13. Quantum electrodynamics of spin 1/2 fermions

the Lagrangian

$$\mathcal{L}_{QED} = \overbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\xi}{2} (\partial_\mu A^\mu)^2 + \sum_f \bar{f} (i\not{\partial} - m_f) f}^{\mathcal{L}_0}$$

$$\underbrace{- e \sum_f q_f \bar{f} \gamma^\mu f A_\mu}_{\mathcal{L}_{int}}$$

describes the electromagnetic interaction of charged spin 1/2 particles,

$$f = \overbrace{e^-, \mu^-, \tau^-}^{\text{leptons}}, \overbrace{u_i, d_i, c_i, s_i, t_i, b_i}^{\text{quarks} \ (i=1,2,3) \ \text{colour}},$$

$$q_f = -1, -1, -1, \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3},$$

and their associated antiparticles

this is a funny world: weak and strong interactions are turned off → heavy leptons and heavy quarks are stable; there is no confinement and consequently free quarks do exist, but there are no protons or neutrons →

hydrogen does not exist in this model,
but bound states like $\mu^+ e^-$ or $u e^-$
are stable

compared to the real world, electromagnetic processes of electrons and positrons are described correctly in this model (apart from tiny corrections induced by weak and strong interactions)

examples:

$e^- e^- \to e^- e^-$ Møller scattering

$e^+ e^- \to e^+ e^-$ Bhabha scattering

$e^- \gamma \to e^- \gamma$ Compton scattering

$e^+ e^- \to \gamma \gamma$ pair annihilation

$\gamma \gamma \to e^+ e^-$ pair creation our model allows also a decent description of reactions like

$$e^+ e^- \to \mu^+ \mu^-$$

although muons are <u>not</u> associated with asymptotic fields in the real world, they decay so slowly (predominantely via $\mu^- \to e^- \bar{\nu}_e \nu_\mu \to \tau_\mu \simeq 2.2 \times 10^{-6} s$) compared to the characteristic time of the electromagnetic interaction that the <u>approximation</u> of muons as asymptotic states is justified for all practical purposes (muons leave long tracks in particle detectors)

<u>tree level</u> amplitude of $e^+ e^- \to f \bar{f}$

$$f \neq e$$

relevant four-point Green's function:

$$\langle 0 | T\, f_d(x_4)\, \overline{f}_c(x_3)\, e_b(x_2)\, \overline{e}_a(x_1) | 0 \rangle$$

$$= \frac{\langle\langle e^{iS_{int}} f_d(x_4)\, \overline{f}_c(x_3)\, e_b(x_2)\, \overline{e}_a(x_1) \rangle\rangle}{\langle\langle e^{iS_{int}} \rangle\rangle}$$

where $\langle\langle \ldots \rangle\rangle = \dfrac{\int [dA\, d\psi\, d\overline{\psi}]\, e^{iS_0[\psi, \overline{\psi}, A]} \ldots}{\int [dA\, d\psi\, d\overline{\psi}]\, e^{iS_0[\psi, \overline{\psi}, A]}}$

denotes collectively all fermion fields

$$\langle\langle e^{iS_{int}} f_d(x_4)\, \overline{f}_c(x_3)\, e_b(x_2)\, \overline{e}_a(x_1) \rangle\rangle$$

$$= \langle\langle \left(1 + i S_{int} + \frac{i^2}{2!} S_{int}^2 + \ldots\right) f_d(x_4)\, \overline{f}_c(x_3)\, e_b(x_2)\, \overline{e}_a(x_1) \rangle\rangle$$

$\langle\langle A_\mu \rangle\rangle = 0$

relevant term:

$$\frac{i^2}{2!} \left\langle\!\left\langle S_{int}^2\, f_d(x_4)\, \bar{f}_c(x_3)\, e_b(x_2)\, \bar{e}_a(x_1) \right\rangle\!\right\rangle_c =$$

$$= \frac{i^2 e^2}{2!} \int d^4x\, d^4y \left\langle\!\left\langle [+\bar{e}(x)\gamma^\mu e(x) - q_f \bar{f}(x)\gamma^\mu f(x)]\, A_\mu(x) \right.\right.$$

$$[+\bar{e}(y)\gamma^\nu e(y) - q_f \bar{f}(y)\gamma^\nu f(y)]\, A_\nu(y)$$

$$\left.\left. f_d(x_4)\, \bar{f}_c(x_3)\, e_b(x_2)\, \bar{e}_a(x_1) \right\rangle\!\right\rangle_c$$

$$= -q_f i^2 e^2 \int d^4x\, d^4y \left\langle\!\left\langle A_\mu(x)\, A_\nu(y) \right\rangle\!\right\rangle$$

$$\left\langle\!\left\langle \bar{e}_m(x)(\gamma^\mu)_{mn}\, e_n(x)\, \bar{f}_r(y)(\gamma^\nu)_{rs}\, f_s(y) \right.\right.$$

$$\left.\left. f_d(x_4)\, \bar{f}_c(x_3)\, e_b(x_2)\, \bar{e}_a(x_1) \right\rangle\!\right\rangle$$

$$= -q_f i^2 e^2 \int d^4x\, d^4y \left\langle\!\left\langle A_\mu(x)\, A_\nu(y) \right\rangle\!\right\rangle$$

$$\left\langle\!\left\langle \bar{e}_m(x)\, e_n(x)\, e_b(x_2)\, \bar{e}_a(x_1) \right\rangle\!\right\rangle$$

$$\left\langle\!\left\langle \bar{f}_r(y)\, f_s(y)\, f_d(x_4)\, \bar{f}_c(x_3) \right\rangle\!\right\rangle$$

$$(\gamma^\mu)_{mn}\, (\gamma^\nu)_{rs}$$

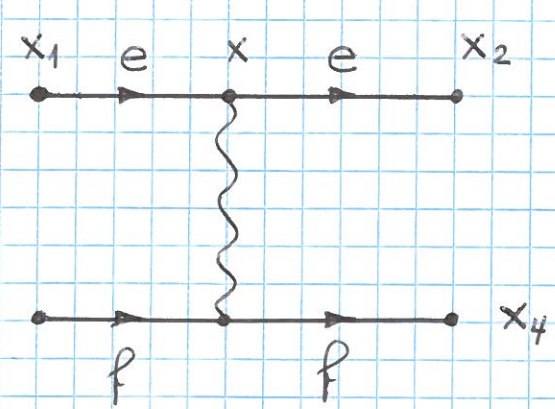
$$= - q_f \, i^2 e^2 \int d^4x \, d^4y \; \langle\!\langle A_\mu(x) \, A_\nu(y) \rangle\!\rangle$$

$$\langle\!\langle e_b(x_2) \, \overline{e}_m(x) \rangle\!\rangle \; \langle\!\langle e_n(x) \, \overline{e}_a(x_1) \rangle\!\rangle$$

$$\langle\!\langle f_d(x_4) \, \overline{f}_r(y) \rangle\!\rangle \; \langle\!\langle f_s(y) \, \overline{f}_c(x_3) \rangle\!\rangle$$

$$(\gamma^\mu)_{mn} \; (\gamma^\nu)_{rs}$$

all other contractions (pairings) lead to disconnected graphs



$$= - q_f (ie)^2 \int d^4x \, d^4y \; i \, D_{\mu\nu}(x-y)$$

$$\left[\frac{1}{i} S_e(x_2 - x_1) \, \gamma^\mu \, \frac{1}{i} S_e(x - x_1) \right]_{ba}$$

$$\left[\frac{1}{i} S_f(x_4 - y) \, \gamma^\nu \, \frac{1}{i} S_f(y - x_3) \right]_{dc}$$

$$= -q_f (ie)^2 \int d^4x\, d^4y \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{-ig_{\mu\nu}}{k^2}$$

$$\left[\frac{1}{i}\int \frac{d^4q_2}{(2\pi)^4} \frac{e^{-iq_2(x_2-x)}}{m_e - \not{q}_2} \gamma^\mu \frac{1}{i}\int \frac{d^4q_1}{(2\pi)^4} \frac{e^{-iq_1(x-x_1)}}{m_e - \not{q}_1}\right]_{ba}$$

$$\left[\frac{1}{i}\int \frac{d^4q_2}{(2\pi)^4} \frac{e^{-iq_4(x_4-y)}}{m_f - \not{q}_4} \gamma^\nu \frac{1}{i}\int \frac{d^4q_3}{(2\pi)^4} \frac{e^{-iq_3(y-x_3)}}{m_f - \not{q}_3}\right]_{dc}$$

integration over x, y

$$\overset{\downarrow}{=} -q_f (ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4}$$

$$(2\pi)^4 \delta^{(4)}(-k + q_2 - q_1) \quad (2\pi)^4 \delta^{(4)}(k + q_4 - q_3)$$

$$\frac{-ig_{\mu\nu}}{k^2} \left[\frac{1}{i} \frac{e^{-iq_2x_2}}{m_e - \not{q}_2} \gamma^\mu \frac{1}{i} \frac{e^{iq_1x_1}}{m_e - \not{q}_1}\right]_{ba}$$

$$\left[\frac{1}{i} \frac{e^{-iq_4x_4}}{m_f - \not{q}_4} \gamma^\nu \frac{1}{i} \frac{e^{+iq_3x_3}}{m_f - \not{q}_3}\right]_{dc}$$

integration over k

$$\overset{\downarrow}{=} \int \prod_{i=1}^{4} \frac{d^4q_i}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(q_2 - q_1 + q_4 - q_3)$$

$$\left[\frac{1}{i} \frac{e^{-iq_2x_2}}{m_e - \not{q}_2} (+ie\gamma^\mu) \frac{1}{i} \frac{e^{+iq_1x_1}}{m_e - \not{q}_1}\right]_{ba} \frac{-ig_{\mu\nu}}{(q_2-q_1)^2}$$

$$\left[\frac{1}{i} \frac{e^{-iq_4x_4}}{m_f - \not{q}_4} (-ieq_f\gamma^\nu) \frac{1}{i} \frac{e^{+iq_3x_3}}{m_f - \not{q}_3}\right]_{dc}$$

$q_1 = k_1,\ q_2 = -k_2,\ q_3 = -k_3,\ q_4 = k_4$

$$= \int \prod_{i=1}^{4} \frac{d^4 k_i}{(2\pi)^4} \ (2\pi)^4 \ \delta^{(4)}(k_1 + k_2 - k_3 - k_4)$$

$$\left[\frac{1}{i} \frac{e^{ik_2 x_2}}{m_e + \not{k}_2} (ie\gamma^\mu) \frac{1}{i} \frac{e^{ik_1 x_1}}{m_e - \not{k}_1} \right]_{ba}$$

$$\frac{-i g_{\mu\nu}}{(k_1 + k_2)^2}$$

$$\left[\frac{1}{i} \frac{e^{-ik_4 x_4}}{m_f - \not{k}_4} (-ieq_f \gamma^\nu) \frac{1}{i} \frac{e^{-ik_3 x_3}}{m_f + \not{k}_3} \right]_{cd}$$

<u>recipe</u> for the determination of the corresponding S-matrix element

for a particle (antiparticle) with momentum p ($p^2 = m^2$) and polarization s in the <u>initial state</u> :

write the corresponding <u>external</u> propagator in the form $\frac{1}{i} \frac{e^{ikx}}{m \mp \not{k}}$

replace $\frac{1}{i} \frac{e^{ikx}}{m - \not{k}}$ by $u(p,s)$ and

$\frac{1}{i} \frac{e^{ikx}}{m + \not{k}}$ by $\bar{v}(p,s)$, respectively for a particle (antiparticle) with momentum p ($p^2 = m^2$) and polarization s in the final state:

external propagator $\rightarrow$ $\frac{1}{i} \frac{e^{-ikx}}{m \mp \not{k}}$

then

$$\frac{1}{i} \frac{e^{-ikx}}{m + \not{k}} \rightarrow v(p,s) \quad \text{and}$$

$$\frac{1}{i} \frac{e^{-ikx}}{m - \not{k}} \rightarrow \bar{u}(p,s) \text{, respectively}$$

finally, omit $\int \prod_{i=1}^{4} \frac{d^4 k_i}{(2\pi)^4}$

remark: in the form given here, the amputation prescription applies only to tree diagrams (in higher orders, also the wave function renormalization has to be taken into account) this recipe follows from the reduction formula for spin 1/2 fermions

first step:

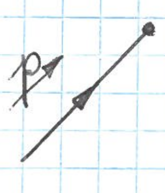
$$\langle \beta\, out|\alpha, ps\, in\rangle = \frac{i}{\sqrt{Z}}\int d^4x\, \langle \beta\, out|\bar\psi(x)|\alpha\, in\rangle\,(i\overleftarrow{\not\partial} + m_{ph})\,u(p,s)\,e^{-ipx} + \dots$$

$$\langle \beta\, out|\alpha, \overline{ps}\, in\rangle = \frac{i}{\sqrt{Z}}\int d^4x\, e^{-ipx}\,\bar v(p,s)\,(i\not\partial - m_{ph})\,\langle \beta\, out|\psi(x)|\alpha\, in\rangle + \dots$$

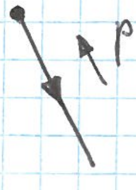
$$\langle \beta, ps\, out|\alpha\, in\rangle = -\frac{i}{\sqrt{Z}}\int d^4x\, e^{ipx}\,\bar u(p,s)\,(i\not\partial - m_{ph})\,\langle \beta\, out|\psi(x)|\alpha\, in\rangle + \dots$$

$$\langle \beta, \overline{ps}\, out|\alpha\, in\rangle = -\frac{i}{\sqrt{Z}}\int d^4x\, \langle \beta\, out|\bar\psi(x)|\alpha\, in\rangle\,(i\overleftarrow{\not\partial} + m)\,v(p,s)\,e^{ipx} + \dots$$

Feynman rules for the computation of $i\mathcal{M}_{fi}$



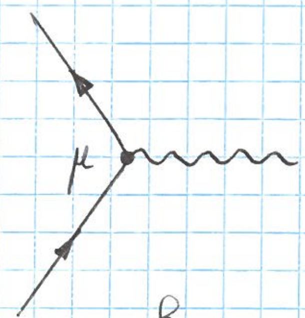
$u(p,s)$ particle with momentum p in the initial state



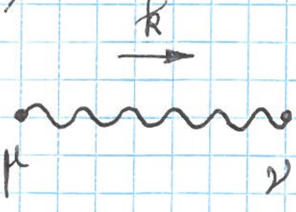
$\bar v(p,s)$ antiparticle with momentum p in the initial state

$\bar{u}(p,s)$ — particle with momentum p in the final state

$v(p,s)$ — antiparticle with momentum p in the final state

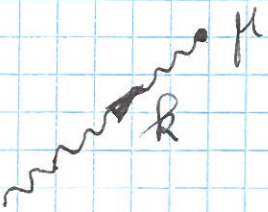


$-iqe\gamma^\mu$ — interaction vertex



$\dfrac{-ig_{\mu\nu}}{k^2+i\varepsilon}$ — internal photon line

$\dfrac{1}{i}\,\dfrac{1}{m-\not{p}-i\varepsilon}$ — internal fermion line



$\varepsilon^\mu(k,\lambda)$ — photon with momentum k and polarization λ in the initial state

$\varepsilon^\mu(k,\lambda)^*$ — photon with momentum k and polarization λ in the final state

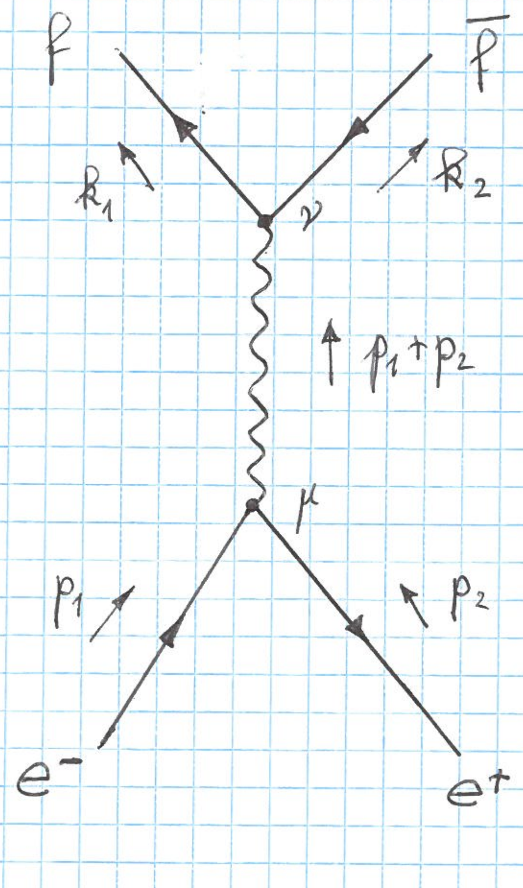
$$\varepsilon^\mu(k,\lambda) = (0, \vec{\varepsilon}(\vec{k},\lambda)) \quad , \quad \vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{k} = 0$$

$$\vec{\varepsilon}(\vec{k},\lambda)^* \cdot \vec{\varepsilon}(\vec{k},\lambda') = \delta_{\lambda\lambda'}$$

$$\underline{e^-(p_1)\ e^+(p_2) \longrightarrow f(k_1)\ \bar{f}(k_2)} \qquad (f \neq e)$$

$$i\mathcal{M} = \bar{v}(p_2, s_2; m_e)\, ie\gamma^\mu\, u(p_1, s_1; m_e)\ \frac{-ig_{\mu\nu}}{(p_1+p_2)^2}$$

$$\bar{u}(k_1, r_1; m_f)\,(-ieq_f\gamma^\nu)\, v(k_2, r_2; m_f)$$



consider the case of <u>unpolarized</u> e^+e^- beams

→ mixed state $\downarrow\downarrow, \downarrow\uparrow, \uparrow\downarrow, \uparrow\uparrow$

→ we have to <u>average</u> over the spin polarizations in the <u>initial state</u>:

$$\frac{1}{4}\sum_{s_1, s_2} |\mathcal{M}|^2$$

if the detector is unable to distinguish the spins of the <u>final state</u> particles, we have to <u>sum</u> over the spin polarizations in the final state:

$$\sum_{r_1, r_2} \cdots$$

$$\Rightarrow \frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = \frac{e^4 q_f^2}{4\,[(p_1+p_2)^2]^2} \sum_{spins}$$

$$\bar{v}(p_2, s_2; m_e)\,\gamma^\mu\, u(p_1, s_1; m_e)\; u^\dagger(p_1, s_1; m_e)(\gamma^0)^2(\gamma^\nu)^\dagger\gamma^0 v(p_2, s_2; m_e)$$

$$\bar{u}(k_1, r_1; m_f)\,\gamma_\mu\, v(k_2, r_2; m_f)\; \bar{v}(k_2, r_2; m_f)\,\gamma_\nu\, u(k_1, r_1; m_f)$$

$$= \frac{e^4 q_f^2}{4\,(p_1+p_2)^4} \sum_{spins}$$

$$\mathrm{Tr}\left[v(p_2, s_2; m_e)\, \bar{v}(p_2, s_2; m_e)\, \gamma^\mu u(p_1, s_1; m_e)\, \bar{u}(p_1, s_1; m_e)\, \gamma^\nu \right]$$

$$\mathrm{Tr}\left[u(k_1, r_1; m_f)\, \bar{u}(k_1, r_1; m_f)\, \gamma_\mu v(k_2, r_2; m_f)\, \bar{v}(k_2, r_2, m_f)\, \gamma_\nu \right]$$

$$= \frac{e^4 q_f^2}{4\,(p_1+p_2)^4} \mathrm{Tr}\left[(\not{p}_2 - m_e)\, \gamma^\mu\, (\not{p}_1 + m_e)\, \gamma^\nu \right] \times$$

$$\times \mathrm{Tr}\left[(\not{k}_1 + m_f)\, \gamma_\mu\, (\not{k}_2 - m_f)\, \gamma_\nu \right]$$

$$= \frac{e^4 q_f^2}{4\,(p_1+p_2)^4} \mathrm{Tr}\left(\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu - m_e^2 \gamma^\mu \gamma^\nu \right) \times$$

$$\times \mathrm{Tr}\left(\not{k}_1 \gamma_\mu \not{k}_2 \gamma_\nu - m_f^2 \gamma_\mu \gamma_\nu \right)$$

<u>traces of γ matrices</u>

1) trace of a product of an odd number of γ matrices vanishes

2) $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\, g^{\mu\nu}$

3) $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4\,(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma})$

$$\Rightarrow \frac{1}{4}\sum_{spins}|M|^2 = \frac{4e^4 q_f^2}{(p_1+p_2)^4} \times$$

$$\times \left[p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - p_1\cdot p_2\, g^{\mu\nu} - m_e^2\, g^{\mu\nu} \right] \times$$

$$\times \left[k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} - k_1\cdot k_2\, g_{\mu\nu} - m_f^2\, g_{\mu\nu} \right]$$

$$= \frac{4e^4 q_f^2}{(p_1+p_2)^4} \Big\{ k_1\cdot p_2\; p_1\cdot k_2 + p_2\cdot k_2\; p_1\cdot k_1$$

$$- (k_1\cdot k_2 + m_f^2)\, p_1\cdot p_2 + p_1\cdot k_1\; p_2\cdot k_2 + p_1\cdot k_2\; k_1\cdot p_2$$

$$- (k_1\cdot k_2 + m_f^2)\, p_1\cdot p_2 - (p_1\cdot p_2 + m_e^2)\big[2\, k_1\cdot k_2$$

$$- 4\,(k_1\cdot k_2 + m_f^2) \big] \Big\}$$

$$= \frac{4e^4 q_f^2}{(p_1+p_2)^4} \Big\{ 2\, p_1\cdot k_1\; p_2\cdot k_2 + 2\, p_1\cdot k_2\; k_1\cdot p_2$$

$$+ 2\, m_f^2\, p_1\cdot p_2 + 2\, m_e^2\, k_1\cdot k_2 + 4\, m_e^2 m_f^2 \Big\}$$

$$= \frac{8e^4 q_f^2}{(p_1+p_2)^4} \Big\{ p_1\cdot k_1\; p_2\cdot k_2 + p_1\cdot k_2\; k_1\cdot p_2$$

$$+ m_f^2\, p_1\cdot p_2 + m_e^2\, k_1\cdot k_2 + 2\, m_e^2 m_f^2 \Big\}$$

Kinematics of $e^-(p_1)\, e^+(p_2) \to f(k_1)\, \bar{f}(k_2)$

energy - momentum conservation:

$$p_1 + p_2 = k_1 + k_2$$

invariants: $$s = (p_1+p_2)^2 = 2m_e^2 + 2p_1 \cdot p_2$$

$$= (k_1+k_2)^2 = 2m_f^2 + 2k_1 \cdot k_2$$

$$t = (p_1 - k_1)^2 = m_e^2 + m_f^2 - 2p_1 \cdot k_1$$

$$= (k_2 - p_2) = m_e^2 + m_f^2 - 2p_2 \cdot k_2$$

the other invariants can be expressed through s, t:

$$p_1 + p_2 - k_1 = k_2$$

$$2m_e^2 + m_f^2 + 2p_1 \cdot p_2 - 2p_1 \cdot k_1 - 2p_2 \cdot k_1 = m_f^2$$

$$\Rightarrow \quad s + t - m_e^2 - m_f^2 - 2p_2 \cdot k_1 = 0$$

$$\Rightarrow \quad 2p_2 \cdot k_1 = s + t - m_e^2 - m_f^2$$

analogously:

$$p_1 + p_2 - k_2 = k_1$$

$$\Rightarrow \quad 2 p_1 \cdot k_2 = s + t - m_e^2 - m_f^2$$

$$\Rightarrow \quad \frac{1}{4} \sum_{spins} |\mathcal{M}|^2 = \frac{8 e^4 q_f^2}{s^2} \Big\{ \frac{1}{4} (m_e^2 + m_f^2 - t)^2$$

$$+ \frac{1}{4} (s + t - m_e^2 - m_f^2)^2 + m_f^2 \frac{1}{2} (s - \cancel{2 m_e^2})$$

$$+ m_e^2 \frac{1}{2} (s - \cancel{2 m_f^2}) + \cancel{2 m_e^2 m_f^2} \Big\}$$

$$= \frac{2 e^4 q_f^2}{s^2} \Big\{ (m_e^2 + m_f^2 - t)^2 + (s + t - m_e^2 - m_f^2)^2$$

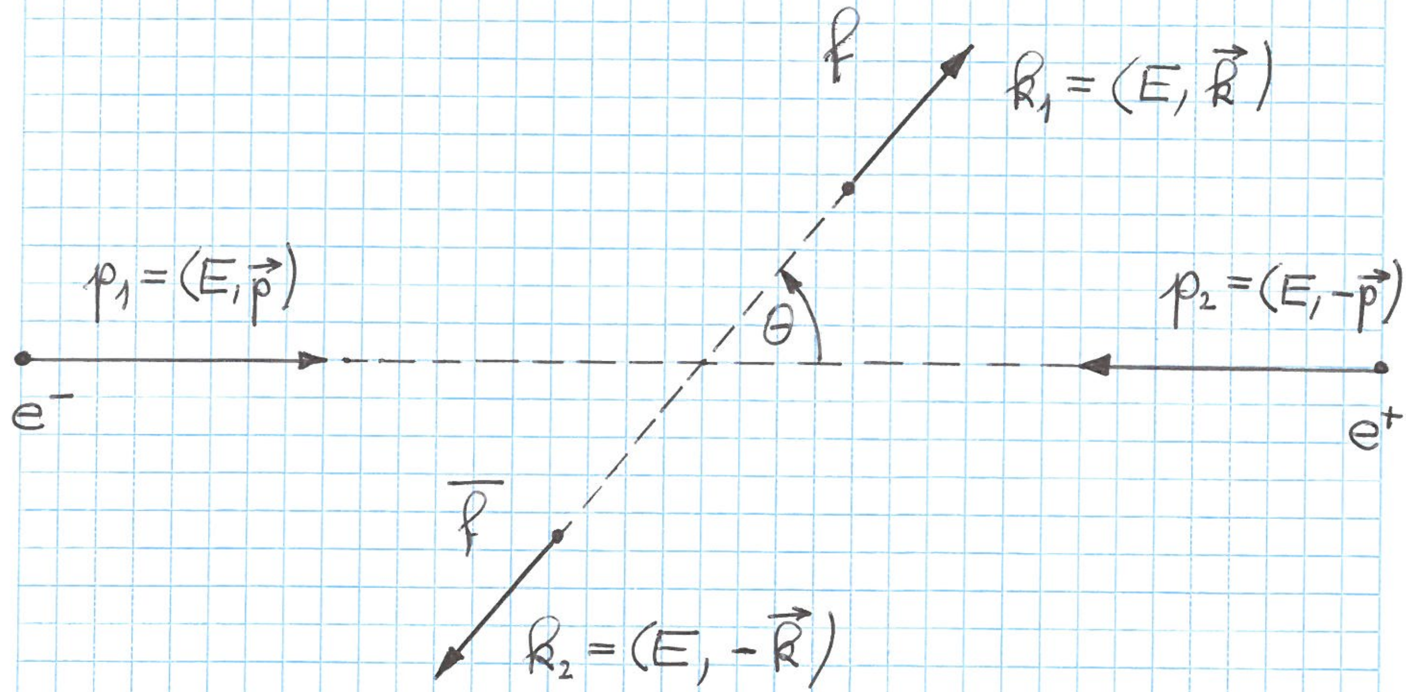
$$+ 2 s (m_e^2 + m_f^2) \Big\}$$

$$= \frac{2 e^4 q_f^2}{s^2} \Big\{ 2 (m_e^2 + m_f^2 - t)^2 + s^2 + 2 s (t - m_e^2 - m_f^2)$$

$$+ 2 s (m_e^2 + m_f^2) \Big\}$$

$$= \frac{2 e^4 q_f^2}{s^2} \Big\{ s^2 + 2 s t + 2 (m_e^2 + m_f^2 - t)^2 \Big\}$$

CMS



$$s = (p_1 + p_2)^2 = (2E)^2 \qquad 2E = \sqrt{s}$$

$$t = (p_1 - k_1)^2 = m_e^2 + m_f^2 - 2\left(E^2 - \vec{p}\cdot\vec{k}\right)$$

$$= m_e^2 + m_f^2 - \frac{s}{2} + 2|\vec{p}|\,|\vec{k}|\cos\theta$$

$$\Rightarrow \frac{1}{4}\sum_{spins}|M|^2 = \frac{2e^4 q_f^2}{s^2}\Big\{\cancel{s^2} + 2s\left(m_e^2 + m_f^2 - \cancel{\frac{s}{2}} + \cancel{2|\vec{p}|\,|\vec{k}|\cos\theta}\right)$$

$$+ \frac{s^2}{2} + 8|\vec{p}|^2|\vec{k}|^2\cos^2\theta - \cancel{4s|\vec{p}|\,|\vec{k}|\cos\theta}\Big\}$$

$$= \frac{2e^4 q_f^2}{s^2}\left\{\frac{s^2}{2} + 2s\left(m_e^2 + m_f^2\right) + 8\underbrace{|\vec{p}|^2}_{E^2 - m_e^2}\,\underbrace{|\vec{k}|^2}_{E^2 - m_f^2}\cos^2\theta\right\}$$

cross section

$$d\sigma = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - m_e^4}} \frac{d^3k_1}{(2\pi)^3 2k_1^0} \frac{d^3k_2}{(2\pi)^3 2k_2^0} (2\pi)^4 \delta^{(4)}(p_1+p_2-k_1-k_2)$$

$$\times \frac{1}{4}\sum_{spins} |M|^2$$

$$(p_1\cdot p_2)^2 - m_e^4 = \left(\frac{s}{2} - m_e^2\right)^2 - m_e^4 = \frac{s^2}{4} - s\, m_e^2$$

$$\Rightarrow \quad 4\sqrt{(p_1\cdot p_2)^2 - m_e^4} = 2\sqrt{s(s-4m_e^2)}$$

$$\int \frac{d^3k_1}{2k_1^0}\frac{d^3k_2}{2k_2^0}\,\delta^{(4)}(p_1+p_2-k_1-k_2)\ldots$$

$$= \int \frac{d^3k_1}{2k_1^0}\, d^4k_2\, \Theta(k_2^0)\,\delta(k_2^2 - m_f^2)\,\delta(p_1+p_2-k_1-k_2)\ldots$$

$$= \int \frac{d^3k_1}{2k_1^0}\,\Theta(p_1^0+p_2^0-k_1^0)\,\delta\left[(p_1+p_2-k_1)^2 - m_f^2\right]\ldots$$

$$\overset{\text{CMS}}{=} \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\theta \int_0^{\infty} \frac{d|\vec{k}_1|\,|\vec{k}_1|^2}{2k_1^0}\,\Theta(\sqrt{s} - k_1^0)\,\times$$

$$\times\,\delta(s - 2\sqrt{s}\,k_1^0)\ldots$$

$$= \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\Theta \int_{m_f}^{\infty} \frac{dk_1^0 \, \cancel{k_1^0} \, |\vec{k}_1|}{2 \cancel{k_1^0}} \, \Theta(\sqrt{s} - k_1^0) \times$$

$$\times \frac{1}{2\sqrt{s}} \, \delta\left(k_1^0 - \frac{\sqrt{s}}{2}\right) \ldots$$

$$= \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\Theta \; \frac{\sqrt{\frac{s}{4} - m_f^2}}{4\sqrt{s}} \; \underbrace{\Theta\left(\frac{\sqrt{s}}{2}\right)}_{1} \; \Theta\left(\frac{\sqrt{s}}{2} - m_f\right) \ldots$$

integration over φ

$$\Rightarrow \frac{d\sigma}{d\cos\Theta} = \frac{1}{2\sqrt{s(s-4m_e^2)}} \, \frac{1}{(2\pi)^2} \, 2\pi \, \frac{\sqrt{\frac{s}{4} - m_f^2}}{4\sqrt{s}}$$

$$\Theta\left(\frac{\sqrt{s}}{2} - m_f\right) \frac{2e^4 q_f^2}{s^2} \left\{ \frac{s^2}{2} + 2s\left(m_e^2 + m_f^2\right) \right.$$

$$\left. + 8 \underbrace{\left(\frac{s}{4} - m_e^2\right)}_{|\vec{p}|^2} \underbrace{\left(\frac{s}{4} - m_f^2\right)}_{|\vec{k}|^2} \cos^2\Theta \right\}$$

$$e^2 = 4\pi\alpha$$

$$\frac{d\sigma}{d\cos\Theta} = \frac{2\alpha^2 q_f^2 \pi\, \Theta\left(\frac{\sqrt{s}}{2} - m_f\right)\sqrt{\frac{s}{4} - m_f^2}}{s^3 \sqrt{s - 4m_e^2}}$$

$$\times \left\{ \frac{s^2}{2} + 2s\,(m_e^2 + m_f^2) + 8\left(\frac{s}{4} - m_e^2\right)\left(\frac{s}{4} - m_f^2\right)\cos^2\Theta \right\}$$

special case $\underline{m_e \ll m_f} < \sqrt{s}/2 \rightarrow$ neglect m_e

$$\rightarrow \frac{d\sigma}{d\cos\Theta} \simeq \frac{2\alpha^2 q_f^2 \pi\, \Theta\left(\frac{\sqrt{s}}{2} - m_f\right)\sqrt{\frac{s}{4} - m_f^2}}{s^2 \sqrt{s}}$$

$$\left\{ \frac{s}{2} + 2m_f^2 + 2\left(\frac{s}{4} - m_f^2\right)\cos^2\Theta \right\}$$

$$\sigma_{tot} = \frac{4\alpha^2 q_f^2 \pi \sqrt{\frac{s}{4} - m_f^2}}{s^2\sqrt{s}} \left\{ \frac{s}{2} + 2m_f^2 + \frac{2}{3}\left(\frac{s}{4} - m_f^2\right) \right\}$$

high energy behaviour ($\sqrt{s} \gg 2m_f$)

$$\frac{d\sigma}{d\cos\Theta} \longrightarrow \frac{\alpha^2 q_f^2 \pi}{2s}\left(1 + \cos^2\Theta\right)$$

$$\sigma_{tot} \rightarrow \frac{4\alpha^2 q_f^2 \pi}{3s}$$

$$d\sigma/d\Omega \rightarrow \frac{\alpha^2 q_f^2}{4s}\left(1 + \cos^2\Theta\right)$$

$e^+ e^- \rightarrow$ hadrons

$$e^+ e^- \rightarrow q_i \, \bar{q}_i \qquad i = 1, 2, 3$$

colour index

$u \quad c \quad t \qquad q_f = 2/3$

$d \quad s \quad b \qquad q_f = -1/3$

$$R := \frac{\sigma(e^+ e^- \rightarrow \text{hadrons})}{\sigma(e^+ e^- \rightarrow \mu^+ \mu^-)}$$

<u>naive</u> expectation (strong interaction switched off)

$$R \xrightarrow[\frac{\sqrt{s}}{2} \gg m_\mu, m_q]{} \underset{\substack{\uparrow \\ \text{colour}}}{3} \sum_q q_q^2$$

below $c\bar{c}$ threshold :

$$R \simeq 3 \left[\underset{\substack{\uparrow \\ u}}{\left(\frac{2}{3}\right)^2} + \underset{\substack{\uparrow \\ d}}{\left(-\frac{1}{3}\right)^2} + \underset{\substack{\uparrow \\ s}}{\left(-\frac{1}{3}\right)^2} \right] = 2$$

above $c\bar{c}$ threshold, but below $b\bar{b}$ threshold:

$$R \simeq 3\left[\underbrace{2 \times \left(\frac{2}{3}\right)^2}_{u,c} + \underbrace{2 \times \left(-\frac{1}{3}\right)^2}_{d,s}\right] = \frac{10}{3}$$

above $b\bar{b}$ threshold:

$$R \simeq 3\left[\underbrace{2 \times \left(\frac{2}{3}\right)^2}_{u,c} + \underbrace{3 \times \left(-\frac{1}{3}\right)^2}_{d,s,b}\right] = \frac{11}{3}$$

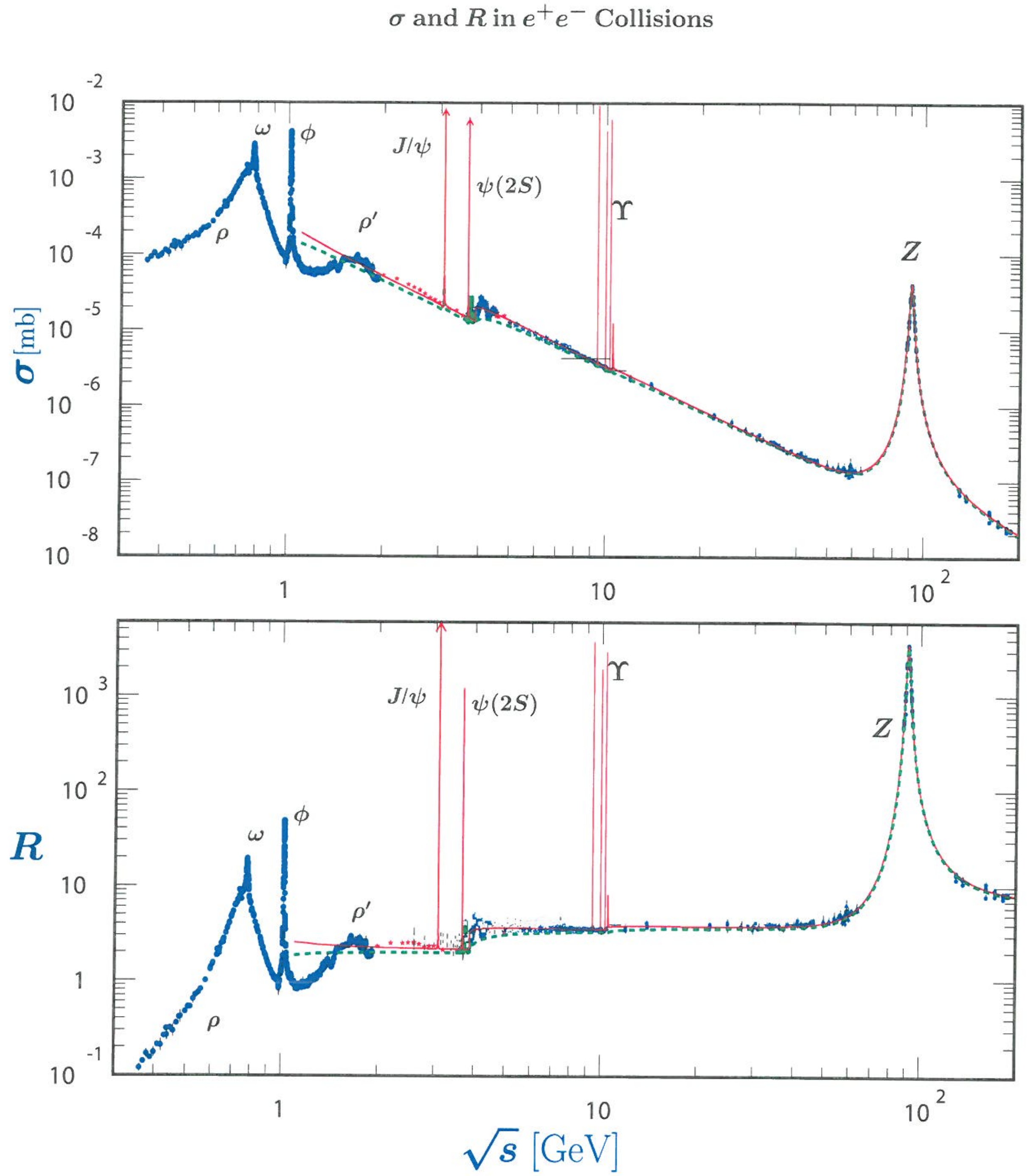


Figure 50.5: World data on the total cross section of $e^+e^- \to hadrons$ and the ratio $R(s) = \sigma(e^+e^- \to hadrons, s)/\sigma(e^+e^- \to \mu^+\mu^-, s)$. $\sigma(e^+e^- \to hadrons, s)$ is the experimental cross section corrected for initial state radiation and electron-positron vertex loops, $\sigma(e^+e^- \to \mu^+\mu^-, s) = 4\pi\alpha^2(s)/3s$. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are an educative guide: the broken one (green) is a naive quark-parton model prediction, and the solid one (red) is 3-loop pQCD prediction (see "Quantum Chromodynamics" section of this *Review*, Eq. (9.7) or, for more details, K. G. Chetyrkin *et al.*, Nucl. Phys. **B586**, 56 (2000) (Erratum *ibid.* **B634**, 413 (2002)). Breit-Wigner parameterizations of J/ψ, $\psi(2S)$, and $\Upsilon(nS), n = 1, 2, 3, 4$ are also shown. The full list of references to the original data and the details of the R ratio extraction from them can be found in `[arXiv:hep-ph/0312114]`. Corresponding computer-readable data files are available at `http://pdg.lbl.gov/current/xsect/`. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, May 2010.)

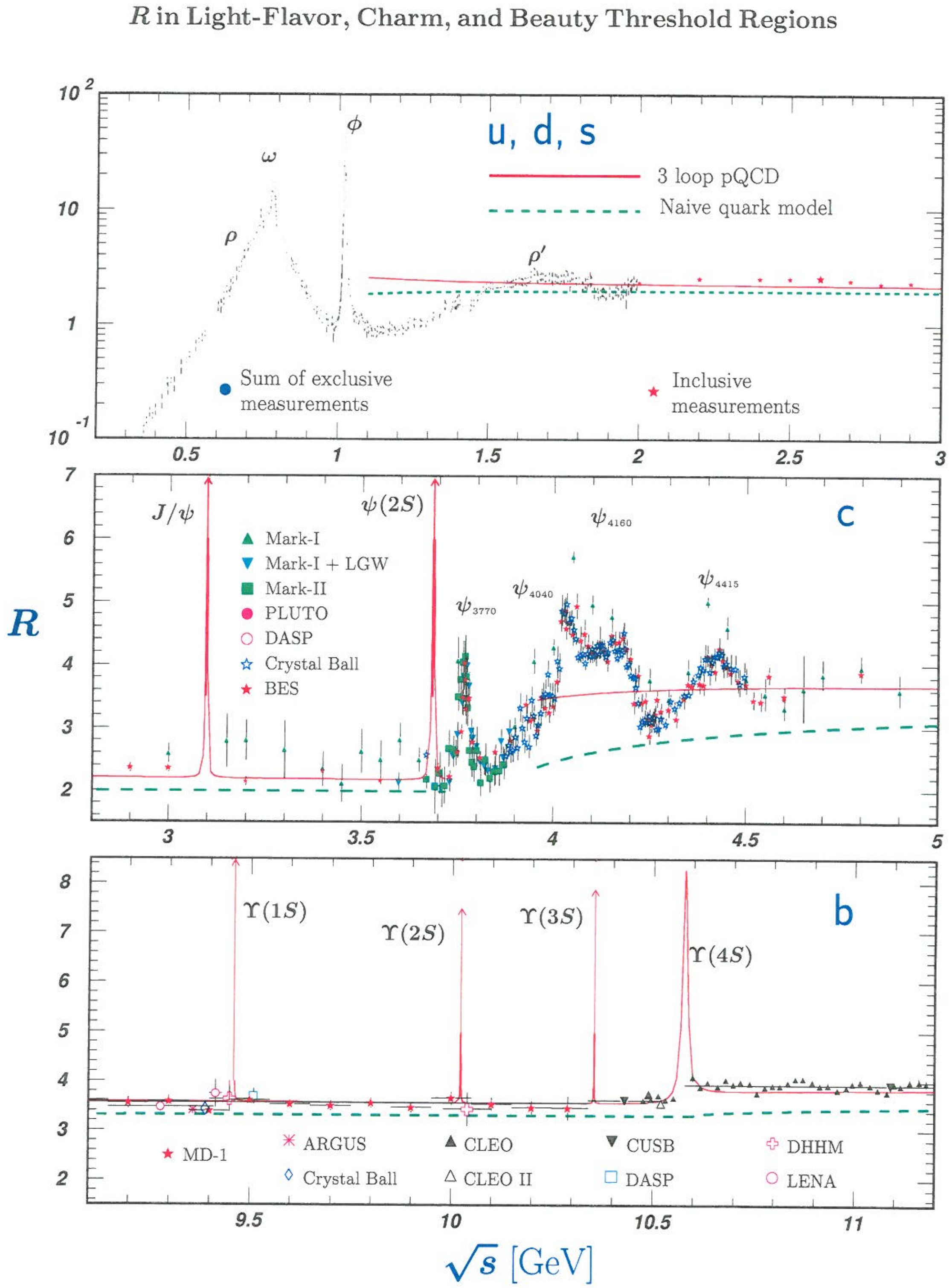


Figure 50.6: R in the light-flavor, charm, and beauty threshold regions. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are the same as in Fig. 50.5. **Note:** CLEO data above $\Upsilon(4S)$ were not fully corrected for radiative effects, and we retain them on the plot only for illustrative purposes with a normalization factor of 0.8. The full list of references to the original data and the details of the R ratio extraction from them can be found in `[arXiv:hep-ph/0312114]`. The computer-readable data are available at `http://pdg.lbl.gov/current/xsect/`. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, May 2010.)