Chapter 2: Elementary Aspects of Quantum Mechanics

We discuss some of the main concepts for the quantum mechanical description of spinless and non-relativistic particles, and the basics of the relevant mathematics.

2.1. The Wave Function

= A particle can be in (infinitely) many states. If the particle is in a quantum mechanically unique state, we say that the particle is in a pure state.

Classical mechanics: A particle in a pure state at time \( t \) is completely determined by quoting its position \( \vec{r} \) and its velocity \( \vec{v} \) at that time \( t \). In a measurement of the position of the particle in this pure state, one will always (with 100% probability) obtain \( \vec{r} \).

Quantum mechanics: A particle in a pure state at some time \( t \) is completely determined by quoting its time-independent wave function \( \psi(\vec{r}, t) \), which satisfies the time-independent Schrödinger equation:

\[ \frac{\partial}{\partial t} \left( \frac{\psi}{\psi^*} \right) = \frac{i}{\hbar} \left( \frac{\partial^2}{\partial x^2} \psi \right) \]

This is the equation that provides the probability to find the particle anywhere in \( \mathbb{R}^3 \) at a given moment of time.

The integral \( \int \psi^* \psi d^3 \vec{r} \) over any area \( S < \mathbb{R}^3 \) gives the probability to find the particle anywhere in the area \( S \).

\[ \int \psi^* \psi d^3 \vec{r} = \int \psi^2 d^3 \vec{r} \quad : \text{probability density for the location of the particle in a location measurement} \]

= We can consider the wave function as a probability amplitude for a particle in the state \( |\psi_0\rangle \) to end up in a pure state located at \( \vec{r} \), i.e.,

\[ \int |\psi(\vec{r}, t)|^2 d^3 \vec{r} = \langle \psi(\vec{r}, t) | \psi(\vec{r}, t) \rangle = 1 \]

= There will be much more to this later...

Experimental realization: probabilistic treatment using a large number \( N \) of equivalently prepared copies of the system being in state \( |\psi_0\rangle \) (or, described by wavelet \( \psi(\vec{r}, t) \)).

1. \( \text{Detector } D_1 \) : signal if particle location happens to be in \( V \)

2. \( \text{Detector } D_2 \) : signal if particle location happens to be in \( V - \mathbb{R}^3 \backslash V \)

= Exactly one of the detectors will always give a signal \( \Rightarrow N_1 + N_2 = N \)

= In the limit \( N \to \infty \) we find

\[ \lim_{N \to \infty} \frac{N_1}{N} = \psi_1^2 \quad \text{(expectation value)} \]

For finite \( N \) we have statistical fluctuations and find \( \frac{N_1}{N} \) to be close to \( \psi_1^2 \) following the gaussian statistical law.

= Indeterministic aspects: For a single copy one cannot predict whether \( D_1 \) or \( D_2 \) will give a signal.

Deterministic aspects: One can give the probabilities for getting a signal from \( D_1 \) and \( D_2 \).
2.2 The Position Measurement

- **Collapse of the state** (also "reduction of the wave packet").

  If detector $D$ gives a signal, the particle is found in area $V$ with 100% probability. While the wave function of the particle, say $\psi(x)$, before that measurement, it changes (collapses) to the new function $\psi'(x) = \frac{\psi(x) \chi(x)}{\sqrt{\int \chi^2(x) dx}}$ after the measurement, where

  \[ \chi(x) = \begin{cases} 1 & x \in V \\ 0 & x \notin V \end{cases} \]

- **Linear operator**: The operation "multiply wave function $\psi(x)$ with $\chi(x)$" is a linear operator acting on wave functions.

  \[ O(\psi(x)) = a(\psi(x)) \]

  An operator $O$ acting on wave fields is linear if

  \[ O(a \phi(x) + b \psi(x)) = a O(\phi(x)) + b O(\psi(x)) \]

  for all wave fields $\phi(x)$ and $\psi(x)$, and $a, b \in \mathbb{C}$.

- **Eigenfunctions**

  A wave field $\phi(x)$ is called an eigenfunction of the linear operator $O$ if

  \[ O(\phi(x)) = \lambda \phi(x) \]

  The set of eigenvalues of a linear operator $O$ is called the spectrum of $O$.

- **Eigenvalues and Eigenfunctions**

  \[ O(\phi(x)) = \lambda \phi(x) \]

  Eigenfunctions:

  - $\lambda = 0$: all wave fields for which $\phi(x) = 0$ for all $x \notin V$
  - $\lambda = 1$: all wave fields for which $\phi(x) = 0$ for all $x \notin V$

  The operator represents the measurement "Is the particle in the area $V$?" which has the possible values yes (1) and no (0).

  The corresponding normalized eigenfunctions are the wave functions for which the particle is 100% inside the area $V$ (1) and outside the area $V$ (0), respectively.

- **Expectation Value**

  The integral $\int (\chi(x)^* O(\chi) \phi(x) dx$ is called the expectation value of the linear operator $O$ in the state described by the wave function $\phi(x)$. It gives the average of the measurement related to the operator $O$.

  \[ \langle \phi | O | \phi \rangle \]

  The expectation value of the operator "multiply wave function $\psi(x)$ with $\chi(x)$" gives the average of the measurement "Is the particle in the area $V$?" which is just $\langle \psi | (\psi \chi) \rangle$.
2.3. The Hilbert Space

The space of complex-valued and square-integrable functions on $\mathbb{R}^3$

$$L^2(\mathbb{R}^3) = \{ \psi: \mathbb{R}^3 \to \mathbb{C} | \int |\psi(x)|^2 \, dx < \infty \}$$

defines an infinite-dimensional vector space over $\mathbb{C}$. $L^2(\mathbb{R}^3)$ is very useful for the quantum mechanical description of a particle in an "infinite square box". Those elements $\psi \in L^2(\mathbb{R}^3)$ for which $\int |\psi(x)|^2 \, dx = 1$ are the wave functions.

One can supplement the inner product $\langle \psi, \phi \rangle$ with the concept of a (complex-valued) scalar product defined by

$$\langle \psi, \phi \rangle = \int \overline{\psi(x)} \cdot \phi(x) \, dx$$

with $\psi, \phi \in L^2(\mathbb{R}^3)$ internal. Defined according to the following properties:

(S1) $\langle \psi, \alpha \psi \rangle = \alpha^2 \langle \psi, \psi \rangle$ for all $\alpha \in \mathbb{C}$, $\psi \in L^2(\mathbb{R}^3)$

(S2) $\langle \psi, \phi + \chi \rangle = \langle \psi, \phi \rangle + \langle \psi, \chi \rangle$ for all $\phi, \chi \in L^2(\mathbb{R}^3)$

(S3) $\langle \psi, \psi \rangle \geq 0$ and $\langle \psi, \psi \rangle = 0$ if and only if $\psi = 0$ for all $\psi \in L^2(\mathbb{R}^3)$

Mathematically, $L^2(\mathbb{R}^3)$ supplemented with the scalar product is a complete inner product space. The scalar product provides the notion of the length ("norm") of a element of the vector space ("vector") and an "angle" between two vectors.

$L^2(\mathbb{R}^3)$ has the property that it is complete with respect to the norm defined

$$||\psi|| = \sqrt{\langle \psi, \psi \rangle}$$

for all $\psi \in L^2(\mathbb{R}^3)$.

Completeness means that each Cauchy sequence $\{\psi_n \}_{n=1}^{\infty}$ for $\psi_n \in L^2(\mathbb{R}^3)$ has a limit for $n \to \infty$ and is also contained inside $L^2(\mathbb{R}^3)$, i.e. $\lim_{n \to \infty} \psi_n = \psi \in L^2(\mathbb{R}^3)$. (Remember Cauchy sequence: For every given real $\varepsilon$, there is an integer $N$ such that $||\psi_n - \psi_m|| < \varepsilon$ for $n,m > N$.)
2.4. Linear Operators

→ Let \( \mathcal{H} \) be a Hilbert space (and we may have in mind \( \mathcal{H} = L^2(\mathbb{R}^n) \)).
The following mathematical considerations are essential for quantum mechanics.

→ Linear operator: An operator \( A : \mathcal{H} \to \mathcal{H} \) that assigns each vector \( \psi \in \mathcal{H} \) the vector \( A\psi \in \mathcal{H} \) is a linear operator, if
\[
A(c\psi_1 + c\psi_2) = cA\psi_1 + cA\psi_2 \quad \forall \psi_1, \psi_2 \in \mathcal{H} \quad \text{and} \quad c \in \mathbb{C}
\]

→ Adjoint operator: The operator \( A^* \) is called the adjoint operator to the linear operator \( A \), if (and only if)
\[
\langle A\psi | \phi \rangle = \langle \psi | A^* \phi \rangle \quad \forall \psi, \phi \in \mathcal{H}
\]

\( A^* \) is called the "dagger" of \( A \).

→ The following rules apply:
\[
\begin{align*}
(A + B)^* &= A^* + B^* \\
(A^*)^* &= A \\
(CA)^* &= C^*A^*
\end{align*}
\]

→ Hermitian (or self-adjoint) operator: A linear operator \( A \) is called Hermitian, if \( A = A^* \).

→ The operator \( \delta(A, \phi) = \phi, \psi \) from 2.2 is Hermitian.

→ The multiplication operator \( F : (\mathcal{H})^2 \to \mathcal{H} \) is Hermitian, if \( \langle \phi | \psi \rangle \) is Hermitian, i.e., as long as \( \langle \phi | \psi \rangle = \langle \psi | \phi \rangle \).

→ Unitary operator: A linear operator is called unitary if \( A^*A = AA^* = 1 \).

→ Eigenvalues and eigenvectors: A \( \psi \in \mathcal{H} \) is called eigenvector to the linear operator \( A \) with the eigenvalue \( \alpha \), if \( \alpha \) and \( \psi \) have the property
\[
A\psi = \alpha \psi \quad \text{for an} \quad \alpha \in \mathbb{C}
\]

→ The set of eigenvalues of a linear operator \( A \) is called the spectrum of \( A \).

→ Eigenvalues of Hermitian operators are real.

→ Orthogonal system (OVS): A set of vectors \( \psi_i \in \mathcal{H} \) \( (i = 1, \ldots, n) \) for which \( \langle \psi_i | \psi_j \rangle = \delta_{ij} \) for all \( i, j \in \{1, \ldots, n\} \)
Completeness of $\mathbb{C}$: Each vector $v \in \mathbb{C}$ can be written as a linear combination of a minimal set of basis vectors \{1, i\}, i.e., there is a unique set of complex numbers $c_1, c_i$ so that \[ v = c_1 \cdot 1 + c_i \cdot i. \]

If a Hermitian operator has several linearly independent eigenvectors in the same eigenvalue (degenerate eigenvalue), one can always create an orthonormal basis to the set of eigenvectors of that eigenvalue.

Spectral Theorem: For each Hermitian operator $A$ with a discrete spectrum, there is a complete orthonormal basis in $\mathbb{C}^n$ made of eigenvectors \{1, i\} to $A$, i.e., $A\xi_n = \lambda_n \xi_n$ for some $\lambda_n \in \mathbb{R}$.

Proof: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The spectral theorem is a highly non-trivial statement which says that the (infinite-dimensional) linear operator $A$ can be made diagonal in a basis that consists of eigenvectors.

For a general linear operator only a block-diagonal form can be achieved.

Alternative wording: Complete orthonormal system (CNS).

Correspondence Principle: Each observable quantity ("observable") in quantum mechanics is represented by a Hermitian operator.

The set of eigenvalues (spectrum) of the Hermitian operator associated to an observable in quantum mechanics is the set of measurable values of the observable.

Projection operator: A Hermitian operator $P$ which also satisfies $P^2 = P = P^*$ is called a projection operator.

The operator \(\Pi = \mathbb{1} - P\) from 2.2. (measurement: "Is the particle at \(y_0 = 0\)?") is a projection operator.

The spectrum of $P$ is \([0, 1]\) and the scalar eigenvalues for both operators are both non-negative. For eigenvalues \(0 \leq \lambda_i \leq 1\), the vectors $\xi_i$ are mutually orthogonal.

Completeness relation: Let \(\{\xi_n, \bar{\xi}_n\}\) be a complete orthonormal system (CNS) of $L^2(\mathbb{R}^3)$. We then have

\[ \sum \xi_n \bar{\xi}_n = \mathbb{1}_{L^2(\mathbb{R}^3)} \]

which we can write as

\[ \sum \xi_n \bar{\xi}_n = \mathbb{1}_{L^2(\mathbb{R}^3)} \]

Completeness relation.

The Dirac $\delta$-function is not an element of $L^2(\mathbb{R})$. It is defined as a functional acting on the functions in $L^2(\mathbb{R})$. $\delta = \delta(x)$ with $\int_{\mathbb{R}} \delta(x) f(x) dx = f(0)$, \(f \in L^2(\mathbb{R})\).

However, it is possible to define it within $L^2(\mathbb{R}^)$ through a sequence $\delta_n \in L^2(\mathbb{R})$ such that the functional defined through $\delta_n$ satisfies $f(0) = \lim_{\delta_n \to \delta} f(0)$ for any function $f \in L^2(\mathbb{R})$.

Such a series can then be taken as a definition for the $\delta$-function. The choice of the limit $\delta_n$ does not need to be unique.

Example: $\delta_n(x) = \frac{1}{\sqrt{\pi n^2}} \exp(-x^2/\pi^2)$, $n = 0, 1, 2, \ldots$.
2.5. Scalar Product and Expectation Values

Consider a physical system in a pure state described by the ket vector $\Psi$ and an observable described by the Hermitian operator $A$ which has $N$ states of eigenvalues $a, a_1, \ldots$ with corresponding eigenstates $\Phi_a, \Phi_{a_1}, \ldots$

So we can "expand" $\Psi$ in the $\Phi_a$ (i.e. write as a linear combination of the $\Phi_a$) as

$$\Psi = \sum_a \alpha_a \langle \Psi | \Phi_a \rangle$$

In this section we will discuss the probability to measure a certain value and the mean value obtained in a large number of measurements made on the state $\Psi$ by applying the Hilbert space formulation.

- If the eigenvalues $a, a_1, \ldots$ are equal, e.g. $a = a_1 = \cdots$ ("degenerate eigenvalue"), then the probability to obtain in a measurement of $A$ the value $a$ is $|\langle \Psi | \Phi_a \rangle|^2$.

- The complex-valued $|\langle \Psi | \Phi_a \rangle|^2$ the corresponding probability amplitude. The state $\Psi$ is completely and uniquely characterized by the knowledge of the probability amplitudes $|\langle \Psi | \Phi_a \rangle|^2$, see Eq. (5).

- If some of the eigenvalues $a, a_1, \ldots$ are equal, e.g. $a = a_1 = \cdots$ ("degenerate eigenvalue"), then the probability to obtain in a measurement of $A$ the value $a$ is

$$|\langle \Psi | \Phi_a \rangle|^2 + |\langle \Psi | \Phi_{a_1} \rangle|^2 + \cdots$$

- The relation $\sum_a |\langle \Psi | \Phi_a \rangle|^2 = \sum_{\Phi} \langle \Psi | \Phi \rangle \langle \Phi | \Psi \rangle = \langle \Psi | \Psi \rangle = 1$ means that the probability of obtaining in a measurement any of the values $a, a_1, \ldots$ is unity, which is directly connected to the norm of the state vector being unity.

- The average of the measured values of the observable $A$ (with a discrete spectrum) is given by

$$\sum_a |\langle \Psi | \Phi_a \rangle|^2 a = \sum_a \alpha_a \langle \Psi | \Phi_a \rangle \langle \Phi_a | \Phi_a \rangle = \sum_a \alpha_a \langle \Phi_a | A \Phi_a \rangle = \sum_a \alpha_a \langle \Psi | \Phi_a | A \Phi_a \rangle = \langle \Psi | A \Psi \rangle$$

and is called the expectation value of the observable $A$.

- For the case of an observable operator with a continuous spectrum there are some modifications.

Example: Location operator in 1 dimension: $X \Psi(x) = x \Psi(x)$.

2. Eigenfunctions: $\delta(x) = \delta(x-y)$, $x \delta(x) = x \delta(x)$ for all $x \in \mathbb{R}$

Problem: $\delta$ are not normalizable. Since they are distributions and thus not directly contained in $L^2(\mathbb{R})$

- However, they are orthogonal in the sense $\langle \delta | \delta \rangle = \int \delta(x) \delta(x') \, dx \, dx' = -\delta(x-y)$

- and therefore form a complete orthogonal basis because each function $\psi \in L^2(\mathbb{R})$ can be written as a linear combination of the $\delta$

$$\psi(x) = \int \delta(x-y) \psi(y) \, dy$$

where

$$\psi(0) = \int \delta(x) \psi(x) \, dx$$

$$\psi(x) = \int \delta(x-y) \psi(y) \, dy - \int \delta(x) \psi(x) \, dx$$

and

$$\psi(x) = \int \delta(x-y) \psi(y) \, dy - \int \delta(x) \psi(x) \, dx$$

are

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It is very convenient to write \( \langle x | H | x \rangle = \langle x | x \rangle \), so that \( H \equiv \langle x | x \rangle \).

We see that \( H(x) = \{ \langle x | x \rangle \} \) is the location probability density of the state, so \( \langle x \rangle \equiv \langle x | x \rangle \) is the probability to location the location of the system (e.g. particle) in the interval \( [a, b] \). We usually interpret this as the interval in which the particle is detected.

The average location of the particle is the expectation value \( \langle x | x \rangle = \int dx \langle x | x \rangle = \int dx \langle x | x \rangle = \int dx \langle x | x \rangle \).

Now consider a mathematical point of view. \( x \) is an example for an unbounded operator, which means that there is at \( x = L^2(\mathbb{R}) \) for which \( x^* \notin L^2(\mathbb{R}) \).

Formally one calls on operator \( A : \mathbb{R} \rightarrow \mathbb{R} \) a bounded operator if there is a real \( C > 0 \), such that \( \| A(x) \| \leq C \| x \| \) for all \( x \in \mathbb{R} \). One can then define the norm of the operator \( A \) as \( \| A \| = \sup_{x \neq 0} \frac{\| A(x) \|}{\| x \|} \), which is finite.

Mathematically, the issue of an operator being unbounded is directly connected with its spectrum containing a continuous interval of eigenvalues, which is get directly connected that \( \delta \)-functions are involved for the eigenfunctions. In analogy to the \( \delta \)-function, which one can define (formally) within \( L^2(\mathbb{R}) \) using the concepts of "sequence" and functional, one can also generalize the concepts of Hermitian bounded operators (for which we return most of the unbounded walk in introducing quantum physics courses) to Hermitian unbounded operators (such as the \( \hat{X} \) and \( \hat{P} \)-operators, which are actually used in quantum physics much of the time).

This formalism is quite involved but turns out to be of little practical use in actual applications of quantum mechanics because the outcome is that the basic calculational rules for bounded and unbounded operators are essentially the same. Once one deals with \( \delta \)-functions, the regular functions generalized sums to integrals and consider the existence of which disappear and for continuous spectra.

We will therefore not mathematically dwell on unbounded operators any deeper.


Educational exercises:

(a) Show that an operator which has a spectrum containing a continuous interval is indeed unbounded.

(b) One can generalize the concept of the "spectrum" from bounded to unbounded operators by defining the spectrum of an operator \( A \) by the set of values \( \lambda \in \mathbb{C} \) for which the operator \( R_A(\lambda) = \frac{1}{\lambda - A} \) is unbounded.

Show that the spectrum of \( x \) is indeed \( \mathbb{R} \).

(c) Show that the spectrum of the momentum operator \( \hat{P} \in \mathbb{R} \).
2.6. The Momentum

From classical mechanics we know that there is a connection between spatial translations and the momentum of a physical system. We use this connection to motivate the form of the momentum operator in quantum mechanics.

So we consider a 4-dimensional system for simplicity: unit of time, 4-position, 4-momentum.

Translation operator $T_a$. $T_a$ shift a wave function by distance $a$, so

$$(T_a \psi)(x) = \psi(x-a).$$

So $T_a$ is a unitary (unit shift) operator.

$L$ infinitesimal translation $T_a \psi(x) = \psi(x-a) = \psi(x-a + \frac{a}{2} \frac{\partial}{\partial x} + \frac{a}{2} \frac{\partial}{\partial x}) = \psi(x)$

We apply the $P$ operator to a de Broglie wave:

$$P \exp \left[ \frac{i}{\hbar} \left( p \cdot x - Et \right) \right] = \frac{\hbar}{i} \frac{\partial}{\partial p} \exp \left[ \frac{i}{\hbar} \left( p \cdot x - Et \right) \right] = P \exp \left[ \frac{i}{\hbar} \left( p \cdot x - Et \right) \right]$$

which should be an eigenfunction of the momentum operator.

Let $\Theta$ be a Hermitian operator:

$$\Theta = \int dx \frac{\hbar}{i} \frac{\partial}{\partial p} \exp \left[ \frac{i}{\hbar} \left( p \cdot x - Et \right) \right]$$

The quantum mechanical operator representation of the momentum $\Theta$ of a system is

$$\Theta = \frac{i}{\hbar} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right)$$

Eigenfunctions and eigenvalues: $P \psi(x) = \frac{\hbar}{i} \frac{\partial}{\partial p} \psi(x)$ with $p \in \mathbb{R}$

Gives the unique solution $\psi_p(x) = \frac{1}{\sqrt{2\pi}} e^{ipx}$

The momentum operator is unbounded, its spectrum is continuous, and its eigenfunctions $\psi_p$ cannot be normalized, but the $\psi_p$ are orthonormal to each other in the sense that

$$\langle \psi_q, \psi_p \rangle = \int dx \, \psi_q^*(x) \psi_p(x) = \int dx \, e^{-ipx} e^{iqx} = \int dx \, e^{iq-x} = \delta(q-p)$$

This is in analogy to the eigenfunctions of the location operator $X$.

We use the Fourier transformation formalism to write the wave function $\psi(x)$ as a superposition (which is the continuous generalization of the notion of linear combination) of the $\psi_p(x)$:

$$\psi(x) = \int dp \, \psi_p(x) \phi(p) = \int dp \, \frac{1}{\sqrt{2\pi}} e^{-ipx} \phi(p)$$
Because of $\langle \phi | p \phi \rangle = \langle \phi | <p | \phi \rangle$ we have

$$\Phi(p) = \langle \phi | p \phi \rangle = \frac{i}{\hbar} \int dx \frac{\partial}{\partial x} \Phi(x) \text{ (from wave function)}$$

4. We see $\Phi(p) = \langle \phi | p \phi \rangle$ is the probability amplitude for measuring the momentum $p$ in the state $\Phi$. 

4.1 $\Phi(p)^* \Phi(p)$: probability density for the measurement of the particle/system in a momentum inaccessible

4.2 $\Phi(p)^* dp$ : the probability to measure a momentum in the interval $[p, p+dp]$ in a momentum inaccessible

4.3 We call $\Phi(p) = \langle \phi | p \phi \rangle$ the momentum space wave function or momentum space probability amplitude

4.4 The knowledge of $\Phi(p) = \Phi(x)$ is physically equivalent to the knowledge of $\langle \phi | x \rangle = \psi(x)$. So we can interpret using $\langle \phi | p \rangle = \langle \phi | x \rangle$ simply as different representations of the same state $\Phi$, in the basics that we either use a momentum state basis or a configuration space basis.

4.5 We can even go one step further and consider $\langle \phi | p \rangle$, $\langle \phi | x \rangle$ as well as operators as abstract objects that do exist independent of the choice of representation.

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### Configuration space and momentum space representations:

4.6 $\Phi(x) = \langle \phi | x \rangle = \int \frac{dp}{\sqrt{2\pi \hbar}} \Phi(p) \langle \phi | p \rangle = \int \frac{dx}{\sqrt{2\pi \hbar}} \Phi(x) \langle \phi | x \rangle$

4.7 $\Phi(p) = \langle \phi | p \rangle = \int \frac{dx}{\sqrt{2\pi \hbar}} \Phi(x) \langle \phi | x \rangle$

4.8 $\Phi(x) = \int \frac{dp}{\sqrt{2\pi \hbar}} \Phi(p) \langle \phi | p \rangle = \int \frac{dx}{\sqrt{2\pi \hbar}} \Phi(x) \langle \phi | x \rangle$

4.9 $\Phi(p) = \int \frac{dx}{\sqrt{2\pi \hbar}} \Phi(x) \langle \phi | x \rangle$

4.10 $\Phi(x) = \int \frac{dp}{\sqrt{2\pi \hbar}} \Phi(p) \langle \phi | p \rangle$

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This is the abstract representation, free momentum operator.

4.11 We see $\langle \phi | p \phi \rangle = \int dp \langle \phi | p \rangle \langle p | \phi \rangle = \int dp \langle \phi | p \rangle \Phi(p) \langle \phi | p \rangle = \int dp \langle \phi | p \rangle \Phi(p) \langle \phi | p \rangle = \int dp \Phi(p) \langle \phi | p \rangle \langle p | \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \Phi(p)$$

4.12 So the momentum operator $\hat{p}$ is a simple multiplication operator in momentum space.

4.13 $\langle \phi | p \phi \rangle = \int dp \langle \phi | p \rangle \Phi(p) \langle \phi | p \rangle = \int dp \Phi(p) \langle \phi | p \rangle \langle p | \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \Phi(p)$$

4.14 $\langle \phi | p \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \langle p | \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \Phi(p)$$

4.15 $\langle \phi | p \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \langle p | \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \Phi(p)$$

4.16 So the momentum operator $\hat{p}$ is a simple multiplication operator in momentum space.

4.17 $\langle \phi | p \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \langle p | \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \Phi(p)$$

4.18 $\langle \phi | p \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \langle p | \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \Phi(p)$$

4.19 $\langle \phi | p \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \langle p | \phi \rangle = \int dp \Phi(p) \langle \phi | p \rangle \Phi(p)$$

4.20 So the momentum operator $\hat{p}$ is a simple multiplication operator in momentum space.
\[ [X, P] = XP - PX - i\hbar \frac{\partial}{\partial x} \]

1. Check in \( x \)-space:
\[ (\frac{\partial}{\partial x} - i\frac{\partial}{\partial p}) \psi_n = \frac{\partial}{\partial x} \psi_n - i \frac{\partial}{\partial p} \psi_n = i \hbar \psi_n \quad \text{for all } \psi_n \]

2. Check in \( p \)-space:
\[ (i\hbar \frac{\partial}{\partial x} - p) \psi_n = i \hbar \psi_n + i \hbar \psi_n - p \frac{\partial}{\partial p} \psi_n = i \hbar \psi_n \quad \text{for all } \psi_n \]

Generalization to 3 spatial dimensions:

\[ \Psi(\mathbf{x} + \mathbf{x}) = \Psi(\mathbf{x}) \exp \left[ -i \frac{\hbar}{m} \frac{\partial}{\partial \mathbf{x}} \right] \exp \left[ i \frac{\hbar}{m} \frac{\partial}{\partial \mathbf{x}} \right] \]

\[ \Psi(\mathbf{x'}) = e^{i \mathbf{k} \cdot \mathbf{x'}} \]

The momentum operator is:

\[ \hat{p} \psi(\mathbf{x}) = \int d^3 x \left( -i \hbar \nabla \right) \psi(\mathbf{x}) \]

\[ \langle \hat{p} \rangle = \frac{\int d^3 x \hat{p} \psi(\mathbf{x}) \overline{\psi(\mathbf{x})}}{\int d^3 x \overline{\psi(\mathbf{x})} \psi(\mathbf{x})} \]

\[ \langle \hat{p} \rangle = \int d^3 x \left( -i \hbar \nabla \right) \psi(\mathbf{x}) \overline{\psi(\mathbf{x})} \]

\[ \langle \hat{p} \rangle = \int d^3 x \nabla \psi(\mathbf{x}) \overline{\psi(\mathbf{x})} \]

\[ \langle \hat{p} \rangle = \int d^3 x \mathbf{p} \psi(\mathbf{x}) \overline{\psi(\mathbf{x})} \]

\[ \langle \hat{p} \rangle = \int d^3 x \mathbf{p} \psi(\mathbf{x}) \overline{\psi(\mathbf{x})} \]

\[ \hat{X} \hat{P}_x = i \hbar \delta(x - \frac{\partial}{\partial x}) \quad \text{for different spatial dimensions commutes!} \]