

Chapter 3: Problems in One Dimension

→ Application of concept of quantum mechanics to simple systems in 1 dimension:

We consider a particle in

- * an impenetrable box → only discrete energy eigenvalues exist
- * a harmonic oscillator potential → zero-point energy & algebraic solution & coherent states
- * a potential with steps → reflection from, transmission into a potential well & tunneling
- * δ -function potential → bound states and scattering eigenstates of the hamilton operator

3.1. Particle in an Impenetrable Box

→ Consider particle "trapped" in a potential $V(x) = \begin{cases} 0, & 0 \leq x \leq L \\ \infty, & x < 0 \text{ or } x > L \end{cases}$

We need to determine all possible energy eigenvalues and the corresponding eigenfunctions from the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) + V(x) \psi(x) = E \psi(x)$$

→ Infinite potential region: Consider $V(x) = \infty$ for $|x| \rightarrow \infty$. The term $\frac{\partial^2}{\partial x^2} \psi$ is finite almost everywhere (i.e. everywhere except at exceptional and separated points) and we have $E < \infty$.

So $\psi(x)$ must vanish almost everywhere.

→ Infinite potential region: Consider $V(x) = \Lambda$ for $\Lambda \rightarrow \infty$. The term $\frac{\partial^2 \psi}{\partial x^2}$ is finite almost everywhere (i.e. everywhere except at exceptional and separated points) and we have $E < \infty$. So $\psi(x)$ must vanish almost everywhere. We can therefore conclude that $\psi(x) = 0$ in the region where $V(x) = \infty$.

→ Smoothness of $\psi(x)$ at boundary points $x=0, L$: For any finite Λ (with $\Lambda \rightarrow \infty$) $\psi(x)$ must be smooth, because if $\psi(x)$ were discontinuous (e.g. $\psi(x) \sim \Theta(x)$ for x close to zero) we had $(\frac{\partial^2}{\partial x^2} \psi(x)) \sim \delta(x)$ and $\frac{\partial^2}{\partial x^2} \psi(x) \sim \delta'(x)$, which does not appear in the time-independent Schrödinger equation. So we must have $\psi(0) = \psi(L) = 0$. $\psi(x)$ is also continuous everywhere in the region $0 \leq x \leq L$.

→ Zero potential region: We have to solve the following eigenvalue problem with boundary conditions:

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \phi(x) = E \phi(x), \quad \phi(0) = \phi(L) = 0$$

$$e^{ix} = \cos(x) + i \sin(x)$$

↳ This is the differential equation for trigonometric functions and the general solution without yet considering the boundary conditions reads $\phi_E(x) = N \exp(i(kx - a))$ with $E = \frac{\hbar^2 k^2}{2m}$ and $k, a, N \in \mathbb{R}$. The boundary conditions further restrict the possible eigenvalues and the form of the eigenfunctions.

$$\phi(0) = 0 : \quad \phi_E(x) = N e^{ia} \sin(kx), \quad E = \frac{\hbar^2 k^2}{2m}$$

$$\phi(L) = 0 : \quad \sin(k_n L) = 0 \Rightarrow k_n L = n\pi \quad (n = 1, 2, \dots) \quad [n=0 \text{ leads to } \phi(x)=0]$$

We can further set $e^{ia} = 1$ ($a=0$) since an overall (constant) phase can be set to zero.

We thus obtain all the energy eigenfunctions and the associated energy eigenvalues:

$$\phi_n(x) = N \sin\left(\frac{n\pi}{L} x\right) \quad \text{with} \quad E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2m L^2}, \quad n \in \mathbb{N}$$

Note: Negative integers n do not lead to additional independent solutions!

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Normalization gives the condition:

five same results
for integrals over half cycles

$$\sin^2 x + \cos^2 x = 1$$

$$1 \stackrel{!}{=} N^2 \int_0^L dx \sin^2\left(\frac{n\pi}{L}x\right) = N^2 \frac{L}{2} \Rightarrow N^2 = \frac{2}{L}$$

↪ Final result: ϕ_n : energy eigenfunctions ; E_n : energy eigenvalues

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin(k_n x), \quad k_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots \quad (n \in \mathbb{N})$$

$$E_n = \frac{\hbar^2 k_n^2}{2m}, \quad \text{so } E_1 = \frac{1}{2m} \left(\frac{n\pi}{L}\right)^2, \quad E_2 = 4E_1, \dots, \quad E_n = n^2 E_1$$

$$\langle \phi_m | \phi_n \rangle = \int_0^L dx \phi_m^*(x) \phi_n(x) = \delta_{nm}$$

↪ Completeness: Every continuous wave function $\psi(x)$, $x \in [0, L]$ with $\psi(0) = \psi(L) = 0$ can be expanded in the ϕ_n written as a superposition of the energy eigenfunctions: (\rightarrow Fourier Analysis!)

$$\psi(x) = \sum_{n=1}^{\infty} \phi_n(x) c_n, \quad c_n = \langle \phi_n | \psi \rangle = \int_0^L dx \phi_n^*(x) \psi(x)$$

This implies the completeness relation

$$\sum_{n=1}^{\infty} \phi_n(x) \phi_n^*(y) = \sum_{n=1}^{\infty} \langle \psi | \phi_n \rangle \langle \phi_n | \psi \rangle = \delta(x-y)$$

$$1 = \sum_{n=1}^{\infty} |\phi_n(x)|^2$$

This implies the completeness relation

$$\sum_{n=1}^{\infty} \phi_n(x) \phi_n^*(y) = \sum_{n=1}^{\infty} \langle x | \phi_n \rangle \langle \phi_n | y \rangle = \delta(x-y) \quad \mathbb{1} = \sum_{n=1}^{\infty} |\phi_n\rangle \langle \phi_n|$$

↳ Time evolution:

$$\psi(x,t) = e^{-\frac{iE_n t}{\hbar}} \sum_{n=1}^{\infty} c_n \phi_n(x) = \sum_{n=1}^{\infty} e^{-\frac{iE_n t}{\hbar}} c_n \phi_n(x)$$

3.2. Harmonic Oscillator

→ Consider particle in the harmonic oscillator potential $V(x) = \frac{m\omega^2}{2}x^2$.

$$\text{Hamilton operator : } H = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2$$

→ We can easily see that the energy spectrum of H contains only real numbers $E \geq 0$, because

$$\begin{aligned}\langle \psi | H | \psi \rangle &= \frac{1}{2m} \langle \psi | P^2 | \psi \rangle + \frac{m\omega^2}{2} \langle \psi | X^2 | \psi \rangle \\ &= \frac{1}{2m} \langle P \psi | P \psi \rangle + \frac{m\omega^2}{2} \langle X \psi | X \psi \rangle = 0\end{aligned}$$

→ The ground state is the energy eigenstate with the lowest energy eigenvalue. Using Heisenberg's uncertainty principle we can show that the ground state's energy is indeed > 0 .

↪ We may assume that the ground state ($|\psi_0\rangle$) has the classic property $\langle X \rangle_{\psi_0} = 0$, $\langle P \rangle_{\psi_0} = 0$ since it should be the state that simply stays at the minimum and does not move.

$$\Rightarrow (\Delta X)_{\psi_0}^2 = \langle X^2 \rangle_{\psi_0}, \quad (\Delta P)_{\psi_0}^2 = \langle P^2 \rangle_{\psi_0}$$

$$\Rightarrow \langle X^2 \rangle_{\psi_0} \langle P^2 \rangle_{\psi_0} = \left(\frac{\pi}{2}\right)^2$$

(Heisenberg's uncertainty principle)

$$\begin{aligned}\Rightarrow \langle H \rangle_{\psi_0} &= \frac{1}{2m} \langle P^2 \rangle_{\psi_0} + \frac{m\omega^2}{2} \langle X^2 \rangle_{\psi_0} \\ &\geq \frac{1}{2m} \left(\frac{\pi}{2}\right)^2 \frac{1}{\langle X^2 \rangle_{\psi_0}} + \frac{m\omega^2}{2} \langle X^2 \rangle_{\psi_0}\end{aligned}$$

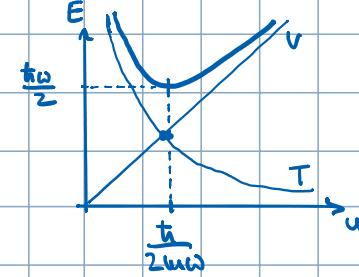


$$\Rightarrow \langle H \rangle_{\phi_0} = \frac{\hbar^2}{2m} \langle P \rangle_{\phi_0} + \frac{1}{2} \langle X \rangle_{\phi_0} \\ \geq \frac{1}{2m} \left(\frac{\pi}{2}\right)^2 \frac{1}{2} \langle X^2 \rangle_{\phi_0} + \frac{m\omega^2}{2} \langle X^2 \rangle_{\phi_0}$$

We can now look for the minimum of the function $E(u) = \frac{\hbar^2}{8mu} \frac{1}{u} + \frac{m\omega^2}{2} u$

$$E'(u) = -\frac{\hbar^2}{8mu} \frac{1}{u^2} + \frac{m\omega^2}{2} = 0$$

$$\hookrightarrow u_{\min} = \frac{\hbar}{2m\omega} \Rightarrow E(u_{\min}) = \frac{\hbar\omega}{2} \text{ with } T(u_{\min}) = V(u_{\min}).$$



So the ground state energy has the property $E_0 = \langle H \rangle_{\phi_0} \geq \frac{\hbar\omega}{2}$.

It turns out that the ground state is the Gauss wave packet from Chap. 2.9. with $p_0 = 0$:

$$\phi_0(x) = (2\pi)^{-1/4} \sigma^{-1/2} \exp\left(-\frac{x^2}{4\sigma^2}\right)$$

normalized

$$\Rightarrow \langle x \rangle_{\phi_0} = 0, \langle P \rangle_{\phi_0} = 0 \quad \checkmark$$

$$\langle X^2 \rangle_{\phi_0}^{1/2} = \sigma, \langle P^2 \rangle_{\phi_0}^{1/2} = \frac{\hbar}{2\sigma} \quad (\text{exercises})$$

$$\text{We check: } -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \phi_0(x) = -\frac{\hbar^2}{2m} \left(\frac{x^2}{4\sigma^4} - \frac{1}{2\sigma^2} \right) \phi_0(x)$$

$$\Rightarrow -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \phi_0(x) + \frac{m\omega^2}{2} x^2 \phi_0(x) = x^2 \left[\underbrace{\frac{m\omega^2}{2} - \frac{\hbar^2}{8m\sigma^4}}_{=0} \right] \phi_0(x) + \frac{\hbar^2}{4m\sigma^2} \phi_0(x)$$

So we must have $\sigma = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$ and get $E_0 = \frac{\hbar\omega}{2}$ as the ground state energy.

Note that $\frac{1}{2m} \langle P^2 \rangle_{\phi_0} = \frac{m\omega^2}{2} \langle X^2 \rangle_{\phi_0}$, so kinetic and potential energy average values are equal in the ground state.

Note: The ground state wave function is unique (up to a global phase factor).

→ Ladder operators:

From Chap. 2.8. (\rightarrow generalized uncertainty relation) and the fact that the ground state $\phi_0(x)$ minimizes the Heisenberg's uncertainty relation it follows that

$$\underbrace{\left(\frac{X}{\Delta X} + i \frac{P}{\Delta P} \right)}_{= 2a} (\phi_0) = 0 \quad \text{with } \Delta X = \left(\frac{\hbar}{2m\omega} \right)^{1/2}, \Delta P = \left(\frac{\hbar m\omega}{2} \right)^{1/2}$$

It turns out to be extremely useful to define the ladder operators

$$a := \left(\frac{m\omega}{2\hbar} \right)^{1/2} X + i \left(\frac{1}{2\hbar m\omega} \right)^{1/2} P, \quad a^+ = \left(\frac{m\omega}{2\hbar} \right)^{1/2} X - i \left(\frac{1}{2\hbar m\omega} \right)^{1/2} P$$

$$\Rightarrow X = \left(\frac{\hbar}{2m\omega} \right)^{1/2} (a + a^+), \quad P = \left(\frac{\hbar m\omega}{2} \right)^{1/2} i (a^+ - a)$$

The commutation relation of the a and a^+ reads :

$$[X, P] = i\hbar = -[P, X], \quad [a, a^+] = -\frac{i}{2\hbar} [X, P] + \frac{i}{2\hbar} [P, X]$$

$$[a, a^+] = 1 \quad \Rightarrow \quad N = a a^+ - 1$$

We also have that the Hermitian operator

$N = a^+ a$, called number operator, can be written as

$$N = \frac{m\omega}{2\hbar} X^2 + \frac{1}{2\hbar m\omega} P^2 + \frac{i}{2\hbar} [X, P] = \frac{1}{\hbar\omega} \left(\overbrace{\frac{1}{2\hbar} P^2 + \frac{m\omega^2}{2} X^2}^{= H} \right) - \frac{1}{2} = \frac{1}{\hbar\omega} H - \frac{1}{2}$$

such that the Hamilton can be expressed as

such that the Hamilton can be expressed as

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) = \hbar\omega(N + \frac{1}{2}).$$

$$[AB, C] = A[B, C] + [A, C]B$$

In addition we have the following commutation relations

$$[N, a^\dagger] = a^\dagger, [N, a] = -a$$

$$\hookrightarrow [H, a^\dagger] = \hbar\omega a^\dagger, [H, a] = -\hbar\omega a$$

→ Algebraic construction of the energy eigenstates:

↪ We know that $a|\phi_0\rangle = 0$ with $|\phi_0\rangle$ being unique with $\langle\phi_0|\phi_0\rangle = 1$. and so we also have $N|\phi_0\rangle = 0$, i.e. $|\phi_0\rangle$ is eigenstate to N with eigenvalue 0. We already know $H|\phi_0\rangle = \frac{1}{2}\hbar\omega|\phi_0\rangle$, i.e. $|\phi_0\rangle$ is eigenstate to H with eigenenergy $E_0 = \frac{1}{2}\hbar\omega$.

$$\hookrightarrow N a^\dagger |\phi_0\rangle = [N, a^\dagger]|\phi_0\rangle + a^\dagger N |\phi_0\rangle = a^\dagger |\phi_0\rangle$$

$$H a^\dagger |\phi_0\rangle = [H, a^\dagger]|\phi_0\rangle + a^\dagger H |\phi_0\rangle = (E_0 + \hbar\omega) a^\dagger |\phi_0\rangle$$

$$\langle a^\dagger \phi_0 | a^\dagger \phi_0 \rangle = \langle \phi_0 | a a^\dagger \phi_0 \rangle = \langle \phi_0 | (N+1) |\phi_0\rangle = \langle \phi_0 | \phi_0 \rangle = 1$$

$$\begin{aligned} [N, a a^\dagger] &= a[N, a^\dagger] + [N, a]a^\dagger \\ &= a a^\dagger - a a^\dagger = 0 \end{aligned}$$

$$N a a^\dagger |\phi_0\rangle = [N, a a^\dagger]|\phi_0\rangle + a a^\dagger N |\phi_0\rangle = (a a^\dagger - a a^\dagger) |\phi_0\rangle = 0 \Rightarrow a a^\dagger |\phi_0\rangle \propto |\phi_0\rangle$$

$$\langle a a^\dagger \phi_0 | a a^\dagger \phi_0 \rangle = \langle \phi_0 | \underbrace{a a^\dagger}_{N} a^\dagger \phi_0 \rangle = \langle \phi_0 | a a^\dagger \phi_0 \rangle = \langle \phi_0 | (1+N) |\phi_0\rangle = \langle \phi_0 | \phi_0 \rangle = 1$$

So $|\phi_1\rangle := a^\dagger |\phi_0\rangle$ is a normalized eigenstate of N to the eigenvalue 1 and energy eigenstate with the eigenvalue $\frac{3}{2}\hbar\omega$, and we have $a|\phi_1\rangle = |\phi_0\rangle$.

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$$\hookrightarrow \underbrace{N a^+ |\phi_1\rangle}_{=a^+} = [N, a^+] |\phi_1\rangle + a^+ N |\phi_1\rangle = 2 a^+ |\phi_1\rangle$$

$$H a^+ |\phi_1\rangle = [H, a^+] |\phi_1\rangle + a^+ H |\phi_1\rangle = (E_1 + \hbar\omega) a^+ |\phi_1\rangle$$

$$\langle a^+ \phi_1 | a^+ \phi_1 \rangle = \langle \phi_1 | a a^+ \phi_1 \rangle = \langle \phi_1 | (N+1) |\phi_1\rangle = 2 \langle \phi_1 | \phi_1 \rangle = 2$$

$$N a a^+ |\phi_1\rangle = a a^+ N |\phi_1\rangle = a a^+ |\phi_1\rangle = (1+N) |\phi_1\rangle = 2 |\phi_1\rangle$$

$$\langle a a^+ \phi_1 | a a^+ \phi_1 \rangle = \langle \phi_1 | a N a^+ |\phi_1\rangle = \langle \phi_1 | (a a^+ + a a^+ N) |\phi_1\rangle = 2 \langle \phi_1 | (1+N) |\phi_1\rangle = 4 \langle \phi_1 | \phi_1 \rangle = 4$$

So $|\phi_2\rangle := \frac{1}{\sqrt{2}} a^+ |\phi_1\rangle = \frac{1}{\sqrt{2}} (a^+)^2 |\phi_0\rangle$ is a normalized eigenstate of N to the eigenvalue 2 and energy eigenstate to the eigenvalue $\frac{5}{2}\hbar\omega$, and we have $a |\phi_2\rangle = \sqrt{2} |\phi_1\rangle$.

proof: exercises



\hookrightarrow With this method we can (indeed!) construct a CONS of eigenstates of the number operator N and the Hamilton operator H :

$$a |\phi_0\rangle = 0, \langle \phi_0 | \phi_0 \rangle = 1$$

$$|\phi_n\rangle = \frac{1}{\sqrt{n!}} (a^+)^n |\phi_0\rangle,$$

$$a^+ |\phi_n\rangle = \sqrt{n+1} |\phi_{n+1}\rangle, \quad a |\phi_n\rangle = \sqrt{n} |\phi_{n-1}\rangle \quad (*)$$

$$\langle \phi_m | \phi_n \rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |\phi_n\rangle \langle \phi_n| = 1$$

$$N |\phi_n\rangle = n |\phi_n\rangle$$

← "N: number operator"

$$N = a^+ a$$

$$H = \hbar\omega(a^+ a + 1)$$

$$\langle \Phi_m | \Phi_n \rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |\Phi_n \times \Phi_n| = 1$$

$$N|\Phi_n\rangle = n|\Phi_n\rangle \quad \leftarrow \text{"N: number operator"} \quad N = a^\dagger a, \quad H = \hbar\omega(a^\dagger a + 1) \\ H|\Phi_n\rangle = \hbar\omega(n + \frac{1}{2})|\Phi_n\rangle, \quad (n = 0, 1, 2, \dots)$$

Due to property (*) we call a^\dagger creation operator and a annihilation operator.

The state $|\Phi_n\rangle$ is called the n -th excited state.

↳ We can obtain the configuration space energy eigenfunctions by applying the creation operator a^\dagger

$$\Phi_0(x) = (\sqrt{\pi} \tilde{x})^{-1/2} \exp\left\{-\frac{1}{2}\left(\frac{x}{\tilde{x}}\right)^2\right\} \quad \tilde{x} = \left(\frac{\hbar}{m\omega}\right)^{1/2} = \sqrt{2} \Delta x$$

$$\Phi_n(x) = (n! \sqrt{\pi} \tilde{x})^{-1/2} (a^\dagger)^n \exp\left\{-\frac{1}{2}\left(\frac{x}{\tilde{x}}\right)^2\right\}, \quad a^\dagger = \frac{1}{\sqrt{2}} \left(\frac{x}{\tilde{x}} - \tilde{x} \frac{d}{dx} \right), \quad a = \frac{1}{\sqrt{2}} \left(\frac{x}{\tilde{x}} + \tilde{x} \frac{d}{dx} \right)$$

$$= (2^n n! \sqrt{\pi} \tilde{x})^{-1/2} \exp\left\{-\frac{1}{2}\left(\frac{x}{\tilde{x}}\right)^2\right\} H_n\left(\frac{x}{\tilde{x}}\right) \quad \text{"Hermite polynomials"}$$

↳ Hermite polynomials:

$$H_n(x) = e^{x^2/2} (\sqrt{2} a^\dagger)^n \Big|_{x=0} e^{-x^2/2} = e^{x^2} \underbrace{e^{-x^2/2} (x - \frac{d}{dx})^n}_{(-1)^n \frac{d^n}{dx^n}} e^{x^2/2} e^{-x^2} = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

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Orthogonality:

$$\int_{-\infty}^{+\infty} dx e^{-x^2} H_n(x) H_m(x) = \sqrt{\pi} 2^n n! \delta_{nm}$$

Generating function:

$$e^{-t^2 + 2tx} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n H_n(x)$$

Differential equation:

$$\left[\frac{d^2}{dx^2} - 2x \frac{d}{dx} + 2n \right] H_n(x) = 0$$

→ Multiparticle state interpretation:

The energy eigenstates $| \psi_n \rangle$ with $E_n = \hbar \omega (\frac{1}{2} + n)$ also provide the quantum mechanical description of states with n identical Bose particles each having energy $\hbar \omega$.

Example: Photons (light particles) with a fixed direction and a fixed frequency ω

$| \psi_0 \rangle$: zero photon state, "vacuum" with vacuum energy $E_0 = \frac{1}{2} \hbar \omega$

$| \psi_n \rangle$: n photon state with energy $E_n = E_0 + n \hbar \omega$

a : operator that annihilates one photon

a^\dagger : operator that creates one photon

→ Coherent States:

The energy eigenstates $| \psi_n \rangle$ do not at all resemble at all a classical particle oscillating in the harmonic oscillator potential with $x(t) = x_0 \cos(\omega t - \phi)$ because we have at any time t

The energy eigenstates $|\psi_n\rangle$ do not at all resemble at all a classical particle oscillating in the harmonic oscillator potential with $x(t) = x_0 \cos(\omega t - \delta)$ because we have at any time t
 $\langle \psi_n | x | \psi_n \rangle = 0$ and $\langle \psi_n | P | \psi_n \rangle = 0$ (recall: $X = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^\dagger)$, $P = \left(\frac{\hbar m\omega}{2}\right)^{1/2}i(a^\dagger - a)$)

We want to construct a solution of the time-dependent Schrödinger equation that describes an oscillating behavior.

↳ A solution with $\langle x \rangle = 0$ at $t=0$ may have the correct property, so we may look at a wave function that at $t=0$ satisfies the equation

$$\frac{1}{2} \left(\frac{X - x_0}{\Delta X} + i \frac{P - p_0}{\Delta P} \right) |\Psi_z\rangle = 0, \text{ where } \Delta X = \left(\frac{\hbar}{2m\omega}\right)^{1/2}, \Delta P = \left(\frac{\hbar m\omega}{2}\right)^{1/2} \text{ from the ground state } |\psi_0\rangle$$

and $x_0, p_0 \in \mathbb{R}$ arbitrary

$$\Leftrightarrow a |\Psi_z\rangle = \frac{1}{2} \left(\frac{x_0}{\Delta X} + i \frac{p_0}{\Delta P} \right) |\Psi_z\rangle = z |\Psi_z\rangle, \quad z \in \mathbb{C}$$

↳ The eigenvalue equation $a |\Psi_z\rangle = z |\Psi_z\rangle, z \in \mathbb{C}$, is an example of a coherent state:

- * $|\Psi_z\rangle$ is obviously a superposition of states with different particle number
- * destroying one photon from state $|\Psi_z\rangle$ gives $z |\Psi_z\rangle$. $\Rightarrow |\Psi_z\rangle$ is superposition of multiparticle states
- * important for the quantum mechanical description of coherent light.

We define for convenience: $|n\rangle := |\psi_n\rangle$, $n \in \mathbb{N}_0$ and $|z\rangle := |\Psi_z\rangle$.

We can make the ansatz $|z\rangle = f(a^\dagger)|0\rangle$ and because of $[a, f(a^\dagger)] = f'(a^\dagger)$ we obtain

$$a |z\rangle = a f(a^\dagger) |0\rangle = [a, f(a^\dagger)] |0\rangle + f(a^\dagger) a |0\rangle = f'(a^\dagger) |0\rangle \stackrel{!}{=} z f(a^\dagger) |0\rangle$$

$$\Rightarrow f'(a^\dagger) = z f(a^\dagger) \Rightarrow f(a^\dagger) = c e^{z a^\dagger} = c \left(1 + za^\dagger + \frac{1}{2} z^2 (a^\dagger)^2 + \frac{1}{3!} z^3 (a^\dagger)^3 + \dots \right), \quad c \in \mathbb{C}.$$

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$$\text{We see that } |z\rangle = c \sum_{n=0}^{\infty} \frac{(za^+)^n}{n!} |0\rangle = c \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle$$

$$\text{Norm: } 1 \doteq \langle z|z \rangle = |c|^2 \sum_{n,m=0}^{\infty} \frac{(z^*)^n z^m}{\sqrt{n!} \sqrt{m!}} \underbrace{\langle m|n \rangle}_{\delta_{mn}} = |c|^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} = |c|^2 e^{|z|^2} \Rightarrow |c| = c^{-|z|^2/2}$$

We set $c = |c|$.

↳ Time-dependent solution:

$$E_n = \hbar \omega (n + \frac{1}{2})$$

$$\begin{aligned} \varphi_z(x,t) &= c^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{(z \bar{e}^{i\omega t})^n}{\sqrt{n!}} \phi_n(x) e^{-i\omega t/2} \\ &= \varphi_{z(t)}(x) e^{-i\omega t/2} \quad \text{with} \quad z(t) = z \bar{e}^{i\omega t} \end{aligned}$$

Location and momentum expectation value:

$$\begin{aligned} \langle x \rangle &= \langle \varphi_{z(t)} | x \varphi_{z(t)} \rangle = \frac{\tilde{x}}{\sqrt{2}} \langle \varphi_{z(t)} | (a + a^+) \varphi_{z(t)} \rangle = \frac{\tilde{x}}{\sqrt{2}} (z(t) + z^*(t)), \quad \tilde{x} = \left(\frac{\hbar}{\omega m}\right)^{1/2} \\ &= \sqrt{2} \tilde{x} |z| \cos(\omega t - \delta), \quad z = |z| e^{i\delta}, \quad z(t) = |z| e^{-i(\omega t - \delta)} = |z| (\cos(\omega t - \delta) - i \sin(\omega t - \delta)) \end{aligned}$$

$$\begin{aligned} \langle p \rangle &= \frac{-i\hbar}{\sqrt{2} \tilde{x}} \langle \varphi_{z(t)} | (a - a^+) \varphi_{z(t)} \rangle = \frac{-i\hbar}{\sqrt{2} \tilde{x}} (z(t) - z^*(t)) = \sqrt{2} \frac{\hbar}{\tilde{x}} |z| \sin(\omega t - \delta) \\ &= \sqrt{2} m \tilde{x} |z| \omega \sin(\omega t - \delta) = m \frac{d}{dt} \langle x \rangle \end{aligned}$$

→ Exactly corresponds to the motion of a classic particle in the harmonic oscillator potential with the amplitude $x_0 = \sqrt{2} \tilde{x} |z|$.

→ Exactly corresponds to the motion of a classic particle in the harmonic oscillator potential with the amplitude $x_0 = \sqrt{2}|\tilde{x}|z|$.

↳ Explicit expression for wave function:

$$\varphi_z(x, t) = e^{-i\omega t/2} \sum_{n=0}^{\infty} \frac{(\tilde{z}(t) a^+)^n}{n!} e^{-\frac{1}{2}|\tilde{z}|^2} \phi_o(x) = e^{-i\omega t/2} e^{\tilde{z}(t)a^+} e^{-\frac{1}{2}|\tilde{z}|^2} \phi_o(x)$$

$$= e^{-i\omega t/2} e^{\tilde{z}(t)a^+} e^{-\tilde{z}^*(t)a} e^{-\frac{1}{2}[\tilde{z}(t)a^+, -\tilde{z}^*(t)a]} \phi_o(x)$$

gives 1 on b.

$$= e^{-i\omega t/2} e^{\tilde{z}(t)a^+ - \tilde{z}^*(t)a} \phi_o(x)$$

$$\rightarrow -i\frac{\sqrt{2}|\tilde{z}|x}{\tilde{x}} \sin(\omega t - \delta) - i\sqrt{2}|\tilde{z}| \tilde{x} \cos(\omega t - \delta) \frac{d}{dx}$$

$$= e^{-i\omega t/2} e^{-i\frac{\sqrt{2}|\tilde{z}|x}{\tilde{x}} \sin(\omega t - \delta)} e^{i\frac{1}{2}|\tilde{z}|^2 \underbrace{\sin(\omega t - \delta) \cos(\omega t - \delta)}_{\frac{1}{2}\sin(2(\omega t - \delta))}} e^{-i\sqrt{2}|\tilde{z}| \tilde{x} \cos(\omega t - \delta) \frac{d}{dx}} \phi_o(x)$$

$$= \frac{1}{\pi^{1/4} x^{1/2}} \exp \left\{ -i \left[\frac{\omega t}{2} - \frac{1}{2} |\tilde{z}|^2 \sin(2(\omega t - \delta)) + \frac{\sqrt{2}|\tilde{z}|x}{\tilde{x}} \sin(\omega t - \delta) \right] \right\} \exp \left\{ -\frac{1}{2x^2} (x - x_0 \cos(\omega t - \delta))^2 \right\}$$

time-dependent part phase

$$-\frac{1}{2}|\tilde{z}|^2 [a^+, a] = -\frac{1}{2} [\tilde{z}(t)a^+, \tilde{z}^*(t)a]$$

valid if $[A, B]_A = [[A, B], I] = 0$

$$\text{use: } e^{A+B} = e^A e^B e^{-[A,B]/2} \quad (*)$$

$$a^+ = \frac{1}{\sqrt{2}} \left(\frac{x}{\tilde{x}} - \tilde{x} \frac{d}{dx} \right), \quad a = \frac{1}{\sqrt{2}} \left(\frac{x}{\tilde{x}} + \tilde{x} \frac{d}{dx} \right)$$

(shift operator)

$$= T_{-\sqrt{2}|\tilde{z}| \tilde{x} \cos(\omega t - \delta)}$$

$\left. \begin{array}{l} (*) \\ & \& \end{array} \right] \& \left[x, \frac{d}{dx} \right] = -1$

$$e^{-i\sqrt{2}|\tilde{z}| \tilde{x} \cos(\omega t - \delta) \frac{d}{dx}} \phi_o(x)$$

04/24/2017 (7)

$$\Rightarrow |\psi_z(x, t)|^2 = \frac{1}{\pi \kappa^2} \exp \left\{ -\frac{1}{\kappa^2} (x - x_0 \cos(\omega t - \delta))^2 \right\}, \quad x_0 = \sqrt{2} \kappa |z|, \quad \kappa = \left(\frac{\hbar}{\omega m} \right)^{1/2}$$

This is exactly a Gaussian wave packet oscillating with frequency ω , that does not broaden with time.

This does completely agree with our classic notion of a coherent light beam!

↳ Energy expectation value:

$$\begin{aligned} \langle H \rangle &= \hbar \omega \langle a^\dagger a + \frac{1}{2} \rangle = \frac{\hbar \omega}{2} + \hbar \omega \langle \psi_{z(t)} | a^\dagger a | \psi_{z(t)} \rangle = \frac{\hbar \omega}{2} + \hbar \omega \langle a | \psi_{z(t)} | a^\dagger | \psi_{z(t)} \rangle \\ &= \frac{\hbar \omega}{2} + \hbar \omega |z(t)|^2 = \hbar \omega (|z|^2 + \frac{1}{2}) = \hbar \omega \left(\frac{x_0^2}{2 \kappa^2} + \frac{1}{2} \right) \quad \kappa = \left(\frac{\hbar}{\omega m} \right)^{1/2} \\ &= \frac{m \omega^2}{2} x_0^2 + \frac{\hbar \omega}{2} \quad x_0: \text{"classic" amplitude} \end{aligned}$$

We see that the classic limit is obtained for $|z| \gg 1$.

This is consistent with the following observation:

We see that the classic limit is obtained for $|z| \gg 1$.

This is consistent with the following observation:

$$\text{We have } \langle \varphi_z | \varphi_z \rangle = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} \underbrace{\langle \Phi_n | \Phi_n \rangle}_{=1}.$$

The function $f_z(n) = \frac{|z|^{2n}}{n!}$ has its maximum at $n = |z|$, so the states $|\Phi_n\rangle$ with $n = |z|$ dominate the coherent state for $|z| \gg 1$.

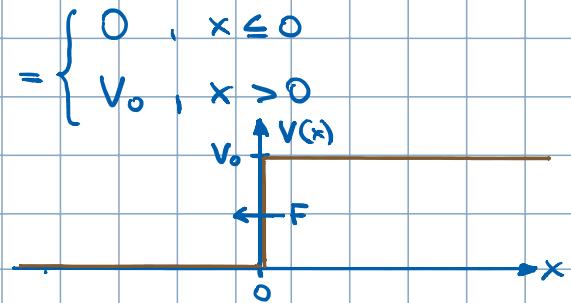
05/01/2016 ①

3.3. Potential With a Step

→ Consider a particle that experiences a very localized force at $x=0$ of the form $F(x) = -V_0 \delta(x)$.

So the particle moves in a potential of the form $V(x) = V_0 \Theta(x) = \begin{cases} 0, & x \leq 0 \\ V_0, & x > 0 \end{cases}$
and the force acts to the left.

$$\text{Hamilton operator : } H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_0 \Theta(x)$$



We have to find all solutions of the eigenvalue problem $H \phi(x) = E \phi(x)$.

We have $\langle H \rangle \geq 0$ for any state, so $E \geq 0$ as well.

→ Region $x < 0$: We have to solve the free particle Schrödinger equation $-\frac{\hbar^2}{2m} \phi''(x) = E \phi(x)$

$$\text{The solution has the form : } Ae^{ikx} + Be^{-ikx}, \quad p = \hbar k, \quad E = \frac{\hbar^2 k^2}{2m} \geq 0, \quad k \in \mathbb{R}$$

Note that $Pe^{\pm ikx} = \pm p e^{\pm ikx}$, so $e^{\pm ikx}$ describe a right/left moving particle.

Region $x > 0$: We solve the eigenvalue equation $-\frac{\hbar^2}{2m} \phi''(x) = (E - V_0) \phi(x)$

$(E - V_0)$ can be positive or negative, so we must consider 2 cases.

Case $0 \leq E \leq V_0$: Solution has the form: $C e^{-k_x x}, \quad k_x > 0$

$(E - V_0)$ can be positive or negative, so we must consider 2 cases.

Case $0 \leq E \leq V_0$: Solution has the form: $C e^{-kx}$, $k > 0$

$$\text{with } \frac{\hbar^2 k^2}{2m} = \frac{V_0 - E}{\geq 0} = V_0 - \frac{\hbar^2 k^2}{2m} \Rightarrow k = \sqrt{\frac{2mV_0}{\hbar^2} - k^2}, \quad k \leq \frac{\sqrt{2mV_0}}{\hbar}$$

Note that the solution e^{+kx} is not allowed since it diverges for large x .

Case $E > V_0$: Solution has the form: $C e^{ikx} + D e^{-ikx}$

$$\text{with } \frac{\hbar^2 k^2}{2m} = \frac{E - V_0}{> 0} = \frac{\hbar^2 k^2}{2m} - V_0 \Rightarrow k = \sqrt{k^2 - \frac{2mV_0}{\hbar^2}}, \quad |k| \geq \frac{\sqrt{2mV_0}}{\hbar}$$

Continuity at $x = 0$: $\phi(x)$ and $\phi'(x)$ must be continuous at $x = 0$, because otherwise $\phi''(x)$ would contain terms $\sim \delta(x)$ or $\sim \delta'(x)$, which however do not appear in the Schrödinger equation.

→ Construction of a complete set of energy eigenfunctions

We want to identify the eigenfunctions $\phi_k(x)$ with: $H \phi_k(x) = \frac{\hbar^2 k^2}{2m} \phi_k(x)$

(1) Right-moving incoming particle with $E(k) < V_0$:

$$\phi_k^1(x) = \begin{cases} e^{ikx} + R e^{-ikx}, & x \leq 0 \\ C e^{-kx}, & x > 0 \end{cases} \quad \text{with } 0 < k < \frac{\sqrt{2mV_0}}{\hbar}$$
$$k = \sqrt{\frac{2mV_0}{\hbar^2} - k^2} > 0$$

incoming part

05/01/2016 (2)

Continuity of $\Phi_e(x)$ at $x=0$: $1+R=C$

$$\text{Continuity of } \Phi_e'(x) = \begin{cases} ik(e^{ikx} - Re^{-ikx}), & x \leq 0 \\ -kC e^{-kx}, & x > 0 \end{cases}$$

at $x=0$ gives $ik(1-R) = -kC$

$$\Rightarrow C = \frac{2k}{k+ik} \quad R = \frac{k-ik}{k+ik} \quad \text{with} \quad |C| = \frac{2k}{\sqrt{k^2+k^2}} \quad |R| = 1$$

$$\hookrightarrow C = \frac{2k}{\sqrt{k^2+k^2}} e^{i\delta_c} = \sqrt{\frac{2}{mV_0}} tk e^{i\delta_c}, \quad \delta_c = -\arctan\left(\frac{k_y}{k}\right) \quad \left(-\frac{\pi}{2} \leq \delta_c \leq 0\right)$$

$$R = e^{i\delta_R}, \quad \delta_R = -2 \arctan\left(\frac{k_y}{k}\right) \quad (-\pi \leq \delta_R \leq 0)$$

↳ Physical interpretation:

with sharp momentum $p=tk$

$$\overleftrightarrow{\partial_x} = \overrightarrow{\partial_x} - \overleftarrow{\partial_x}$$

* $e^{ikx} \Theta(-x)$: incoming wave coming from left with probability current $j_{in} = \frac{t}{2im} e^{-ikx} \overleftrightarrow{\partial_x} e^{ikx} = \frac{tk}{m}$

* $Re^{-ikx} \Theta(x)$: totally reflected wave going left with probability current $j_{refl} = \frac{t(R)^2}{2im} e^{-ikx} \overleftrightarrow{\partial_x} e^{-ikx} = -\frac{tk}{m} \underbrace{|R|^2}_{=1}$

* $C e^{-kx} \Theta(x)$: wave function penetrating into the classically forbidden zone $x>0$ with probability current $j_{penet} = \frac{t|C|^2}{2im} e^{-kx} \overleftrightarrow{\partial_x} e^{-kx} = 0$

Quantum effect!

Does not exist classically.

Note: The reflected wave (as well as the penetrating wave)

obtains an additional phase shift $e^{i\delta_R}$ ($e^{i\delta_c}$) which can be

Quantum effect!
Does not exist classically.

Note: The reflected wave (as well as the penetrating wave) obtains an additional phase shift $e^{i\theta_R}$ ($e^{i\theta_C}$) which can be physically observed by a time delay of the reflected particle.

Important: This physical interpretation is further supported by the fact that the interference between incoming and reflected waves vanishes:

$$\begin{aligned} j_{x \rightarrow 0} &= \frac{\hbar}{2im} [(e^{-ikx} + R^* e^{ikx})(ik)(e^{ikx} - Re^{-ikx}) - (e^{-ikx} - R^* e^{ikx})(-ik)(e^{ikx} + Re^{-ikx})] \\ &= \frac{\hbar}{2im} [ik(1 - |R|^2 - Re^{-2ikx} + R^* e^{2ikx}) + ik(1 - |R|^2 - R^* e^{-2ikx} + Re^{-2ikx})] \\ &= \frac{\hbar k}{m} (1 - |R|^2) \quad \leftarrow \text{The net current } j_{x \rightarrow 0} = 0, \text{ due to } |R| = 1. \end{aligned}$$

\hookrightarrow Limit of infinitely high potential wall ($V_0 \rightarrow \infty$): $k \rightarrow \infty$, $C \rightarrow 0$, $R \rightarrow -1 = e^{-i\pi}$

↑
phase shift of reflected wave!
(also for $V_0 < \infty$)

(2) Right-moving incoming particle with $E(k) > V_0$:

$$\phi_k(x) = \begin{cases} e^{ikx} + R e^{-ikx}, & x \leq 0 \\ T e^{ikx}, & x > 0 \end{cases} \quad \text{with } k > \frac{\sqrt{2mV_0}}{\hbar} \\ k' = \sqrt{k^2 - \frac{2mV_0}{\hbar^2}} < k$$

Continuity of $\phi_k(x)$ and $\phi_{k'}(x)$ at $x=0$ gives: $1+R=T$ and $ik(1-R)=ik'T$

$$\Rightarrow T = \frac{2k}{k+k'}, \quad R = \frac{k-k'}{k+k'} \quad \text{with } 1 < T < 2 \quad \text{and } 0 < R < 1$$

05/01/2016 (3)

Physical interpretation:

with sharp momentum $p = \hbar k$

* $e^{ikx} \theta(-x)$: incoming wave coming from left as in (1) with $v_{\text{in}} = \frac{\hbar k}{m}$

* $R e^{-ikx} \theta(-x)$: partially reflected wave going left with $v_{\text{refl}} = -|R|^2 \frac{\hbar k}{m}$ ← Quantum effect!
Does not exist classically.

* $T e^{ik'x} \theta(x)$: transmitted wave going right with $v_{\text{trans}} = |T|^2 \frac{\hbar k'}{m}$

with sharp momentum $p' = \hbar k'$

↳ Probability that incoming particle is reflected: $|v_{\text{refl}}| = |R|^2$
 Probability that incoming particle is transmitted: $|v_{\text{trans}}| = |T|^2 \frac{k'}{k}$

We indeed have $|R|^2 + |T|^2 \frac{k'}{k} = 1$:
 } PROBABILITY CONSERVATION
 } PARTICLE NUMBER CONSERVATION

↳ Limit of infinite energy ($E \gg V_0$): $k' \rightarrow k$, $R \rightarrow 0$, $T \rightarrow 1$

(3) Left-moving incoming particle with $E(k) > V_0$: \rightarrow only $k > \frac{\sqrt{2mV_0}}{\hbar}$ is possible

$$\phi_k^3(x) = \begin{cases} T' e^{-ikx} & x \leq 0 \\ e^{-ik'x} + R' e^{ik'x} & x > 0 \end{cases} \quad \text{with } k > \frac{\sqrt{2mV_0}}{\hbar}$$

$$k' = \sqrt{k^2 - \frac{2mV_0}{\hbar^2}} < k$$

Continuity of $\phi_L(x)$ and $\phi_R(x)$ at $x=0$ gives: $1 + R' = T'$ and $ik'(E + R') = -ikT$

$$1 e^{-ikx} + R e^{ikx}, x > 0$$

$$k' = \sqrt{k^2 - \frac{2mV_0}{\hbar^2}} < k$$

Continuity of $\phi_k(x)$ and $\phi'_k(x)$ at $x=0$ gives: $1+R'=T'$ and $ik'(-1+R') = -ikT$

$$\Rightarrow T' = \frac{2k}{k+k'}, R' = \frac{k-k'}{k+k'}$$

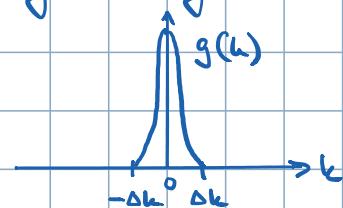
Exactly the same result for the reflected and the transmitted wave amplitudes as for case (2)!

→ Construction of scattered wave packets:

- * Starts at $x=x_0$ at $t=0$

- * has group velocity $v_g = \frac{p_0}{m}$ (right-moving incoming)

↳ We take a superposition of ϕ_k states with a strongly peaked wave number function real-valued $g(k)$ shifted to $k_0 = p_0/\hbar = mV_0/\hbar$ and an additional phase such that the wave packet is located at x_0 at $t=0$ ($x_0 < 0$).



$$\Psi(x, t) = \int_{-\infty}^{+\infty} dk g(k-k_0) e^{-ikx_0} e^{-i\omega(k)t} \phi_k(x), \quad \omega(k) = \frac{E(k)}{\hbar} = \frac{\hbar^2 k^2}{2m}$$

phase shifting packet at x_0 at $t=0$

Case $E(k) < V_0$: We have to take $\phi_k(x)$ from case (1)

$$\phi_k(x) = \begin{cases} e^{ikx} + R e^{-ikx}, & x \leq 0 \\ C e^{-kx}, & x > 0 \end{cases}$$

with $0 < k < \frac{\sqrt{2mV_0}}{\hbar}$

$$k = \sqrt{\frac{2mV_0}{\hbar^2} - k^2} > 0$$

$$C = \frac{2k}{\sqrt{k^2 + k^2}}, \quad \delta_c = -\arctan\left(\frac{k_0}{k}\right)$$

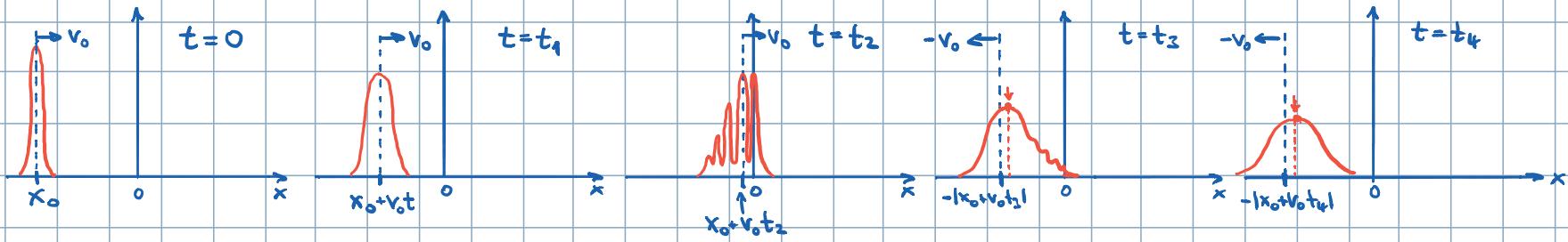
$$\left(-\frac{\pi}{2} \leq \delta_c \leq 0 \right)$$

velocity
 \downarrow
 $v_g = \frac{p_0}{m} = \frac{\hbar k_0}{m}$

$$R = e^{i\delta_R}, \quad \delta_R = -2 \arctan\left(\frac{k_0}{k}\right)$$

$$\left(-\pi \leq \delta_R \leq 0 \right)$$

05/01/2016 (4)



$t=0$: Wave packet located at x_0 , moves right with $v_0 = \frac{p_0}{m}$

$t=t_1$: Wave packet broadens, located at $x_0 + v_0 t_1$.

$t=t_2$: Packet hits potential barrier, penetration into zone $x > 0$, continued broadening.

Packet pauses a bit at barrier due to k -dependence of complex phases δ_R and δ_C .

$t=t_3$: Reflected wave moves left with $v_0 = \frac{p_0}{m}$, further broadening. Wave is a bit behind the classic reflected particle due to k -dependence of complex phase δ_R . \rightarrow "phase shift".

$t=t_4$: Further broadening, phase shift remains constant

Case $E(k) > V_0$: We have to take $\phi_k(x)$ from case (2)

$$\phi_k(x) = \begin{cases} e^{ikx} + R e^{-ikx}, & x \leq 0 \\ T e^{ik'x}, & x > 0 \end{cases}$$

with $k > \frac{\sqrt{2mV_0}}{\hbar}$

$k' = \sqrt{k^2 - \frac{2mV_0}{\hbar^2}}$

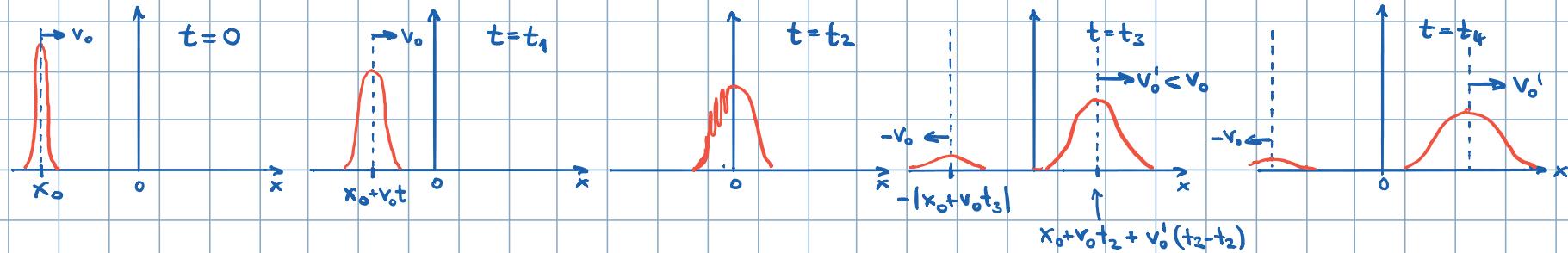
$$, V_0 = \frac{p_0}{m} = \frac{\hbar k_0}{m}, V'_0 = \frac{\hbar k'_0}{m} < V_0$$

$$\phi_w(x) = \begin{cases} e^{ikx} + R e^{-ikx}, & x \leq 0 \\ T e^{ik'x}, & x > 0 \end{cases}$$

with $k > \frac{\sqrt{2mV_0}}{\hbar}$, $V_0 = \frac{P_0}{m} = \frac{\hbar k_0}{m}$, $V'_0 = \frac{\hbar k'_0}{m} < V_0$

$$k' = \sqrt{k^2 - \frac{2mV_0}{\hbar^2}} < k$$

$$T = \frac{2k}{k+k'}, \quad R = \frac{k-k'}{k+k'} \quad \text{with } 1 < T < 2 \text{ and } 0 < R < 1, \quad T, R \in \mathbb{R}!$$



$t=0$: Wave packet located at x_0 , moves right with $v_0 = \frac{P_0}{m}$

$t=t_1$: Wave packet broadens, located at $x_0 + v_0 t_1$.

$t=t_2$: Packet hits potential barrier, continued broadening, transmitted and reflected wave packets separate.

$t=t_3$: Reflected wave moves left with $-v_0 = \frac{-P_0}{m}$, further broadening. \leftarrow bounced back without delay
 Transmitted wave moves right with $v'_0 = \frac{\hbar k'_0}{m} < v_0$, further broadening \leftarrow follows classic motion

$t=t_4$: Further broadening.

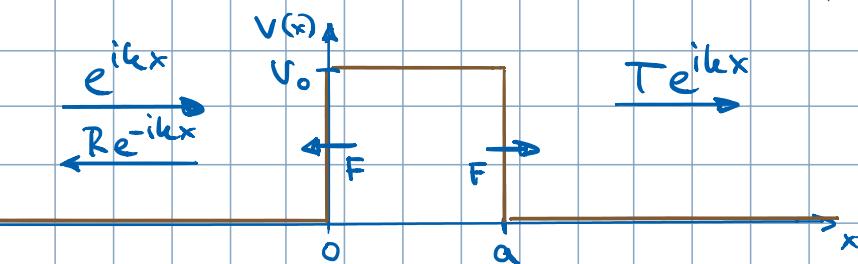
No Phase Shifts!

05/01/2016 (5)

3.4. Potential Wall

→ Consider a particle experiencing a force related to a potential wall of the form

$$V(x) = V_0 \Theta(x) \Theta(a-x) = \begin{cases} 0, & x < 0 \text{ or } x > a \\ V_0, & 0 \leq x \leq a \end{cases}, \quad V_0 > 0$$



From our considerations for the potential step we expect that an incoming particle with $E < V_0$ (coming from the left) can penetrate into the classically forbidden zone $0 \leq x \leq a$ and with some finite probability emerge for $x > a$.

→ We only consider energy eigenfunctions representing incoming particles from the left

(1) Right-moving incoming particle with $E(h) < V_0$:

$$\Phi_L(x) = \begin{cases} e^{ikx} + R e^{-ikx}, & x < 0 \\ A e^{kx} + B e^{-kx}, & 0 \leq x \leq a \\ T e^{ikx}, & x > a \end{cases} \quad \text{with } 0 < k < \frac{\sqrt{2mV_0}}{\hbar}$$

$$k = \sqrt{\frac{2mV_0}{\hbar^2}} > 0$$

$$\phi_k(x) = \begin{cases} e^{ikx} + R e^{-ikx}, & x \leq 0 \\ A e^{ikx} + B e^{-ikx}, & 0 \leq x \leq a \\ T e^{ikx}, & x > a \end{cases} \quad \text{with } 0 < k < \frac{\sqrt{2mV_0}}{\hbar} \quad k = \sqrt{\frac{2mV_0}{\hbar^2} - k^2} > 0$$

We have:

- * partial reflection towards $x < 0$, phase shift
- * penetration into classically forbidden zone $0 \leq x \leq a$
- * classically forbidden transmission into $x > a$

} "tunnel effect"
 → quantum effect that does not exist classically

(2) Right-moving incoming particle with $E(k) > V_0$:

$$\phi_k(x) = \begin{cases} e^{ikx} + R e^{-ikx}, & x \leq 0 \\ A e^{ikx} + B e^{-ikx}, & 0 \leq x \leq a \\ T e^{ikx}, & x > a \end{cases} \quad \text{with } k > \frac{\sqrt{2mV_0}}{\hbar} \quad k' = \sqrt{k^2 - \frac{2mV_0}{\hbar^2}} < k$$

We have:

- * partial reflection towards $x < 0$ ← quantum effect
- * transmission into $x > 0$, like classic particle

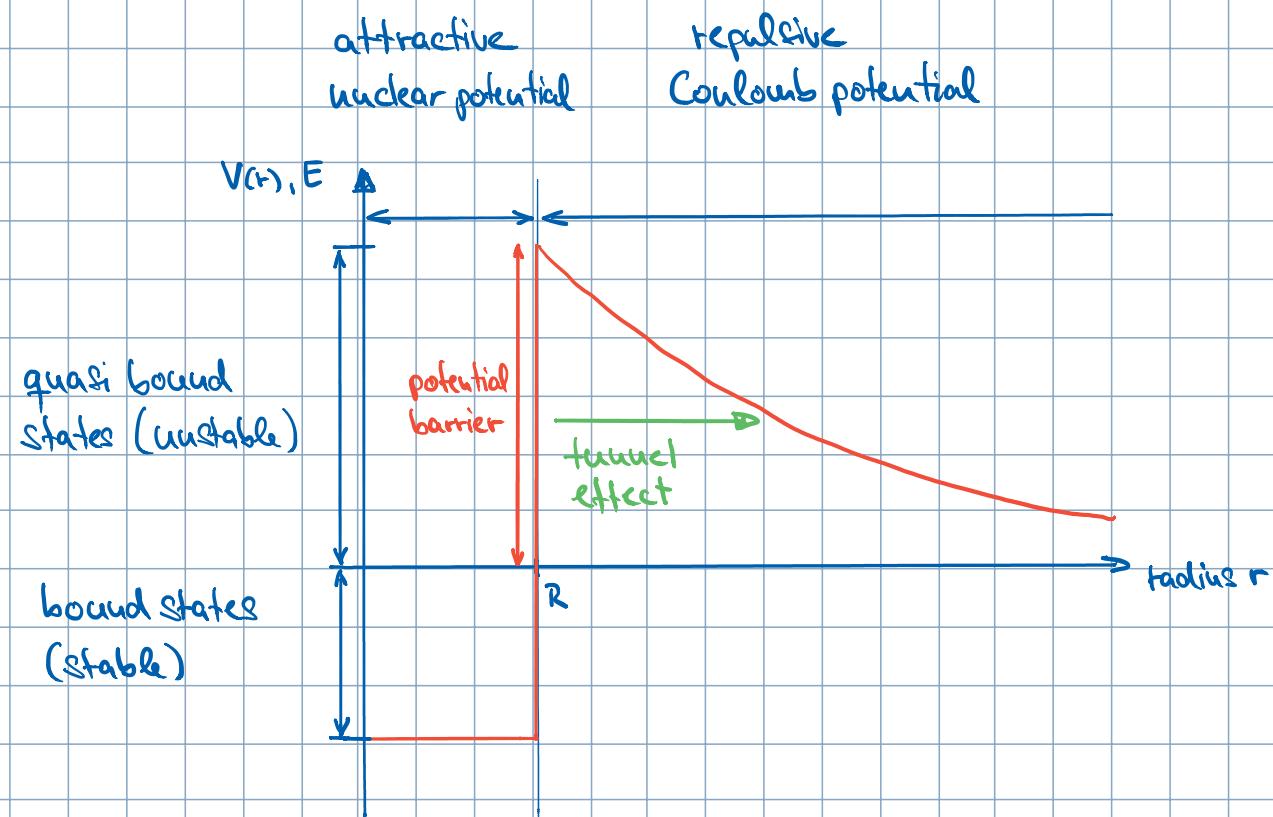
→ Nature's application of the tunnel effect: **α -Decay** → α -particle = ${}^4\text{He}$ nucleus = 

↳ Heavy nuclei are unstable and can decay by the emission of α -particles

Mechanism: Inside the nucleus ($R \lesssim 10^{-15} \text{ m } \text{Å}^{1/2}$) the nuclear force provides an attractive potential for the α -particle generated by the other protons and neutrons.

Outside the nucleus $r > R$ the nuclear force is screened and the Coulomb-force provides a repulsive potential $V(r) \sim \frac{2(z-2)e^2}{r}$ for the α -particle.

05/01/2016 (6)



$^{232}_{90}\text{Th}$ (Thorium)

$$E_{\alpha} \approx 4.0 \text{ MeV}$$

$$T_{1/2} \approx 1.4 \cdot 10^{10} \text{ a}$$

$^{226}_{88}\text{Ra}$ (Radium)

$$E_{\alpha} = 4.8 \text{ MeV}$$

$$T_{1/2} \approx 1600 \text{ a}$$

$^{212}_{84}\text{Po}$ (Polonium)

$$E_{\alpha} \approx 8.8 \text{ MeV}$$

$$T_{1/2} \approx 3 \cdot 10^{-7} \text{ s}$$

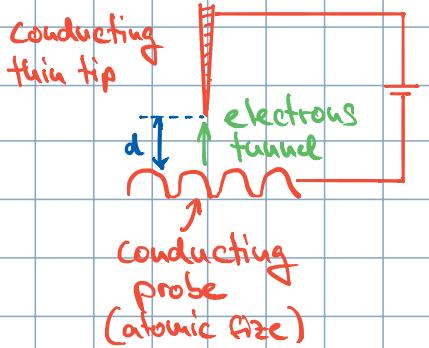
^{84}Po (Polonium)

$E_\alpha \approx 8.8$ MeV

$T_{1/2} \approx 5 \cdot 10^5$ s

→ Physical application of the tunnel effect: Scanning Tunneling Microscopy ("Raster-tunnelmikroskopie")

↳ Nobel prize 1986 for Gerd Binnig and Heinrich Rohrer



Tunnel current depends exponentially on distance d .
One can picture the form of the conducting surface
by scanning the surface (by moving the tip).