Chapter 6: Angular Momentum

6.1. Unitary Transformations

\[ T_x = \exp(-i \frac{\vec{p} \cdot \vec{x}}{\hbar}) \]

This can easily cause confusion. The momentum operator \( \vec{p} \) is the generator of translations.

Translation acting on a wave function in \( \xi \)-space:

\[ \psi(\xi) \rightarrow T_x \psi(\xi) = \exp(-i \frac{\vec{p} \cdot \vec{x}}{\hbar}) \psi(\xi) = \exp(-i \vec{p} \cdot \vec{x}/\hbar) \psi(\xi) = \psi(\xi - \vec{a}) \]

Translation acting on location eigenstates:

\[ \langle \xi | T_x | \psi \rangle = \langle \xi - \vec{a} | \exp(-i \vec{p} \cdot \vec{x}/\hbar) | \psi \rangle = \langle \xi - \vec{a} | \psi \rangle \]

\[ \langle \vec{x} | T_x | \vec{x} \rangle = \exp(-i \vec{p} \cdot \vec{x}/\hbar) \langle \vec{x} | \vec{x} \rangle = \langle \vec{x} + \vec{a} | \vec{x} \rangle \]

Translation acting on momentum eigenstates:
Translation acting on momentum eigenstates:

\[ |q\rangle \rightarrow T_{\tilde{x}} |q\rangle = \exp\left(-i\frac{\tilde{x}\cdot\hat{p}}{\hbar}\right) |q\rangle = \exp\left(-i\frac{\tilde{x}}{\hbar}\right) |q\rangle \]

Translation acting on an operator:

Let \( |q\rangle \) be a state and \( A \) a linear operator \( \Rightarrow A |q\rangle \) is also a state

\[ \Rightarrow T_{\tilde{x}} A |q\rangle = T_{\tilde{x}} A T_{\tilde{x}}^{-1} T_{\tilde{x}} |q\rangle \]

translational operator translated state

\[ \Rightarrow A \rightarrow T_{\tilde{x}} A T_{\tilde{x}}^{-1} = \exp\left(-i\frac{\tilde{x}\cdot\hat{p}}{\hbar}\right) A \exp\left(+i\frac{\tilde{x}\cdot\hat{p}}{\hbar}\right) \]

Translation acting on a function of the \( \tilde{x} \) location operator:

use \( [X_i, p_j] = i\hbar \delta_{ij} \) \[ [f(X_i), p_j] = i\hbar \delta_{ij} f'(X_i) \]

\[ f(\tilde{x}) \rightarrow \exp\left(-i\frac{\tilde{x}\cdot\hat{p}}{\hbar}\right) f(\tilde{x}) \exp\left(+i\frac{\tilde{x}\cdot\hat{p}}{\hbar}\right) = f(\tilde{x} - \tilde{x}_1) \]

There is also a translation in momentum space that is generated by the \( (-\tilde{x}) \) operator:

\[ T_{\tilde{p}} = \exp\left(i\beta\tilde{p}\right) \]

This can be seen from the fact that \( [f(\tilde{p}_i), X_j] = -i\hbar f'(\tilde{p}_i) \delta_{ij} \) or from the momentum space representation of the \( \tilde{x} \) operator:

\[ (X)_{\text{momentum}} = +i\hbar \frac{\partial}{\partial \tilde{p}} \]
So we have: \[ \Phi(p) \rightarrow \exp(iq\bar{x}/t) \Phi(p) = \Phi(p-q) \]

\[ |\tilde{p}\rangle \rightarrow \exp(iq\bar{x}/t) |\tilde{p}\rangle = |\tilde{p}+q\rangle \]

\[ |\tilde{x}\rangle \rightarrow \exp(iq\bar{x}/t) |\tilde{x}\rangle = \exp(iq\bar{x}/t) |\tilde{x}\rangle \]

\[ A \rightarrow \exp(iq\bar{x}/t) A \exp(-iq\bar{x}/t) \]

\[ f(\tilde{p}) \rightarrow \exp(iq\bar{x}/t) f(\tilde{p}) \exp(-iq\bar{x}/t) = f(\tilde{p}+q) \]

**6.2. Orbital Angular Momentum and Spatial Rotations**

→ Orbital angular momentum operator: \[ \bar{L} = \bar{X} \times \bar{P} \] , \[ L_x = \text{even } x P_x \] → generator for rotations

There is no issue with ordering of \( \bar{X} \) and \( \bar{P} \) operators because \( \text{even } x P_x = \text{even } P_x x \) due to \( l+m \).

→ Unitary operator for spatial rotation by angle \( l \Delta \bar{L}/t \) around axis \( \frac{\bar{L}}{|\bar{L}|} \): \[ \exp(-i\bar{L}\bar{L}/t) \]

Check by an infinitesimal rotation: \[ \exp(-i\bar{L}\bar{L}/t) = 1 - i\bar{L}\bar{L}/t \] \( l \ll 1 \)

\[ L_0 \left( 1 - i\bar{L}\bar{L}/t \right) |\tilde{x}\rangle = \left( 1 - i\bar{L}\bar{L}/t \right) |\tilde{x}\rangle = \left( 1 - i\bar{L}\bar{L}/t \right) |\tilde{x}\rangle \]
Check by an infinitesimal rotation: \( \exp(-i \frac{\vec{L}}{\hbar}) = I - i \frac{\vec{L}}{\hbar} \) \( \| \vec{L} \| \ll \hbar \)

\[ \langle \vec{x} | (I - i \frac{\vec{L}}{\hbar}) | \vec{x} \rangle = \langle \vec{x} | (I - i e_{\mu\nu\lambda} J^\mu \epsilon^{\nu\lambda} \vec{P} \hbar) | \vec{x} \rangle = \langle \vec{x} | (I - i \vec{e} \times \vec{x}) \vec{P} \hbar | \vec{x} \rangle = \langle \vec{x} | \exp(-i (\vec{e} \times \vec{x}) \vec{P} \hbar) | \vec{x} \rangle \]

Ch. 6.1.

\[ | \vec{x} + \vec{e} \times \vec{x} \rangle \leftarrow \text{indeed correctly rotated state (Ch. 5.2)} \]

\[ \Rightarrow \exp(-i \frac{\vec{L}}{\hbar}) | \vec{x} \rangle = | R(\vec{x}) \rangle \]

\[ R(\vec{x}) : \text{spatial rotation matrix}, \text{see Ch. 5.2} \]

The same should also happen to the \( | \vec{p} \rangle \) state because a rotation treats all vectors equally:

\[ \langle \vec{p} | (I - i \frac{\vec{L}}{\hbar}) | \vec{p} \rangle = \langle \vec{p} | (I - i e_{\mu\nu\lambda} J^\mu \epsilon^{\nu\lambda} \vec{P} \hbar) | \vec{p} \rangle = \langle \vec{p} | (I - i \vec{e} \times \vec{p}) \vec{P} \hbar | \vec{p} \rangle = \exp(i (\vec{e} \times \vec{p}) \vec{P} \hbar) | \vec{p} \rangle \]

Ch. 6.1.

\[ | \vec{p} + \vec{e} \times \vec{p} \rangle \leftarrow \text{yes, works!} \]

\[ \Rightarrow \exp(-i \frac{\vec{L}}{\hbar}) | \vec{p} \rangle = | R(\vec{p}) \rangle \]

\[ R(\vec{x}) : \text{spatial rotation matrix} \]

The rotation matrices \( R(\vec{x}) \) form the group \( SO(3) \), which are the set of orthogonal \( 3 \times 3 \) real matrices with determinant 1.

\[ \Rightarrow \quad R(\vec{x}) = R^T(\vec{x}) \]
We also check the action of the rotation operator on the $\hat{X}$ and $\hat{P}$ operators:

\[
(1 - i \frac{\hat{L}}{\hbar}) \hat{X} (1 + i \frac{\hat{L}}{\hbar}) = (1 - i \frac{\hat{L}}{\hbar})(1 + i \frac{\hat{L}}{\hbar}) \hat{X} (1 + i \frac{\hat{L}}{\hbar}) e^{i \hbar \kappa_x \hat{X}} P_0 (\hbar) \]

\[
= (1 - i \frac{\hat{L}}{\hbar})(1 + i \frac{\hat{L}}{\hbar}) \hat{X} + (1 - i \frac{\hat{L}}{\hbar}) (-\hat{Z} \times \hat{X}) \quad \leftarrow \text{neglect } O(\hat{L}) \text{ terms}
\]

\[
= \hat{X} - \hat{Z} \times \hat{X} \quad \leftarrow \text{rotation in negative direction}
\]

\[
\Rightarrow \exp(-i \frac{\hat{L} \hbar}{\hbar}) \hat{X} \exp(i \frac{\hat{L} \hbar}{\hbar}) = R^x (\alpha) \hat{X}
\]

\[
\exp(-i \frac{\hat{L} \hbar}{\hbar}) f(\hat{X}) \exp(i \frac{\hat{L} \hbar}{\hbar}) = f(R^x (\alpha) \hat{X}) \quad \downarrow \text{obvious to see}
\]

\[
\exp(-i \frac{\hat{L} \hbar}{\hbar}) \hat{P} \exp(i \frac{\hat{L} \hbar}{\hbar}) = R^x (\alpha) \hat{P} \quad \downarrow \text{same considerations}
\]

\[
\exp(-i \frac{\hat{L} \hbar}{\hbar}) f(\hat{P}) \exp(i \frac{\hat{L} \hbar}{\hbar}) = f(R^x (\alpha) \hat{P}) \quad \downarrow \text{same considerations}
\]

\[
\exp(-i \frac{\hat{L} \hbar}{\hbar}) \hat{L} \exp(i \frac{\hat{L} \hbar}{\hbar}) = R^x (\alpha) \hat{L}
\]

\[
\exp(-i \frac{\hat{L} \hbar}{\hbar}) f(\hat{L}) \exp(i \frac{\hat{L} \hbar}{\hbar}) = f(R^x (\alpha) \hat{L})
\]

This makes sense since e.g. $f(R^x (\alpha) \hat{X})$ means that we have to rotate $\hat{X}$ with $R^x (\alpha)$ to obtain the same result $f(\hat{X})$ as before the rotation. So the function $f$ is indeed rotated by angle $\frac{\alpha}{\hbar}$ around axis $\frac{\hat{L}}{\hbar}$. Compare to $f(\hat{x}) = f(\hat{x} \pm \frac{\alpha}{\hbar})$ which was translation of $f$ by $\frac{\alpha}{\hbar}$. 

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This makes sense since e.g. \( f(R(x)\mathbf{x}) \) means that we have to rotate \( \mathbf{x} \) with \( R(x) \) to obtain the same result \( f(x) \) as before the rotation. So the function \( f \) is indeed rotated by angle \( \frac{\theta}{|\mathbf{r}|} \) around axis \( \frac{\mathbf{r}}{|\mathbf{r}|} \). Compare to \( f(\mathbf{x} + \mathbf{a}) \) which was translation of \( f \) by \( \mathbf{a} \).

The above relations tell that \( \mathcal{L} \) generates the same kind of rotations on any vector operator. This can also be expressed in the following commutation relations:

\[
\begin{align*}
\left[ \mathcal{L}_\mu, \mathbf{X}_\nu \right] &= i\hbar \epsilon_{\mu
u\rho} \mathbf{X}_\rho, \\
\left[ \mathcal{L}_\mu, \mathbf{P}_\nu \right] &= i\hbar \epsilon_{\mu
u\rho} \mathbf{P}_\rho, \\
\left[ \mathcal{L}_\mu, \mathcal{L}_\nu \right] &= i\hbar \epsilon_{\mu
u\rho} \mathcal{L}_\rho.
\end{align*}
\]

Finally, we also check the action of the rotation operator on a wave function:

\[
\langle x | \exp(-i2\mathcal{L}/\hbar)|\Psi \rangle = \langle \mathcal{L}(x)|\Psi \rangle - \frac{i}{\hbar} (\mathcal{L}_x|\Psi \rangle) = \text{indeed function rotated by } \mathcal{L}(x)
\]

In simplified form:

\[
\langle x | \exp(-i2\mathcal{L}/\hbar)|\Psi \rangle = \langle x - \mathcal{L}_x|\Psi \rangle = \langle x - \mathcal{L}_x|\Psi \rangle = \frac{i}{\hbar} (\mathcal{L}_x|\Psi \rangle)
\]

\[
\Rightarrow \langle x | \mathcal{L}/\hbar |\Psi \rangle = \frac{i}{\hbar} \mathbf{L} \cdot \mathbf{p} \langle x |\Psi \rangle = \frac{i}{\hbar} \mathbf{L} \cdot \mathbf{p} \langle x |\Psi \rangle
\]
6.3. General Theory of the Angular Momentum

For any quantum mechanical system, spatial rotations are described by the unitary operators

\[ \exp(-i2\vec{J}(\psi)) \]

where \( \vec{J} \) is the spatial angular momentum operator, which is Hermitian.

\[ \vec{J} = (\vec{J}_x, \vec{J}_y, \vec{J}_z) = (J_x, J_y, J_z) \]

"SU(2) structure constants"

These operators satisfy the angular momentum commutation relation

\[ [\vec{J}_k, \vec{J}_l] = i\epsilon_{klm} \vec{J}_m \]

\( \vec{J}_k \) and \( \vec{J}_l \) are the angular momentum operators. These operators satisfy the commutation relations (SU(2) commutation relations).

It is convenient to consider the operators \( T_k = J_k/\hbar \) to avoid factors of \( \hbar \) which satisfy

\[ [T_k, T_l] = i\epsilon_{klm} T_m \]

Different realizations of the angular momentum operators \( \vec{J} \) (or \( \vec{I} \)) are called representations.

Examples:
1. Spin-\( \frac{1}{2} \) systems: \( T_k = \sigma_k/2 \)
2. Spin-less particle in \( \mathbb{R}^2 \): \( T_k = -i\epsilon_{klm} x_l \nabla_m \)
3. Spin-\( \frac{1}{2} \) particle in \( \mathbb{R}^3 \): \( T_k = (\sigma_k/2) \otimes (-i\epsilon_{klm} x_l \nabla_m) \) \( \otimes \) direct product

(1) and (2) are examples for irreducible representations, which are representations that cannot be reduced to smaller representations.
(3) Spin-$\frac{1}{2}$ particle in $\mathbb{R}^3$: $T_\nu = \left(\sigma_\nu / 2\right) \otimes \left(-i \varepsilon_{\mu \lambda \nu} X^\mu Y^\lambda \right)$ by direct product.

(1) and (2) are examples for irreducible representations, which are representations that cannot be written as a direct product of lower elementary representations.

It is possible to classify the irreducible representations by the eigenvalue of the operator $\overline{T}^2 = T_x^2 + T_y^2 + T_z^2$, which commutes with each $T_\mu$, i.e., $[\overline{T}^2, T_\mu] = 0$, and one of the $T_\mu$.

---

### General Structure of the Irreducible Representations of the Rotation Group

We derive the general structure of irreducible representations where we use the eigenvalue of $\overline{T}^2$ to classify the representation and one of the $T_\mu$ to label each state in the representation. (We take $T_z$.)

(2) Spin-$\frac{1}{2}$: $\overline{T}^2 = s(s+1)$ with $s = \frac{1}{2}$, $T_z$ has eigenvalues $\pm \frac{1}{2}$.

We define: $T_\pm = T_\mu \pm iT_\nu = T_x \pm iT_y$

- $(T_\pm)^* = T_\mp$ (a)
- $[T_\pm, T_\pm] = i T_\nu \pm T_x = \pm T_\pm$ (b)
- $[T_\pm, T_\pm] = -2i [T_x T_y] = 2 T_2$ (c)
- $[\overline{T}^2, T_\pm] = 0$ (d)
- $T_+, T_- = T_x^2 + T_y^2 - i [T_x T_y] = T_x^2 - T_y^2 + T_2$ (e)
- $\overline{T}^2 = T_+^2 + T_-^2 + T_2^2 = T_+ T_- - T_\pm^2 + T_\mp^2 = T_+ T_- + T_\pm^2 - T_\mp^2$ (f)
Step 1: The eigenvalues of $\tilde{T}^2$ are $\geq 0$, which we therefore can unit as $j(j+1)$, $j \geq 0$.

On the eigenspace of $\tilde{T}^2$ to the eigenvalue $j(j+1)$ the eigenvalues $m$ of $T_z$ satisfy $m^2 \leq j(j+1)$. We can use $j$ and $m$ to label each angular momentum state. So we define states $|\tilde{4}_{j_{\mu}}\rangle$ with

$$\tilde{T}^2 |\tilde{4}_{j_{\mu}}\rangle = j(j+1) |\tilde{4}_{j_{\mu}}\rangle, \quad T_z |\tilde{4}_{j_{\mu}}\rangle = m |\tilde{4}_{j_{\mu}}\rangle, \quad \langle \tilde{4}_{j_{\mu}} | \tilde{4}_{j_{\nu}} \rangle = 1$$

For all states $|\tilde{4}_{j_{\mu}}\rangle$:

$$\langle \tilde{4}_{j_{\mu}} | \tilde{T}^2 |\tilde{4}_{j_{\nu}}\rangle = \langle j+1 | j+1 \rangle + \langle j | j \rangle + \langle j-1 | j \rangle \geq 0 \quad \text{(also true for eigenstates)}$$

with $\langle j+1 | j+1 \rangle = \frac{2}{j+1}$.

Step 2: The eigenspace of $\tilde{T}^2$ to the eigenvalue $j(j+1)$ is closed w.r. to the operators $T_x, T_y, T_z$.

So the eigenspace is also closed w.r. to rotations, which are functions of $T_x, T_y, T_z$.

Let $|\tilde{4}_{j_{\mu}}\rangle$ be an eigenstate to $\tilde{T}^2$ with eigenvalue $j(j+1)$, then $\tilde{T}^2 |\tilde{4}_{j_{\mu}}\rangle = T_z \tilde{T}^2 |\tilde{4}_{j_{\mu}}\rangle = j(j+1) T_z |\tilde{4}_{j_{\mu}}\rangle$.

Step 3: The operators $T_\pm$ raise/lower the eigenvalue of $T_z$ by one unit.

We have

$$T_\pm |\tilde{4}_{j_{\mu}}\rangle = \sqrt{j(j+1) - m(m+1)} |\tilde{4}_{j_{\mu}}\rangle$$

$$T_\pm T_\pm |\tilde{4}_{j_{\mu}}\rangle = \pm T_\pm |\tilde{4}_{j_{\mu}}\rangle + T_\pm T_\pm |\tilde{4}_{j_{\mu}}\rangle = (m \pm 1) T_\pm |\tilde{4}_{j_{\mu}}\rangle$$

$$\langle T_\pm |\tilde{4}_{j_{\mu}} \mid T_\pm \tilde{4}_{j_{\nu}} \rangle = \langle \tilde{4}_{j_{\mu}} \mid T_\pm T_\pm \tilde{4}_{j_{\nu}} \rangle = \langle \tilde{4}_{j_{\mu}} \mid (\tilde{T}^2 - T_z T_z) \tilde{4}_{j_{\nu}} \rangle = j(j+1) - m(m+1)$$

Step 4: The maximal size of eigenvalues $m$ on a $j$-eigenstate is $j$. So $|m| = j$. 

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Step 4: The maximal size of eigenvalues \( \mu \) on a \( j \)-eigenspace is \( j \): \( |\mu| = j \).

Let \( \mu_{\text{max/min}} \) be the largest/smallest eigenvalue of \( T_z \) on the \( j \)-eigenspace, the relation

\[
T_z (\mu_{\text{max/min}}) = \left\{ \begin{array}{ll} j(j+1) - \mu_{\text{max/min}} (\mu_{\text{max/min}} \pm 1) & \mu_{\text{max/min}} \neq \pm j \\ \pm j(j+1) & \mu_{\text{max/min}} = \pm j \end{array} \right.
\]

does lead to a contradiction unless \( \mu_{\text{max}} = j \) and \( \mu_{\text{min}} = -j \).

Step 5: The results from steps 1-4 are only consistent if

\[ j = 0, 1, 2, \ldots \quad (\text{integer spin}) \]

or

\[ j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \quad (\text{half integer spin}) \]

and \( \mu = j, j+1, \ldots, j-1, -j \) \( 2j+1 \) values.

For each allowed value of \( j \) the \( 2j+1 \) orthonormal states \( |4j, \mu \rangle \) \( (\mu = -j, \ldots, j) \)
form the basis of a subspace of \( \mathcal{L} \) that is invariant under rotations and represents a system with spin \( j \).

Each spin \( j \) system forms an irreducible representation of the \( SU(2) \) rotation group.

For the angular momentum operators \( \mathbf{S} \) we have:

\[ |4j, \mu \rangle = |lijm \rangle \]

\[ S_z |lijm \rangle = \mu |lijm \rangle, \quad S_+ |lijm \rangle = \sqrt{j(j+1)-\mu(\mu+1)} |lij, m+1 \rangle \]

\[ S_- |lijm \rangle = \sqrt{j(j+1)-\mu(\mu-1)} |lij, m-1 \rangle \]
6.4. Examples for Irreducible Representations

- **Spin - 0 (Trivial) Representation**
  \[ j = 0, \quad T = 0 \]  
  -> Particles without spin: e.g., Higgs boson, mesons (Pions, kaons, ...)

- **Spin - \( \frac{1}{2} \) Representation**  
  "fundamental representation" of SU(2)
  \[ j = \frac{1}{2}, \quad T = \frac{1}{2} \sigma, \quad m = \frac{1}{2}, -\frac{1}{2} \]

  \[ T_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \]
  \[ T_+ = T_1 + iT_2 = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \]
  \[ T_- = T_1 - iT_2 = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} \]

  \[ |\frac{1}{2}, +\frac{1}{2}\rangle \iff (0) \quad T_2 |\frac{1}{2}, +\frac{1}{2}\rangle = \frac{1}{2} |\frac{1}{2}, +\frac{1}{2}\rangle \quad T_+ |\frac{1}{2}, +\frac{1}{2}\rangle = 0 \quad T_- |\frac{1}{2}, +\frac{1}{2}\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle \]

  \[ |\frac{1}{2}, -\frac{1}{2}\rangle \iff (0) \quad T_2 |\frac{1}{2}, -\frac{1}{2}\rangle = \frac{1}{2} |\frac{1}{2}, -\frac{1}{2}\rangle \quad T_+ |\frac{1}{2}, -\frac{1}{2}\rangle = 1\frac{1}{2}, +\frac{1}{2}\rangle \quad T_- |\frac{1}{2}, -\frac{1}{2}\rangle = 0 \]

- **Spin - 1 Representation**  
  2-dimensional representation, "fundamental representation" of SO(3)
  \[ j = 1, \quad T = 2 \sigma = 1 (1+1) \sigma \quad m = +1, 0, -1 \]

  Spin basis:  \[ \{ |1, m\rangle \} = \{ |1, 1\rangle, |1, 0\rangle, |1, -1\rangle \} \]  
  \[ \text{with} \quad T_2 |1, m\rangle = m |1, m\rangle \]
Spin basis: \{ |j, m\> \} = \{ |1, +1\>, |1, 0\>, |1, -1\> \} \quad \text{with} \quad T_z |j, m\> = m |j, m\>

\[
T_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad T_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \quad T_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

The spin basis is useful for the description of spin-\( \frac{1}{2} \) particles in analogy to the description of spin-\( \frac{1}{2} \).

In the spin basis, the 3 basis states \(| j, m\> \) (\( j = 1, 2, 3 \)) do not describe the 3 spatial directions \((x, y, z)\). However, there is an alternative basis where the 3 basis states indeed correspond to the spatial directions. This basis is called the "adjoint representation of SU(2)."

**Adjoint Representation:**

The adjoint representation is obtained from the \( SU(2) \) structure constants and can be derived from the infinitesimal rotations of spatial 3-vectors. The generators of the adjoint rep

\[
L^a \text{ Rotation by } \| \mathbf{A} \| \ll 1 \text{ around axis } \frac{\mathbf{A}}{\| \mathbf{A} \|} : \mathbf{x} \mapsto \mathbf{x} + \mathbf{A} \times \mathbf{x} = (1 - i \frac{\mathbf{A}}{\| \mathbf{A} \|}) \mathbf{x}
\]

\[
x_n \mapsto x_n + \epsilon_{nkl} \epsilon_{k} \mathbf{x}_l = (\delta_{nk} - i \frac{\mathbf{A}}{\| \mathbf{A} \|}) \mathbf{x}_k
\]

\[
\Rightarrow \quad t_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix} \quad t_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad t_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
One can show that: \[ T_k = U t_k U^+ \] with \[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i & 0 \\ 0 & 0 & 1 \\ 0 & i & 0 \end{pmatrix} \]

This means that a spatial 3-vector \( \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) has the form \( U \vec{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} -x + iy \\ z \\ z + iy \end{pmatrix} \) in the spin-\( 1 \) basis.

Comment:

All representations of angular momentum with half-integer \( j \) (i.e. \( j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \)) are not representations of the spatial rotation group \( SO(3) \). This is only possible for integer \( j \) (i.e. \( j = 0, 1, 2, \ldots \)).

6.5. Spherical Harmonic Functions

The spherical harmonic functions are the angular momentum analogue to the eigenfunctions \( \langle \vec{x}|\vec{P}\rangle = \frac{e^{i\vec{P}\cdot\vec{x}}}{(2\pi)^{3/2}} \) of the momentum operator \( \vec{P} \).

The spherical harmonic functions are an explicit realization of the irreducible representations of angular momentum and the rotation group for all integer values of \( j \), i.e. for \( j = 0, 1, 2, \ldots \).

We have to switch from cartesian to spherical (polar) coordinates:

\[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta \]
We have to switch from Cartesian to Spherical (polar) coordinates:

\[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta \]

In spherical coordinates the basis vectors at each point \( \mathbf{\hat{x}} = r \mathbf{\hat{e}}_r \) in space depend on the \((r, \theta, \phi) = (\text{radius}, \text{polar angle}, \text{azimuthal angle})\) values of that point \( \mathbf{\hat{x}} \):

\[ \mathbf{\hat{e}}_r = \mathbf{\hat{e}}_x \sin \theta \cos \phi + \mathbf{\hat{e}}_y \sin \theta \sin \phi + \mathbf{\hat{e}}_z \cos \theta \]

\[ \mathbf{\hat{e}}_\theta = \mathbf{\hat{e}}_x \cos \theta \cos \phi + \mathbf{\hat{e}}_y \cos \theta \sin \phi - \mathbf{\hat{e}}_z \sin \theta \]

\[ \mathbf{\hat{e}}_\phi = -\mathbf{\hat{e}}_x \sin \phi + \mathbf{\hat{e}}_y \cos \phi \]

Form of the nabla:

\[ \nabla = \mathbf{\hat{e}}_r \frac{\partial}{\partial r} + \mathbf{\hat{e}}_\theta \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} + \mathbf{\hat{e}}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \]

\[ \mathbf{\hat{e}}_r \times \mathbf{\hat{e}}_\theta = \mathbf{\hat{e}}_\phi \]

\[ \mathbf{\hat{e}}_r \times \mathbf{\hat{e}}_\phi = -\mathbf{\hat{e}}_\theta \]

Angular momentum operator:

\[ \mathbf{L} = \mathbf{\hat{x}} \times \mathbf{\hat{r}} \frac{\partial}{\partial \mathbf{\hat{r}}} = \frac{\mathbf{\hat{r}}}{i} \times \mathbf{\hat{r}} \frac{\partial}{\partial \mathbf{\hat{r}}} = \frac{\mathbf{\hat{r}}}{i} \left( \mathbf{\hat{e}}_\theta \frac{\partial}{\partial \theta} - \mathbf{\hat{e}}_\phi \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) \]

\[ L_x = L_\phi = \frac{\hbar}{i} \left( -\sin \phi \frac{\partial}{\partial \phi} - \cos \phi \cot \theta \frac{\partial}{\partial \theta} \right) \]

\[ L_y = L_\theta = \frac{\hbar}{i} \left( \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \cot \theta \frac{\partial}{\partial \phi} \right) \]

\[ L_z = L_\phi = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \]

\[ L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \]

\[ d\mathbf{S} = \sin \theta \, d\phi \, d\theta \, d\phi \]
The angular momentum operators $\hat{L}_z$ only act on the polar and azimuthal angles. So they act on functions $Y(\theta, \phi)$ defined on the surface of the unit sphere (called $S^2$).

The set of complex-valued functions defined on the surface of the unit sphere

$$L^2(S^2) = \{ Y(\theta, \phi) : S^2 \to \mathbb{C} \mid \int_0^{2\pi} \int_0^\pi |Y(\theta, \phi)|^2 \sin \theta d\theta d\phi \leq \infty \}$$

with the scalar product $\langle Y_1, Y_2 \rangle = \int_0^{2\pi} \int_0^\pi \overline{Y_1(\theta, \phi)} Y_2(\theta, \phi) \sin \theta d\theta d\phi$, $\mathbb{D} = (\theta, \phi)$ is a Hilbert space.

The spherical harmonics functions $Y_{lm}(\theta, \phi) \in L^2(S^2)$ are the simultaneous eigenfunctions of the operators $\hat{L}_z$ and $\hat{L}_z$ with

$$\hat{L}_z Y_{lm}(\theta, \phi) = \hbar m Y_{lm}(\theta, \phi), \quad m = -l, -l+1, \ldots, l-1, l$$

and they form a CONS of $L^2(S^2)$.

We determine the $Y_{lm}(\theta, \phi)$ by using the separation ansatz $Y_{lm}(\theta, \phi) = P_{lm}(\cos \theta) \, e^{\im \phi}$ and starting with the eigenvalue equation

$$\hat{L}_z Y_{lm} = \hbar m Y_{lm} \quad \Rightarrow \quad \frac{d}{d\phi} b_n(\phi) = \im \hbar b_n(\phi) \quad \Rightarrow \quad b_n(\phi) = e^{\im \phi}.$$
We determine the \( Y_{lm}(\theta, \phi) \) by using the separation ansatz \( Y_{lm}(\theta, \phi) = P_m(\cos \theta) e^{i \phi} \) and starting with the eigenvalue equation

\[
L^2 \ Y_{lm} = \mu \ n \ Y_{lm} \quad \Rightarrow \quad \frac{\partial^2}{\partial \theta^2} \ b_n(\theta) - i \mu \ b_n(\phi) = 0
\]

Thus the second eigenvalue equation \( L^2 \ Y_{lm} = \hbar^2 \ l(l+1) \ Y_{lm} \) can be rewritten as

\[
\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{\mu^2}{\sin^2 \theta} + l(l+1) \right] P_m(\cos \theta) = 0
\]

We change variables : \( \xi = \cos \theta \)

\[
\frac{\partial}{\partial \theta} \xi = -\frac{\partial}{\partial \xi} \cos \theta = -\sin \theta \frac{\partial}{\partial \xi}
\]

\[
\Rightarrow \quad \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) = \frac{\partial}{\partial \xi} \left( \sin \theta \frac{\partial}{\partial \theta} \right) = \frac{\partial}{\partial \xi} \left( (1-\xi^2) \frac{\partial}{\partial \xi} \right) = (1-\xi^2) \frac{\partial^2}{\partial \xi^2} - 2 \xi \frac{\partial}{\partial \xi}
\]

\[
\Rightarrow \quad \left[ (1-\xi^2) \frac{\partial^2}{\partial \xi^2} - 2 \xi \frac{\partial}{\partial \xi} + l(l+1) \right] P_m(\xi) = 0
\]

This is the defining equation of the associated Legendre polynomials \( P_m(\xi) \)

The \( P_m(\xi) \) can be obtained from the (regular) Legendre polynomials \( P_\ell(\xi) = \frac{1}{2^{\ell+1}} \frac{d^{\ell+1}}{d\xi^{\ell+1}} (\xi^2-1)^\ell \), \( \ell = 0, 1, \ldots \)

by the relation

\[
P_m(\xi) = (-\xi^2)^{\frac{m}{2}} \frac{d^m}{d\xi^m} P_\ell(\xi) \quad \text{with} \quad m > 0
\]

The regular Legendre polynomials form a complete set of orthogonal functions on the interval \([-1, 1]\) with

\[
\int_{-1}^{1} d\xi \ P_\ell(\xi) P_m(\xi) = \frac{2}{2m+1} \delta_{m,\ell}
\]

The satisfy the recursion relations

\[
(l+1) P_{l+1}(\xi) = (2l+1) \xi P_l(\xi) - l P_{l-1}(\xi) \quad , \quad (1-\xi^2) \frac{d}{d\xi} P_\ell(\xi) = \ell (P_{\ell-1}(\xi) - \xi P_\ell(\xi))
\]

Lowe Legendre polynomials: \( P_0(\xi) = 1 \), \( P_1(\xi) = \xi \), \( P_2(\xi) = \frac{1}{2} (3\xi^2-1) \), \( P_3(\xi) = \frac{1}{2} (5\xi^3-3\xi) \), \ldots
The associated Legendre polynomials have the properties

\[ \int_{-1}^{1} P_{lm}(\xi) P_{lm}(\xi) = \frac{2}{2l+1} \frac{(2m)!}{(l-m)!} \delta_{lm} \quad (m \geq 0) \]

\[ P_{lm}(-\xi) = (-1)^{l+m} P_{lm}(\xi) \]

\[ P_{l0}(\xi) = P_{0l}(\xi) \quad , \quad P_{00}(\xi) = (2l-1)!! = (2l-1)(2l-3) \ldots 3 \cdot 1 \]

\[ (2l-1)!! = (2l-1)(2l-3) \ldots 3 \cdot 1 \]

---

Explicit form of the spherical harmonic functions:

\[ Y_{lm}(\theta, \phi) = (-1)^{l} \frac{\Gamma(l+m)}{2l+1 \Gamma(l-m)!} P_{lm}(\cos \theta) e^{i\phi} \]

Orthogonality:

\[ \int d\Omega Y_{lm}^*(\Omega) Y_{lm}(\Omega) = \delta_{ll'} \delta_{mm'} \]

Completeness:

\[ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') = \frac{2l+1}{4\pi} P_{l0}(\cos \alpha) = \frac{1}{4\pi} \delta(\theta-\theta') \delta(\phi-\phi') \]

Addition theorem:

\[ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta, \phi) Y_{lm}(\theta', \phi') = \frac{2l+1}{4\pi} P_{l0}(\cos \alpha), \quad \alpha : \text{angle between directions} (\theta, \phi) \text{ and } (\theta', \phi') \]

\[ \cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi-\phi') \]

Parity transformation:

\[ \hat{P} Y_{lm}(\theta, \phi) = Y_{lm}(\pi-\theta, \phi+\pi) = (-1)^{l} Y_{lm}(\theta, \phi) \]

\[ \cos(\pi-\theta) = -\cos \theta \]
Hankel transformation: \( x \rightarrow \tilde{x} = -x \Rightarrow (\theta, \phi) \rightarrow (\pi - \theta, \phi + \pi) \)

\[
P \ Y_{\ell m}(\theta, \phi) - Y_{\ell m}(\pi - \theta, \phi + \pi) = (-1)^{\ell} Y_{\ell m}(\theta, \phi)
\]

\(
\cos (\pi - \theta) = -\cos \theta
\)

\(m\)-Symmetry: \( Y_{\ell m}^{*}(\theta, \phi) = (-1)^{m} Y_{\ell m}^{*}(\theta, \phi) \)

Some explicit expressions: (use \( m \)-symmetry to obtain expressions for \( m < 0 \))

\[
Y_{00} = \frac{1}{\sqrt{4\pi}}
\]

\[
Y_{11} = -\frac{\sqrt{3}}{8\pi} \sin \theta e^{i\phi}, \quad Y_{10} = \frac{\sqrt{3}}{8\pi} \cos \theta
\]

\[
Y_{22} = \frac{\sqrt{15}}{16\pi} \sin^{2} \theta e^{2i\phi}, \quad Y_{21} = -\frac{\sqrt{15}}{16\pi} \sin \theta \cos \theta e^{i\phi}, \quad Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^{2} \theta - 1)
\]

The completeness property of the spherical harmonic functions \( Y_{\ell m} \) allows to unit energy wave function \( \psi(x) \in L^2(\mathbb{R}^3) \) as

\[
\psi(x) = \Phi(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell m}(r) Y_{\ell m}(\theta, \phi)
\]

where

\[
u_{\ell m}(r) = \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin \theta d\theta \ Y_{\ell m}^{*}(\theta, \phi) \ \Phi(r, \theta, \phi) \ \Phi(r, \theta, \phi) \ Y_{\ell m}(\theta, \phi) \Psi(x)
\]

angular/multiple spectrum
Abb. 5.6. Polardiagramme der Bahndrehimpulseigenfunktionen $Y_{lm}$ mit $l = 0, 1, 2, 3$

$\rho = |Y_{lm}(\vartheta, \varphi)|^2$

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Namenschema:

- $l = 0$ steht für $s$-Orbitale
- $l = 1$ steht für $p$-Orbitalen
- $l = 2$ steht für $d$-Orbitalen
- $l = 3$ steht für $f$-Orbitalen

CMR = Cosmic Microwave Background

$T = 2.725^\circ K$, $\frac{\delta T}{T} \sim 10^{-5}$

The CMB temperature map, obtained by WMAP experiment [cite{WMAP}]. The brighter a region is, the hotter radiation comes from it.

The angular spectrum of the CMB temperature anisotropy [cite{ACBAR}]. The line is a prediction of the standard $\Lambda CDM$ model. The quantity in vertical axis is $D_\ell$ defined by (???).