Chapter 8 : Adding Angular Momenta

→ We consider systems where different angular momenta arise. The hydrogen atom is an example where the orbital angular momentum and the spin of the electron have to be considered.

In this context the total angular momentum operator (the sum of the individual angular momentum operators) becomes important and the combined representation is in general not any more irreducible.

In this chapter we want to discuss which irreducible representations result from combining two angular momenta in such a way.

8.1. Problem Setting

→ Let $\vec{J}_1$ and $\vec{J}_2$ be two angular momentum operators that belong to different and independent degrees of freedom. $\Rightarrow [(J_{1/2})_i, (J_{1/2})_j] = i\hbar\, \epsilon_{ijk}\, (L_{1/2})_k$

Example : - Orbital angular momenta of two different particles.
- Orbital angular momentum and spin of an electron.

↳ This means that $[\vec{J}_1, \vec{J}_2] = 0$ or $[(J_1)_i, (J_2)_j] = 0$, $i,j = 1,2,3$

We define the total angular momentum operator : $\vec{J} = \vec{J}_1 + \vec{J}_2$ ← What does that mean? See below!

The operator $\vec{J}$ is also a regular angular momentum operator because

$$[J_i, J_j] = i\hbar\, \epsilon_{ijk} J_k .$$

→ The operators $\vec{J}_1$ and $\vec{J}_2$ are (usually and alway in this context) defined in connection with two independent irreducible representations related to the set of states

$$\{ |j_1, m_1\rangle ;\ m_1 = -j_1, \dots, j_1 \} \quad \text{and} \quad \{ |j_2, m_2\rangle ;\ m_2 = -j_2, \dots, j_2 \}$$

with $\quad \vec{J}_1^2 |j_1 m_1\rangle = \hbar^2 j_1(j_1+1) |j_1 m_1\rangle \quad$ and $\quad \vec{J}_2^2 |j_2 m_2\rangle = \hbar^2 j_2(j_2+1) |j_2 m_2\rangle$

$$J_{1,3} |j_1 m_1\rangle = \hbar m_1 |j_1 m_1\rangle \qquad J_{2,3} |j_2 m_2\rangle = \hbar m_2 |j_2 m_2\rangle$$

→ The combination of these two irreducible representations and the total angular momentum operator $\vec{J} = \vec{J}_1 + \vec{J}_2$ are defined in the context of the set of direct product states

$$\{ |j_1 m_1 ; j_2 m_2\rangle := |j_1 m_1\rangle \otimes |j_2 m_2\rangle \ ,\ m_1 = -j_1, \dots, j_1 ,\ m_2 = -j_2, \dots, j_2 \}$$

Each operator defined in that direct product space is also defined as a direct product operator :

$$A := A_1 \otimes A_2 \quad \text{with} \quad A\, |j_1 m_1 ; j_2 m_2\rangle := A_1 |j_1, m_1\rangle \otimes A_2 |j_2, m_2\rangle$$

The total angular momentum operator $\vec{J} = \vec{J}_1 + \vec{J}_2$ is actually defined as

$$\vec{J} = \vec{J}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \vec{J}_2$$

with $\vec{J}\,|j_1 m_1; j_2 m_2\rangle = \vec{J}_1 |j_1 m_1\rangle \otimes |j_2 m_2\rangle + |j_1 m_1\rangle \otimes \vec{J}_2 |j_2 m_2\rangle$

→ The important observation is that the states $|j_1 m_1; j_2 m_2\rangle$ are eigenstates of J_3 with eigenvalues $\hbar(m_1+m_2)$, BUT in general not eigenstates of $\vec{J}^2 = (\vec{J}_1+\vec{J}_2)^2$.
The latter is because

$$\begin{aligned}
[\vec{J}^2, J_{1,3}] &= [(J_{1,\ell}+J_{2,\ell})(J_{1,\ell}+J_{2,\ell}), J_{1,3}] = (J_{1,\ell}+J_{2,\ell})[J_{1,\ell}, J_{1,3}] + [J_{1,\ell}, J_{1,3}](J_{1,\ell}+J_{2,\ell}) \\
&= (J_{1,\ell}+J_{2,\ell})\, i\hbar\, \epsilon_{\ell 3 m} J_{1,m} + i\hbar\, \epsilon_{\ell 3 m} J_{1,m}(J_{1,\ell}+J_{2,\ell}) \\
&= i\hbar\, \epsilon_{3 m \ell}\, (\cancel{J_{1,\ell} J_{1,m}} + \cancel{J_{1,m} J_{1,\ell}} + J_{2,\ell} J_{1,m} + J_{1,m} J_{2,\ell}) \\
&= 2 i\hbar\, \epsilon_{3 m \ell}\, J_{1,m} J_{2,\ell} \neq 0
\end{aligned}$$

$$[\vec{J}^2, J_{2,3}] = 2 i\hbar\, \epsilon_{3 m \ell}\, J_{2,m} J_{1,\ell} = -2 i\hbar\, \epsilon_{3 m \ell}\, J_{1,m} J_{2,\ell} \neq 0 .$$

This means that the states $|j_1 m_1; j_2 m_2\rangle$ do not form an irreducible representation with respect to $\vec{J}^2$ and J_3.

→ But we have the following vanishing commutators:

$$\begin{aligned}
[\vec{J}^2, \vec{J}_1^2] &= J_{1,k}[\vec{J}^2, J_{1,k}] + [\vec{J}^2, J_{1,k}]J_{1,k} = 2 i\hbar\, \epsilon_{k m \ell}\,(J_{1,k} J_{1,m} J_{2,\ell} + J_{1,m} J_{2,\ell} J_{1,k}) \\
&= 2 i\hbar\, \epsilon_{k m \ell}\,(J_{1,k} J_{1,m} J_{2,\ell} + J_{1,m} J_{1,k} J_{2,\ell}) = 0
\end{aligned}$$

$$[\vec{J}^2, \vec{J}_2^2] = 0$$

$$[\vec{J}^2, J_3] = 0 \;, \quad [\vec{J}_1^2, J_3] = [\vec{J}_2^2, J_3] = 0$$

$\Rightarrow$ $\boxed{\vec{J}^2, \vec{J}_1^2, \vec{J}_2^2, J_3 \text{ form a set of fully commuting operators}}$

→ It is the aim to identify the irreducible representations contained in

$$\{\, |j_1 m_1; j_2 m_2\rangle := |j_1 m_1\rangle \otimes |j_2 m_2\rangle \;,\; m_1 = -j_1, \dots, j_1 \;,\; m_2 = -j_2, \dots, j_2 \,\}$$

For each such irreducible representation $\{\, |j m j_1 j_2\rangle ,\; m = -j, j+1, \dots j-1, j \,\}$ the operators $\vec{J}^2, J_3, \vec{J}_1^2, \vec{J}_2^2$ are fully commuting with the eigenvalues $\hbar^2 j(j+1)$, $\hbar m$, $\hbar^2 j_1(j_1+1)$ and $\hbar^2 j_2(j_2+1)$, respectively.

↳ Each state $|j m \underbrace{j_1 j_2}_{\text{can be dropped if the context is clear}}\rangle$ is a linear combination of the states $|j_1 m_1; j_2 m_2\rangle$ (with $m = m_1+m_2$ because both are eigenstates to $J_3 = J_{1,3}+J_{2,3}$), which we want to identify.

8.2. Adding of Two Spin-$\frac{1}{2}$ Operators

→ We consider $\vec{S} = \vec{S}_1 + \vec{S}_2$.

There are $2 \times 2 = 4$ direct product states, which are

$$|\uparrow\uparrow\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 , \quad |\downarrow\downarrow\rangle = |\downarrow\rangle_1 \otimes |\downarrow\rangle_2$$

$$|\uparrow\downarrow\rangle = |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 , \quad |\downarrow\uparrow\rangle = |\downarrow\rangle_1 \otimes |\uparrow\rangle_2$$

We have:
$$S_3 |\uparrow\uparrow\rangle = \hbar\left(\tfrac{1}{2}+\tfrac{1}{2}\right)|\uparrow\uparrow\rangle = \hbar|\uparrow\uparrow\rangle$$
$$S_3 |\downarrow\downarrow\rangle = -\hbar|\downarrow\downarrow\rangle$$
$$S_3 |\uparrow\downarrow\rangle = 0 , \quad S_3|\downarrow\uparrow\rangle = 0$$

Looks like we are having a $j=1$ and a $j=0$ representation.

→ We have:
$$\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1\cdot\vec{S}_2$$
$$= \tfrac{3}{2}\hbar^2 + 2S_{1,3}S_{2,3} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

$S_\pm = S_1 \pm i S_2$

$S_+|\uparrow\rangle = 0 , \quad S_-|\downarrow\rangle = 0$

↳ $$\vec{S}^2|\uparrow\uparrow\rangle = \left(\tfrac{3}{2}\hbar^2 + 2\hbar^2\tfrac{1}{4}\right)|\uparrow\uparrow\rangle = \hbar^2\cdot 2\,|\uparrow\uparrow\rangle$$
$$\vec{S}^2|\downarrow\downarrow\rangle = \hbar^2\cdot 2\,|\downarrow\downarrow\rangle$$

$|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ indeed belong to a $j=1$ representation with $S_3 = \pm\hbar$

We construct the $j=1$, $S_3=0$ state by applying S_- on $|\uparrow\uparrow\rangle$: (recall $T_\pm|jm\rangle = \sqrt{j(j+1)-m(m\pm1)}\,|j\,m\pm1\rangle$)

$$\frac{1}{\hbar\sqrt{2}}S_-|\uparrow\uparrow\rangle = \frac{1}{\hbar\sqrt{2}}(S_{1-}+S_{2-})|\uparrow\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)$$ ← has norm $=1$

→ The state orthogonal to the $j=1$, $S_3=0$ state is $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

We have:
$$S_3\,\tfrac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = 0$$
$$\vec{S}^2\,\tfrac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = \left(\tfrac{3}{2}\hbar^2 - 2\tfrac{\hbar^2}{4} - \hbar^2\right)\tfrac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = 0$$

$S_{1-}|\uparrow\rangle = |\downarrow\rangle$

$S_{1+}|\downarrow\rangle = |\uparrow\rangle$

→ **Final result:** **When adding two spin-$\frac{1}{2}$ angular momenta (i.e. the direct product space of two spin-$\frac{1}{2}$ representations) the representation consists of two irreducible representations:**

$j=1$ ("triplet"): $|1\,1\rangle = |\uparrow\uparrow\rangle$, $|1\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$, $|1,-1\rangle = |\downarrow\downarrow\rangle$

$j=0$ ("singlet"): $|0\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

8.3. Adding Orbital Angular Momentum and Spin-$\frac{1}{2}$

→ We consider $\vec{J} = \vec{L} + \vec{S}$ $\quad (\ell \geq 1)$

There are $2(2\ell+1)$ direct product states which are:

$|\ell, m\rangle|\uparrow\rangle$ and $|\ell, m\rangle|\downarrow\rangle$, $m = -\ell, \dots, +\ell$

$2\ell+2+2\ell = 2(2\ell+1)$ — o.k.

$2\ell+2$ states $\quad$ 2ℓ states

→ We show that these states contain an irreducible $\ell+\frac{1}{2}$ and an irreducible $\ell-\frac{1}{2}$ representation.

We start by showing that $|\ell+\frac{1}{2}, \ell+\frac{1}{2}\rangle = |\ell,\ell\rangle|\uparrow\rangle$ $\quad$ ($j = \ell+\frac{1}{2}$ state with $J_3 = \hbar(\ell+\frac{1}{2})$)

$$J_3|\ell,\ell\rangle|\uparrow\rangle = (L_3+S_3)|\ell,\ell\rangle|\uparrow\rangle = \hbar(\ell+\tfrac{1}{2})|\ell,\ell\rangle|\uparrow\rangle \checkmark$$

$$\vec{J}^2 = \vec{L}^2 + \vec{S}^2 + 2L_3S_3 + L_{1+}S_{2-} + L_{1-}S_{2+}$$

$$\vec{J}^2|\ell,\ell\rangle|\uparrow\rangle = \hbar^2\left(\ell(\ell+1) + \tfrac{3}{4} + 2\,\tfrac{\ell}{2}\right)|\ell,\ell\rangle|\uparrow\rangle = \hbar^2(\ell+\tfrac{1}{2})(\ell+\tfrac{3}{2})|\ell,\ell\rangle|\uparrow\rangle \checkmark$$

We determine the $|\ell+\frac{1}{2}, \ell-\frac{1}{2}\rangle$ state by applying the J_- operator $\quad J_-|\ell+\frac{1}{2}, \ell+\frac{1}{2}\rangle = \hbar\sqrt{2\ell+1}\,|\ell+\frac{1}{2}, \ell-\frac{1}{2}\rangle$

$$J_-|\ell,\ell\rangle|\uparrow\rangle = \hbar\sqrt{2\ell}\,|\ell,\ell-1\rangle|\uparrow\rangle + \hbar\,|\ell,\ell\rangle|\downarrow\rangle$$

$$\hookrightarrow |\ell+\tfrac{1}{2}, \ell-\tfrac{1}{2}\rangle = \sqrt{\frac{2\ell}{2\ell+1}}\,|\ell,\ell-1\rangle|\uparrow\rangle + \sqrt{\frac{1}{2\ell+1}}\,|\ell,\ell\rangle|\downarrow\rangle$$

Repetition of applying J_- leads to (proof by induction)

$|\ell, \ell+1\rangle = 0 = |\ell, -\ell-1\rangle$

$$|\ell+\tfrac{1}{2}, m\rangle = \sqrt{\frac{\ell+m+\frac{1}{2}}{2\ell+1}}\,|\ell, m-\tfrac{1}{2}\rangle|\uparrow\rangle + \sqrt{\frac{\ell-m+\frac{1}{2}}{2\ell+1}}\,|\ell, m+\tfrac{1}{2}\rangle|\downarrow\rangle \quad \text{with } m = -(\ell+\tfrac{1}{2}), \dots, (\ell+\tfrac{1}{2})$$

→ We can now for $m = -(\ell-\frac{1}{2}), \dots, (\ell-\frac{1}{2})$ define states orthogonal to the $|\ell+\frac{1}{2}, m\rangle$, which turn out to be the $|\ell-\frac{1}{2}, m\rangle$ states:

$$|\ell-\tfrac{1}{2}, m\rangle = -\sqrt{\frac{\ell-m+\frac{1}{2}}{2\ell+1}}\,|\ell, m-\tfrac{1}{2}\rangle|\uparrow\rangle + \sqrt{\frac{\ell+m+\frac{1}{2}}{2\ell+1}}\,|\ell, m+\tfrac{1}{2}\rangle|\downarrow\rangle, \quad m = -(\ell-\tfrac{1}{2}), \dots, (\ell-\tfrac{1}{2})$$

We check: $J_3|\ell-\frac{1}{2}, m\rangle = \hbar m|\ell-\frac{1}{2}, m\rangle$ ✓
$\vec{J}^2|\ell-\frac{1}{2}, m\rangle = \hbar^2(\ell-\frac{1}{2})(\ell+\frac{1}{2})|\ell-\frac{1}{2}, m\rangle$ ✓

→ Final result: When adding an orbital angular momentum ($\ell = 1, 2, \dots$) with a spin-$\frac{1}{2}$ the resulting representation consists of two irreducible representations:

$\underline{j = \ell+\frac{1}{2}:}$ $\quad |\ell+\frac{1}{2}, m\rangle = \alpha_+|\ell, m-\frac{1}{2}\rangle|\uparrow\rangle + \beta_+|\ell, m+\frac{1}{2}\rangle|\downarrow\rangle$ $\quad |\ell,\ell+1\rangle = |\ell,-\ell-1\rangle = 0$

$\underline{j = \ell-\frac{1}{2}:}$ $\quad |\ell-\frac{1}{2}, m\rangle = \alpha_-|\ell, m-\frac{1}{2}\rangle|\uparrow\rangle + \beta_-|\ell, m+\frac{1}{2}\rangle|\downarrow\rangle$

$$\text{with } \alpha_\pm = \pm\beta_\mp = \pm\sqrt{\frac{\ell \pm m + \frac{1}{2}}{2\ell+1}}, \quad \beta_+^2 + \beta_-^2 = 1$$

8.4. The General Case

→ We consider $\vec{J} = \vec{J}_1 + \vec{J}_2$

There are $(2j_1+1)\cdot(2j_2+1)$ direct product states $|j_1 m_1\rangle|j_2 m_2\rangle =: |j_1 m_1 ; j_2 m_2\rangle$ which are eigenstates of $\vec{J}_1^2, J_{1,3}, \vec{J}_2^2, J_{2,3}$.

We want to identify the irreducible spin-j states $|j m j_1 j_2\rangle$ which are eigenstates of $\vec{J}^2, J_3, \vec{J}_1^2, \vec{J}_2^2$.

→ Both set of states form CONS's, so that the $|j m j_1 j_2\rangle$ states can be expanded in terms of the direct product states. As coefficients the following scalar products arise:

$$\langle j_1' m_1 ; j_2' m_2 | j m j_1 j_2\rangle$$

From considering $\langle j_1' m_1 ; j_2' m_2 | A | j m j_1 j_2\rangle$ for $A = \vec{J}_1^2, \vec{J}_2^2, J_3$ one can trivially see that

$$\langle j_1' m_1 ; j_2' m_2 | j m j_1 j_2\rangle \neq 0 \quad \text{for} \quad j_1' = j_1,\ j_2' = j_2,\ m = m_1 + m_2$$

So the expansion of the $|j m j_1 j_2\rangle$ states has the form:

$$|j m j_1 j_2\rangle = \sum_{m_1+m_2=m} |j_1 m_1 ; j_2 m_2\rangle \underbrace{\langle j_1 m_1 ; j_2 m_2 | j m j_1 j_2\rangle}_{\text{Clebsch-Gordon-Coefficients}}$$

Can always chosen real

→ We want to determine which values for j are possible for given j_1 and j_2 $(j_1 \geq j_2)$. Due to $m = m_1 + m_2$ we can consider all possible assignments for the values for m, m_1, m_2

m-values	(m_1, m_2) pairs		number of (m_1, m_2) pairs
j_1+j_2	(j_1, j_2)		1
j_1+j_2-1	$(j_1-1, j_2), (j_1, j_2-1)$		2
j_1+j_2-2	$(j_1-2, j_2), (j_1-1, j_2-1), (j_1, j_2-2)$		3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_1-j_2	$(j_1-2j_2, j_2) \ldots\ldots$	$(j_1, -j_2)$	$2j_2+1$
j_1-j_2-1	$(j_1-2j_2-1, j_2) \ldots\ldots$	$(j_1-1, -j_2)$	$2j_2+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-(j_1-j_2)$	$(-j_1, j_2) \ldots\ldots$	$(-j_1+2j_2, -j_2)$	$2j_2+1$
$-(j_1-j_2)-1$	$(-j_1, j_2-1) \ldots\ldots$	$(-j_1+2j_2-1, -j_2)$	$2j_2$
$\vdots$	$\vdots$		$\vdots$
$-(j_1+j_2)$	$(-j_1, -j_2)$		1

→ We now carry out an accounting of states:

1) Largest possible j-value: $j = j_1 + j_2$ with m-values $m = -(j_1+j_2), \dots, (j_1+j_2)$
⇒ gives the spin-(j_1+j_2) states $|j_1+j_2, m\rangle$

2) The largest available m value is $m = j_1 + j_2 - 1$, so the next largest possible j-value is $j = j_1 + j_2 - 1$ with available m values $m = -(j_1+j_2)+1, \dots, (j_1+j_2)-1$
⇒ gives the spin-(j_1+j_2-1) states $|j_1+j_2-1, m\rangle$

3) ... and so on

→ Final outcome: The possible j-values for irreducible spin-j representations are:

$$j = |j_1-j_2|,\ |j_1-j_2|+1,\ \dots,\ j_1+j_2-1,\ j_1+j_2 \qquad (2j_2+1 \text{ possibilities})$$

Each of the spin-j representation appears exactly once.

Check: We count states

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = \sum_{k=0}^{2j} \left(2(j_1-j_2+k)-1\right) = (2j_1+1)(2j_2+1) \checkmark$$

→ The result we have found is frequently written in the following abbreviated form:

$$j_1 \otimes j_2 = (j_1+j_2) \oplus (j_1+j_2-1) \oplus \dots \oplus |j_1-j_2|$$

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34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

m_1 m_2	J M	J M	...
m_1 m_2	Coefficients		
. .			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

1/2×1/2

m_1 m_2	$J=1$, $M=+1$
+1/2 +1/2	1

m_1 m_2	$J=1$, $M=0$	$J=0$, $M=0$
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	$J=1$, $M=-1$
−1/2 −1/2	1

2×1/2

m_1 m_2	$J=5/2$, $M=+5/2$
+2 +1/2	1

m_1 m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	$J=5/2$, $M=-5/2$
−2 −1/2	1

1×1/2

m_1 m_2	$J=3/2$, $M=+3/2$
+1 +1/2	1

m_1 m_2	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	$J=3/2$, $M=-3/2$
−1 −1/2	1

3/2×1/2

m_1 m_2	$J=2$, $M=+2$
+3/2 +1/2	1

m_1 m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	$J=2$, $M=0$	$J=1$, $M=0$
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	$J=2$, $M=-2$
−3/2 −1/2	1

2×1

m_1 m_2	$J=3$, $M=+3$
+2 +1	1

m_1 m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	$J=3$, $M=-3$
−2 −1	1

3/2×1

m_1 m_2	$J=5/2$, $M=+5/2$
+3/2 +1	1

m_1 m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	$J=5/2$, $M=-5/2$
−3/2 −1	1

1×1

m_1 m_2	$J=2$, $M=+2$
+1 +1	1

m_1 m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	$J=2$, $M=-2$
−1 −1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

3/2×3/2

m_1 m_2	$J=3$, $M=+3$
+3/2 +3/2	1

m_1 m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	$J=3$, $M=-3$
−3/2 −3/2	1

2×3/2

m_1 m_2	$J=7/2$, $M=+7/2$
+2 +3/2	1

m_1 m_2	$J=7/2$, $M=+5/2$	$J=5/2$, $M=+5/2$
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	$J=7/2$, $M=+3/2$	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	$J=7/2$, $M=+1/2$	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	$J=7/2$, $M=-1/2$	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	$J=7/2$, $M=-3/2$	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	$J=7/2$, $M=-5/2$	$J=5/2$, $M=-5/2$
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	$J=7/2$, $M=-7/2$
−2 −3/2	1

2×2

m_1 m_2	$J=4$, $M=+4$
+2 +2	1

m_1 m_2	$J=4$, $M=+3$	$J=3$, $M=+3$
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	$J=4$, $M=+2$	$J=3$, $M=+2$	$J=2$, $M=+2$
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	$J=4$, $M=+1$	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	$J=4$, $M=0$	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	$J=4$, $M=-1$	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	$J=4$, $M=-2$	$J=3$, $M=-2$	$J=2$, $M=-2$
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	$J=4$, $M=-3$	$J=3$, $M=-3$
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	$J=4$, $M=-4$
−2 −2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.