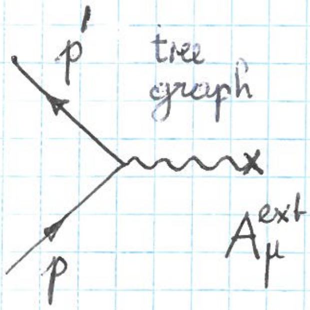
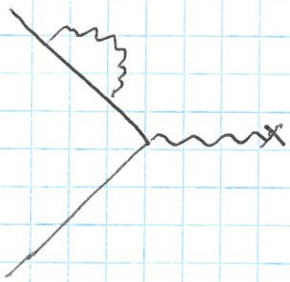
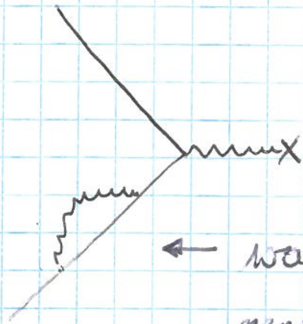
14. Anomalous magnetic moments of $e^\pm$ and $\mu^\pm$

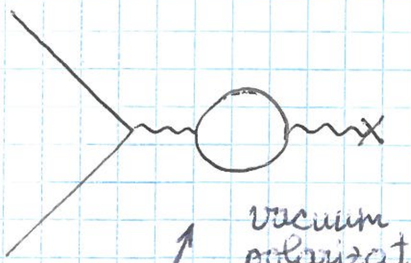
consider scattering of an electron in an external electromagnetic field $A_\mu^{ext}(x)$





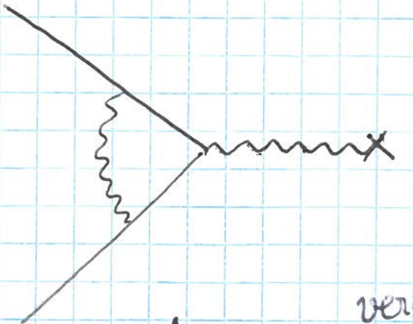


wave function renormalization



vacuum polarization

shielding of "bare" electromagnetic charge by virtual e^+e^- pairs



vertex correction

responsible for anomalous magnetic moment of the electron at one-loop

replace $A_\mu(x) \longrightarrow A_\mu(x) + A_\mu^{ext}(x)$ in functional integral

quantum field (integration variable in the functional integral)

$$S = S_{QED} - \int d^4x\, j^\mu(x)\, A_\mu^{ext}(x)$$

$$\langle p', s' \text{ out} \mid p, s \text{ in} \rangle_{A_\mu^{ext}} =$$

$$= \langle p', s' \mid T\, e^{-i\int d^4x\, j^\mu(x) A_\mu^{ext}(x)} \mid p, s \rangle$$

$$= -i \int d^4x\, A_\mu^{ext}(x)\, \langle p', s' \mid j^\mu(x) \mid p, s \rangle + O[(A^{ext})^2]$$

current conservation, P invariance $\longrightarrow$

$$\rightarrow \langle p', s' \mid j^\mu(0) \mid p, s \rangle =$$

$$= (-e)\ \bar{u}(p', s') \left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}}{2m} q_\nu F_2(q^2) \right] u(p, s)$$

charge of e^-

form factors $\quad F_1(0) = 1$

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu], \quad q = p' - p$$

e, m denote the <u>physical</u> quantities

Gordon decomposition (exercise)

$$\bar{u}(p',s')\gamma^\mu u(p,s) = \bar{u}(p',s')\left[\frac{p^\mu + p'^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p,s)$$

$\rightarrow \quad \langle p',s' | j^\mu(0) | p,s\rangle =$

$$= (-e)\,\bar{u}(p',s')\Big[\underbrace{\frac{p^\mu + p'^\mu}{2m} F_1(q^2)}_{\text{does not shake the spin}} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\big(F_1(q^2)+F_2(q^2)\big)\Big]u(p,s)$$

(corresponds to Lorentz force in the case of spinless particles)

nonrelativistic limit ($\rightarrow$ particle physics I)

$\rightarrow$ magnetic moment

$$\vec{\mu} = -\underbrace{\frac{e}{2m}}_{\mu_B}\Big\{\vec{L} + \underbrace{2\left[1+F_2(0)\right]}_{g}\underbrace{\frac{\vec{\sigma}}{2}}_{\vec{S}}\Big\}$$

Bohr's magneton $\qquad$ Landé factor $\qquad$ spin tree approximation : $g = 2$

higher order corrections $\rightarrow g = 2\,[1 + F_2(0)]$

"anomaly" $$a_e = \frac{g-2}{2} = F_2(0)$$

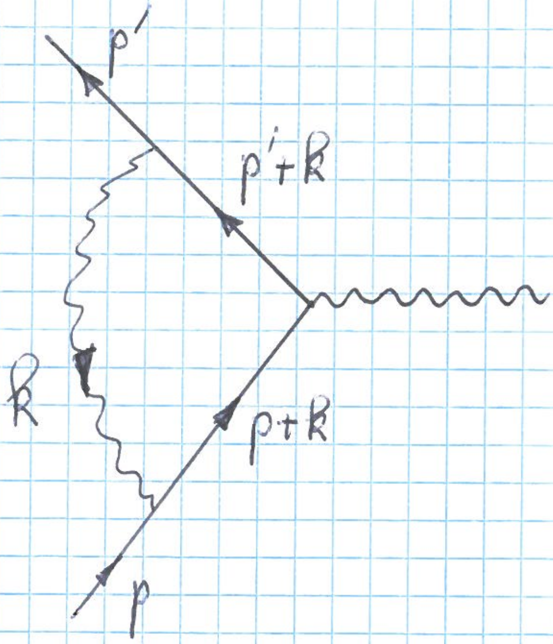
<u>computation</u> of the anomalous magnetic moment of the electron at <u>one-loop</u> (Schwinger 1948):

invoking the Gordon decomposition, we can write

$$\langle p', s' \,|\, j^\mu(0) \,|\, p, s \rangle =$$

$$= (-e)\, \bar{u}(p', s') \left[\gamma^\mu \left(F_1(q^2) + F_2(q^2) \right) - \frac{(p+p')^\mu}{2m} F_2(q^2) \right] u(p,s)$$

we are only interested in $F_2(0)$ $\rightarrow$ throw away any term proportional to γ^μ appearing in our calculation contribution of



to $\langle p', s' | j_\mu(0) | p, s \rangle = (-e)\,(-ie^2)\,\times$

$$\times\,\bar{u}(p',s') \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\alpha (\not{p}'+\not{k}+m)\,\gamma_\mu\,(\not{p}+\not{k}+m)\,\gamma_\alpha}{(k^2+i\varepsilon)\left[(p+k)^2-m^2+i\varepsilon\right]\left[(p'+k)^2-m^2+i\varepsilon\right]}\,u(p,s)$$

$+\ldots$

I use the formula

$$\frac{1}{a\,b\,c} = 2! \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\times$$

$$\times\,\frac{1}{(xa+yb+zc)^3}$$

with $a = k^2+i\varepsilon$, $b = (p+k)^2-m^2+i\varepsilon$, $c = (p'+k)^2-m^2+i\varepsilon$

$$x\,(k^2+i\varepsilon) + y\left[(k+p)^2 - m^2 + i\varepsilon\right] + z\left[(k+p')^2 - m^2 + i\varepsilon\right]$$

$$= k^2 + 2(yp+zp')\cdot k + y\underbrace{(p^2-m^2)}_{0} + z\underbrace{(p'^2-m^2)}_{0} + i\varepsilon$$

$$= \left[k + yp + zp'\right]^2 - (yp+zp')^2 + i\varepsilon$$

$\rightarrow$ shift $k \rightarrow k - yp - zp'$

$$\Rightarrow \langle p', s' | j_\mu(0) | p, s\rangle = (-e)\,(-2ie^2)\int_0^1 dx\,dy\,dz\;\delta(x+y+z-1)$$

$$\times\, \bar{u}(p', s')\int \frac{d^4k}{(2\pi)^4}\,\frac{\gamma^\alpha(\not{k} - y\not{p} - z\not{p}' + \not{p}' + m)\gamma_\mu(\not{k} - y\not{p} - z\not{p}' + \not{p} + m)\gamma_\alpha}{\left[k^2 - (yp+zp')^2 + i\varepsilon\right]^3}\, u(p, s)$$

<u>remark</u>: as long as we are only interested in $F_2(0)$ we don't have to worry about the regularization of the integral ($F_2(0)$ must be <u>finite</u> as QED is a <u>renormalizable</u> QFT)

<u>massage</u> of the numerator of the integrand:

1) terms <u>quadratic</u> in m: $\gamma^\alpha \gamma_\mu \gamma_\alpha \sim \gamma_\mu$

2) terms <u>linear</u> in m: organize by powers of k

→ term linear in k disappears because of symmetric integration

→ we are left with

$$m\,\gamma^\alpha \left[(-y\not{p} - z\not{p}' + \not{p}')\gamma_\mu + \right.$$

$$\left. + \gamma_\mu (-y\not{p} - z\not{p}' + \not{p}) \right] \gamma^\alpha$$

$\gamma^\alpha \not{a} \not{b} \gamma_\alpha = 4 a \cdot b$

$$= 4m \left[(1-2y)\, p_\mu + (1-2z)\, p'_\mu \right]$$

$$\xrightarrow[y \leftrightarrow z]{} 4m\,(1-y-z)\,(p+p')_\mu$$

3) terms <u>without</u> m: organize by powers of k

term <u>quadratic</u> in k: $\gamma^\alpha \gamma^\sigma \gamma_\mu \gamma^\rho \gamma_\alpha k_\sigma k_\rho$

$$\rightarrow \frac{1}{4} k^2 \gamma^\alpha \gamma^\sigma \gamma_\mu \gamma_\sigma \gamma_\alpha \sim \gamma_\mu$$

term <u>linear</u> in k: integrates to $\emptyset$

remaining piece:

$$\gamma^\alpha(-y\not{p}-z\not{p}'+\not{p}')\gamma_\mu(-y\not{p}-z\not{p}'+\not{p})\gamma_\alpha$$

$$\underset{\substack{\uparrow\\ \gamma^\alpha\not{a}\not{b}\not{c}\gamma_\alpha = \\ = -2\not{c}\not{b}\not{a}}}{=} -2(-y\not{p}-z\not{p}'+\not{p})\gamma_\mu(-y\not{p}-z\not{p}'+\not{p}')$$

$$\underset{\bar{u}(p')\dots u(p)}{\longrightarrow} -2\left[(1-y)\not{p}-zm\right]\gamma_\mu\left[(1-z)\not{p}'-ym\right]$$

organize again by powers of m:

m^2 term can be thrown away

m term: $\quad 2m\left[z\,\gamma_\mu(1-z)\not{p}' + y(1-y)\not{p}\gamma_\mu\right]$

$$\longrightarrow 2m\,(p+p')_\mu\left[z(1-z)+y(1-y)\right]$$

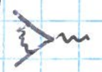
m^0 term: $\quad -2(1-y)(1-z)\,\not{p}\gamma_\mu\not{p}'$

$$\longrightarrow -4m(1-y)(1-z)(p+p')_\mu$$

sum of all relevant terms:

$$2m\,(p+p')_\mu \quad (1-y-z)(y+z)$$

$$\Rightarrow \langle p',s'|\,j_\mu(0)\,|p,s\rangle\Big| = (-e)(-2ie^2)\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$



$$\bar{u}(p',s')\int\frac{d^4k}{(2\pi)^4}\,\frac{2m\,(p+p')_\mu\,(1-y-z)(y+z)}{[k^2-(yp+zp')^2+i\varepsilon]^3}\,u(p,s)$$

$$\Rightarrow F_2(q^2)\Big|_{\text{one-loop}} = 8im^2e^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\times$$

$$\times\,(1-y-z)(y+z)\underbrace{\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{[k^2-(yp+zp')^2+i\varepsilon]^3}}_{\dfrac{-i}{(4\pi)^2\,2\,[(yp+zp')^2-i\varepsilon]}}$$

$$\Rightarrow F_2(0)\Big|_{\text{one-loop}} = \frac{e^2}{4\pi^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{(1-y-z)(y+z)}{(y+z)^2}$$

$$= \frac{\alpha}{\pi} \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\, \frac{x}{1-x}$$

$$= \frac{\alpha}{\pi} \int_0^1 dx\, \frac{x}{1-x} \int_0^{1-x} dy = \frac{\alpha}{\pi} \int_0^1 dx\, x = \frac{\alpha}{2\pi}$$

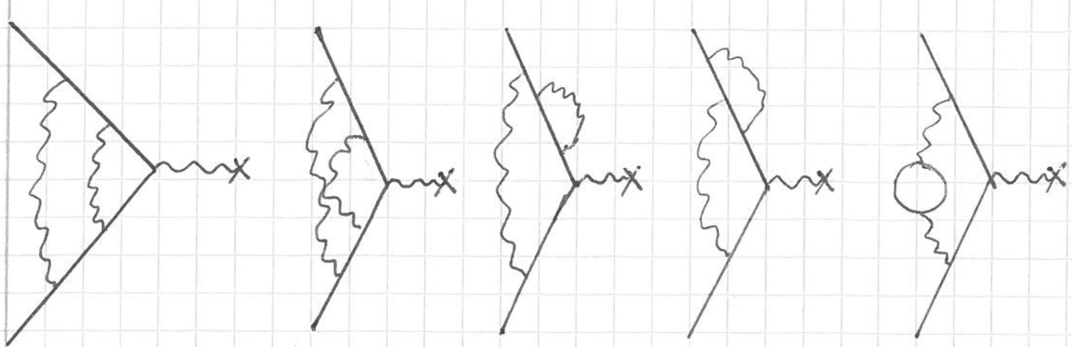
→ one-loop result $\boxed{a_\ell = F_2(0) = \frac{\alpha}{2\pi}}$ Schwinger 1948

$$a_\ell^{SM} = a_\ell^{QED} + a_\ell^{weak} + a_\ell^{had} \qquad (\ell = e, \mu)$$

QED contributions: loops with only photons and leptons

$$a_e^{QED} = 0.5\, \frac{\alpha}{\pi} - 0.328\,478\,444\,002\,90\,(60) \left(\frac{\alpha}{\pi}\right)^2$$
$$+ 1.181\,234\,016\,828\,(19) \left(\frac{\alpha}{\pi}\right)^3$$
$$- 1.912\,07\,(84) \left(\frac{\alpha}{\pi}\right)^4$$
$$+ 7.795\,(336) \left(\frac{\alpha}{\pi}\right)^5$$

two-loop contributions (Sommerfield / Petermann 1957)



3-loop order : 72 diagrams
4 – " – " – : 891 – " –
5 – " – " – : 12 672 – " –

$$a_\mu^{QED} = 0.5\,\frac{\alpha}{\pi} + 0.765\,857\,425(17)\left(\frac{\alpha}{\pi}\right)^2$$
$$+ 24.050\,509\,96(32)\left(\frac{\alpha}{\pi}\right)^3$$
$$+ 130.877\,3(61)\left(\frac{\alpha}{\pi}\right)^4$$
$$+ 751.92(93)\left(\frac{\alpha}{\pi}\right)^5$$

most accurate determination of input parameter α from atomic recoil velocity through photon absorption

$$\alpha^2 = \frac{2R_\infty}{c} \times \frac{M_{atom}}{m_e} \times \frac{h}{M_{atom}}$$

$$\frac{\Delta R_\infty}{R_\infty} = 7 \times 10^{-12} \qquad \frac{\Delta(M_{Rb}/m_e)}{M_{Rb}/m_e} = 4.4 \times 10^{-10}$$

$v_r = \dfrac{\hbar k}{M_{Rb}}$ (when Rb atom absorbs a photon of mom. $\hbar k$)

$\rightarrow \alpha^{-1}(Rb) = 137.035\,999\,037(91)$ [0.66 ppb]

R. Bouchendira et al., Phys. Rev. Lett. 106 (2011) 080801

$\alpha^{-1}(Rb) = 137.035\,999\,049(90)$ P.J. Mohr, B.N. Taylor, D. B. Newell, Rev. Mod. Phys. 84 (2012) 1527

$\rightarrow a_e^{QED} = 1\,159\,652\,179.908\,(25)_{\alpha^4}\,(23)_{\alpha^5}\,(763)_{\alpha(Rb)} \times 10^{-12}$

$a_e^{exp} = (1\,159\,652\,180.73 \pm 0.28) \times 10^{-12}$

D. Hanneke et al., Phys. Rev. Lett. 100 (2008) 120801

$a_e^{exp} - a_e^{QED} = 0.822\,(813) \times 10^{-12}$

$a_e^{weak} = 0.0297\,(5) \times 10^{-12}$

$a_e^{had} = 1.706\,(15) \times 10^{-12}$

$\rightarrow a_e^{SM} = 1\,159\,652\,181.643\,(25)_{\alpha^4}\,(23)_{\alpha^5}\,(16)_{weak + had}\,(763)_{\alpha(Rb)} \times 10^{-12}$

from T. Aoyama et al., Phys. Rev. D91 (2015) 033006

$a_e^{exp} - a_e^{SM} = -0.91 \pm 0.82 \times 10^{-12}$

intrinsic theoretical uncertainty in a_e^{SM} : $\sim 38 \times 10^{-15}$

$\rightarrow$ less than 1/20 of uncertainty due to finestructure constant!

$\rightarrow$ more precise value of α can be obtained by assuming that SM is valid and solving $a_e^{SM} = a_e^{exp}$ for α

$\rightarrow \alpha^{-1}(a_e) = 137.035\,999\,1570\,(29)_{\alpha^4}\,(27)_{\alpha^5}\,(18)_{had}\,(331)_{exp}$

from T. Aoyama et al., Phys. Rev. D91 (2015) 033006

<u>muon</u> anomalous magnetic moment

QED contribution:

$a_\mu^{QED}(Rb) = 1\,165\,847\,189.51(9)_{mass}(19)_{\alpha^4}(7)_{\alpha^5}(77)_{\alpha(Rb)} \times 10^{-12}$

$a_\mu^{QED}(a_e) = 1\,165\,847\,188.46(9)_{mass}(19)_{\alpha^4}(7)_{\alpha^5}(30)_{\alpha(a_e)} \times 10^{-12}$

from T. Aoyama et al., Phys. Rev. Lett. 109 (2012) 111 807

to be compared with

$a_\mu^{exp} = 116\,592\,089(63) \times 10^{-11}$ [0.54 ppm]

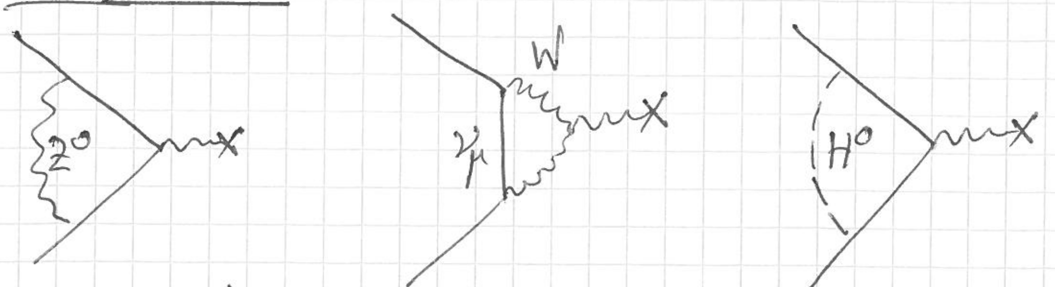
G. W. Bennet et al., Phys. Rev. D73 (2006) 072003

$$a_\mu^{exp} - a_\mu^{QED} = 737.0\,(6.3) \times 10^{-10}$$

a_μ^{QED} differs from a_μ^{exp} by more than $100\,\sigma$!

<u>reason:</u> larger mass of the muon (compared to the electron) leads to <u>large hadronic</u> and <u>sizable weak</u> contributions!

<u>weak contr.:</u>

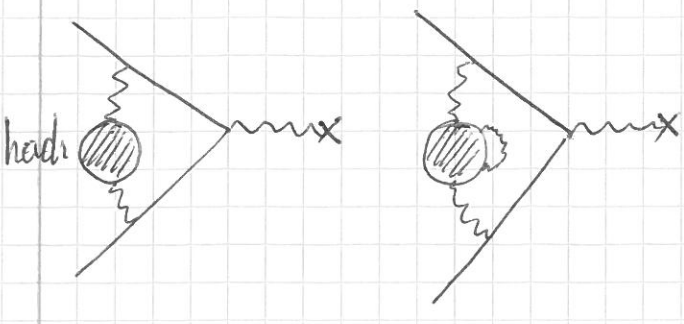


$$a_\mu^{weak(1)} = 19.48 \times 10^{-10}$$

+ 2-loop contributions $\rightarrow$ $a_\mu^{weak} = (15.36 \pm 0.10) \times 10^{-10}$

(recent update from C. Gnendinger et al., Phys. Rev. D88 (2013) 053005)

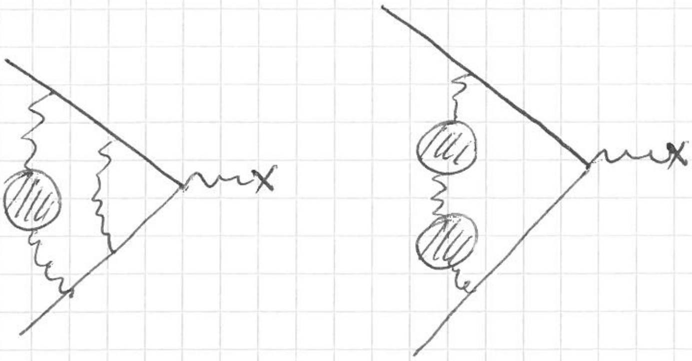
Hadronic contributions:



HVP - LO

lowest-order hadronic vacuum polarization

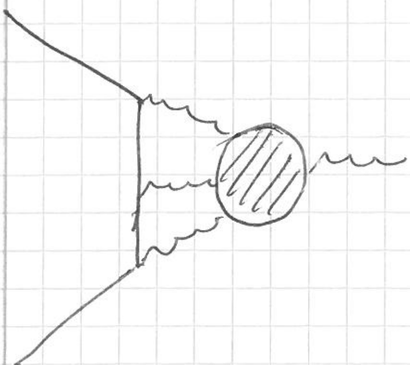
$$a_\mu^{HVP-LO} = (692.3 \pm 4.2) \times 10^{-10}$$



HVP - HO

higher-order hadronic vacuum polarization

$$a_\mu^{HVP-HO} = (-8.60 \pm 0.07) \times 10^{-10}$$



HL×L

hadronic light-by-light scattering

$$a_\mu^{HL\times L} = (10.5 \pm 2.6) \times 10^{-10}$$

$$\rightarrow \quad a_\mu^{exp} - a_\mu^{SM} \sim 30 \times 10^{-10}$$

at present, the SM value for a_μ misses the experimental determination by about 3.5σ

(see also seminar talk by Marc Knecht: particle.univie.ac.at)