

21. Massive vector field

$V_\mu(x)$ describes massive spin 1 field (vector boson)

$$\mathcal{L} = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{M^2}{2} V_\mu V^\mu$$

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \quad , \quad V_\mu \text{ real}$$

$$\Rightarrow \text{field equation } (\square + M^2) V^\mu - \partial^\mu \partial_\nu V^\nu = 0$$

$$\Rightarrow (\square + M^2) \partial_\mu V^\mu - \square \partial_\nu V^\nu = M^2 \partial_\mu V^\mu = 0$$

$$\stackrel{M \neq 0}{\Rightarrow} \partial_\mu V^\mu = 0 \quad (\text{projects out spin 0 component})$$

$$\boxed{\begin{aligned} (\square + M^2) V^\mu &= 0 \\ \partial_\mu V^\mu &= 0 \end{aligned}}$$

plane wave solutions :

$$\varepsilon^\mu e^{\pm i k x} \quad , \quad k^2 = M^2 \quad , \quad k_\mu \varepsilon^\mu = 0$$

rest frame: $k = \begin{pmatrix} M \\ \vec{0} \end{pmatrix} \Rightarrow \varepsilon = \begin{pmatrix} 0 \\ \vec{\varepsilon} \end{pmatrix}$

normalization $|\vec{\varepsilon}| = 1 \Rightarrow \varepsilon^2 = -1$

3 polarizations for given $\vec{k}$: $\varepsilon^\mu(k, \lambda)$, $\lambda = 1, 2, 3$

$$\varepsilon(k, 1) = \begin{pmatrix} 0 \\ \vec{\varepsilon}(k, 1) \end{pmatrix}, \quad \varepsilon(k, 2) = \begin{pmatrix} 0 \\ \vec{\varepsilon}(k, 2) \end{pmatrix}$$

$$\vec{\varepsilon}(k, 1) \cdot \vec{k} = \vec{\varepsilon}(k, 2) \cdot \vec{k} = 0 \quad \text{transversal}$$

$$\varepsilon(k, 3) = \frac{1}{M} \begin{pmatrix} |\vec{k}| \\ \frac{k^\mu \vec{k}}{|\vec{k}|} \end{pmatrix} \quad \text{longitudinal}$$

$$\varepsilon^\mu(k, \lambda) \varepsilon_\mu(k, \sigma) = -\delta_{\lambda\sigma}, \quad \varepsilon^\mu(k, \lambda) k_\mu = 0$$

$$\sum_\lambda \varepsilon^\mu(k, \lambda) \varepsilon^\nu(k, \lambda) = -g^{\mu\nu} + k^\mu k^\nu / M^2$$

→ general solution of field equations:

$$V^\mu(x) = \sum_{\lambda=1}^3 \int d\mu(k) \left[\varepsilon^\mu(k, \lambda) a(k, \lambda) e^{-ikx} + \text{h.c.} \right]$$

quantization: $[a(k, \lambda), a(k', \lambda')^\dagger] = \delta_{\lambda\lambda'} \delta(k, k')$

generating functional of free massive vector field (using the functional integral method):

$$Z[J] = \langle 0 | T e^{-i \int d^4x V^\mu(x) J_\mu(x)} | 0 \rangle$$

$$= \frac{1}{N} \int [dV^\mu] e^{i \int d^4x \left(-\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{M^2 - i\varepsilon}{2} V_\mu V^\mu - V^\mu J_\mu \right)}$$

external current $J_\mu(x)$, normalization $Z[0] = 1$

$$S = \int d^4x \left(-\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{M^2 - i\varepsilon}{2} V_\mu V^\mu - J_\mu V^\mu \right)$$

$$= \int d^4x \left\{ \frac{1}{2} V^\mu [g_{\mu\nu} (\Box + M^2 - i\varepsilon) - \partial_\mu \partial_\nu] V^\nu - J_\mu V^\mu \right\}$$

usual trick: shift of integration variable

$$V^\mu = V'^\mu + W^\mu$$

↑

new integration variable in the functional integral

$$[dV^\mu] = [dV'^\mu] \quad \text{translation invariance of the measure}$$

terms linear in V' disappear, if

$$[g_{\mu\nu} (\Box + M^2 - i\varepsilon) - \partial_\mu \partial_\nu] W^\nu = J_\mu$$

propagator $\Delta^{\nu\sigma}(x)$ = Green's function of the differential operator $g_{\mu\nu} (\Box + M^2 - i\varepsilon) - \partial_\mu \partial_\nu$:

$$[g_{\mu\nu} (\Box + M^2 - i\varepsilon) - \partial_\mu \partial_\nu] \Delta^{\nu\sigma}(x) = \underbrace{\delta_\mu^\sigma}_{g_\mu^\sigma} \delta^{(4)}(x)$$

Fourier representation $\Delta^{\nu\sigma}(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \tilde{\Delta}^{\nu\sigma}(k)$

→ equation in momentum space

$$[g_{\mu\nu} (-k^2 + M^2 - i\varepsilon) + k_\mu k_\nu] \tilde{\Delta}^{\nu\sigma}(k) = g_\mu^\sigma$$

we discuss the more general problem:

find the inverse of

$$T_{\mu\nu} = a(k^2) k_\mu k_\nu + b(k^2) g_{\mu\nu}$$

ansatz for $(T^{-1})^{\nu\rho}$:

$$(T^{-1})^{\nu\rho} = A(k^2) k^\nu k^\rho + B(k^2) g^{\nu\rho}$$

$$(a k_\mu k_\nu + b g_{\mu\nu}) (A k^\nu k^\rho + B g^{\nu\rho}) = \delta_\mu^\rho$$

$$a A k^2 k_\mu k^\rho + a B k_\mu k^\rho + b A k_\mu k^\rho + b B g_\mu^\rho = \delta_\mu^\rho$$

$$\Rightarrow B = \frac{1}{b}, \quad a A k^2 + a B + b A = 0$$

$$A (a k^2 + b) = -\frac{a}{b} \Rightarrow A = -\frac{a}{b(a k^2 + b)}$$

$$\Rightarrow (T^{-1})^{\nu\rho} = \frac{g^{\nu\rho}}{b} - \frac{a k^\nu k^\rho}{b(a k^2 + b)}$$

$$\Rightarrow T^{-1} \text{ exists if } b \neq 0 \text{ and } a k^2 + b \neq 0$$

in the case of the massive vector field we have

$$a = 1, \quad b = -k^2 + M^2 - i\varepsilon$$

$$\Rightarrow \tilde{\Delta}^{\nu\sigma}(k) = \frac{g^{\nu\sigma} - k^\nu k^\sigma / M^2}{M^2 - k^2 - i\varepsilon}$$

$$\Delta^{\nu\sigma}(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \frac{g^{\nu\sigma} - k^\nu k^\sigma / M^2}{M^2 - k^2 - i\varepsilon}$$

$$W^\mu(x) = \int d^4 y \Delta^{\mu\nu}(x-y) J_\nu(y)$$

$$\begin{aligned} Z[j] &= e^{-\frac{i}{2} \int d^4 x J_\mu(x) W^\mu(x)} \\ &= e^{-\frac{i}{2} \int d^4 x d^4 y J_\mu(x) \Delta^{\mu\nu}(x-y) J_\nu(y)} \end{aligned}$$

$$\Rightarrow -\frac{1}{2} \langle 0 | T V^\mu(x) V^\nu(y) | 0 \rangle = -\frac{i}{2} \Delta^{\mu\nu}(x-y)$$

$$\langle 0 | T V^\mu(x) V^\nu(y) | 0 \rangle = i \Delta^{\mu\nu}(x-y)$$

$$= i \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \frac{g^{\mu\nu} - k^\mu k^\nu / M^2}{M^2 - k^2 - i\varepsilon}$$

Complex vector field ($M \neq 0$)

$$\mathcal{L} = -\frac{1}{2} V_{\mu\nu}^* V^{\mu\nu} + M^2 V_\mu^* V^\mu$$

$$V^\mu = (V_1^\mu + i V_2^\mu) / \sqrt{2}, \quad V_{1,2}^\mu \text{ real}$$

generating functional

$$Z[J, J^*] = \langle 0 | T e^{-i \int d^4x (V_\mu^\dagger(x) J^\mu(x) + V^\mu(x) J_\mu^*(x))} | 0 \rangle$$

can be obtained from the generating functional of a real vector field ($J^\mu = (J_1^\mu + i J_2^\mu) / \sqrt{2}$, J_1, J_2 real)

$$\rightarrow Z[J, J^*] = e^{-i \int d^4x d^4y J_\mu^*(x) \Delta^{\mu\nu}(x-y) J_\nu(y)}$$

$$\Rightarrow \langle 0 | T V_\mu(x) V_\nu^\dagger(y) | 0 \rangle = i \Delta^{\mu\nu}(x-y)$$

$$\langle 0 | T V_\mu(x) V_\nu(y) | 0 \rangle = 0$$