of the combined system is $m = \frac{1}{2}$, where $\hbar m$ is the eivenvalue of the spin operator S_z. Which values of m_2 (eigenvalues of spin operator $S_{2,z}$) are in this state possible for particle 2? Determine the probabilities that each of these values is obtained in a measurement. The following relation is useful:

$$|\ell+\tfrac{1}{2}, m\rangle = \sqrt{\frac{\ell+m+\frac{1}{2}}{2\ell+1}}\,|\ell, m-\tfrac{1}{2}\rangle|\uparrow\rangle + \sqrt{\frac{\ell-m+\frac{1}{2}}{2\ell+1}}\,|\ell, m+\tfrac{1}{2}\rangle|\downarrow\rangle$$

Angular momentum and representations III

Consider the properties of the abstract and general angular momentum operators J_k ($k = 1, 2, 3$ or $k = x, y, z$).

1. Write down the SU(2) commutation relations of the angular momentum operators $J_{x,y,z}$.

2. Prove the following statement: If an operator A commutes with two components of the angular momentum operators $\vec{J}$ (lets say J_x and J_y) then A also commutes with the third component.

3. The components of the orbital angular momentum operator L_k can be written as functions of the position operators X_l and the momentum operators P_m, $L_k = \epsilon_{klm} X_l P_m$. Calculate the commutator $[L_n, \vec{L}^2]$ by means of the fundamental commutation relations for X_l, P_n without using any special representation. For efficiency it is useful when you first calculate $[L_n, L_k]$.

4. Consider a system of two particles with spin. Particle 1 has spin $s_1 = \frac{1}{2}$ and particle 2 has spin $s_2 = 3$, where $\hbar^2 s_i(s_i + 1)$ stands for the eigenvalue of the spin operator $\vec{S}_i^2$ ($i = 1, 2$). Which values for s, where $\hbar^2 s(s + 1)$ is the eivenvalue of the total spin operator $\vec{S}^2$ with $\vec{S} = \vec{S}_1 + \vec{S}_2$, are possible for the combined system?

5. Part 4 continued: Assume now that the combined system has spin $s = \frac{5}{2}$, where again $\hbar^2 s(s + 1)$ is the eivenvalue of the total spin operator $\vec{S}^2$ where $\vec{S} = \vec{S}_1 + \vec{S}_2$. The spin in the z-direction of the combined system is $m = \frac{1}{2}$, where $\hbar m$ is the eivenvalue of the spin operator S_z. Which values of m_2 (eigenvalues of spin operator $S_{2,z}$) are in this state possible for particle 2? Determine for each of these m_2 valued the probability that it is obtained in measurement of $S_{2,z}$ in this state. The following relation is useful:

$$|\ell - \tfrac{1}{2}, m\rangle \;=\; -\sqrt{\frac{\ell - m + \frac{1}{2}}{2\ell + 1}}\,|\ell, m - \tfrac{1}{2}\rangle|\uparrow\rangle \;+\; \sqrt{\frac{\ell + m + \frac{1}{2}}{2\ell + 1}}\,|\ell, m + \tfrac{1}{2}\rangle|\downarrow\rangle$$

6. Let the set of states $\{|j, m\rangle \mid j = 0, \frac{1}{2}, 1, \dots ; m = -j, -j + 1, \dots, j - 1, j\}$ be a standard basis of eigenstates of the angular momentum operators $\vec{J}^2$ and J_z. Show by calculation that the standard deviation of measurements of the angular momenta J_x and J_y in each of these eigenstates $|j, m\rangle$ has the form

$$\Delta J_x = \Delta J_y = \hbar\sqrt{\frac{j(j + 1) - m^2}{2}}\,.$$

   To proceed it is useful to first consider the expectation values of the raising and lowering operators $J_\pm = J_x \pm iJ_y$ and of $J_+^2 + J_-^2$, and in addition the relation

$$J_\pm|j, m\rangle = \hbar\sqrt{j(j + 1) - m(m \pm 1)}|j, m \pm 1\rangle$$

   to show that the mean values $\langle J_{x,y}\rangle$ vanish and that $\langle J_x^2\rangle = \langle J_y^2\rangle$. You may then consider the expectation value $\langle J_x^2 + J_y^2\rangle$.