12. Interaction picture

In this section, we want to clarify the physical meaning of the generating functional

$$Z[f] = \langle 0 | T e^{i\int d^dx\, f(x)\phi(x)} | 0 \rangle$$

as vacuum-to-vacuum transition amplitude $\langle 0\,\text{out} | 0\,\text{in} \rangle_f$ and explain the relation between path integral formulae and operator techniques in perturbation theory.

In the Heisenberg picture (used so far), the state vectors are time-independent and the operators (representing observables) are time-dependent. The expectation value of an observable A (represented by the Heisenberg operator $A_H(t)$) at time t in the (pure) state ψ_H is given by

$$\langle \psi_H | A_H(t) | \psi_H \rangle$$

and the equation of motion reads

$$\frac{dA_H(t)}{dt} = i\,[H_H(t), A_H(t)] + \frac{\partial A_H(t)}{\partial t}$$

<u>remark</u>: the term $\partial A_H(t)/\partial t$ only occurs for operators depending explicitly on time, for instance via an external source term

in the case where the Hamiltonian does not depend on time ($H_H(t) =: H$), the solution of the equation of motion for an observable with $\partial A_H(t)/\partial t = 0$ can be written as

$$A_H(t) = e^{iHt} A_H(0) e^{-iHt}$$

and the time dependence of the <u>expectation value</u> is given by

$$\langle \psi_H | e^{iHt} A_H(0) e^{-iHt} | \psi_H \rangle$$

which is the only relevant <u>physical</u> quantity

in the <u>Schrödinger picture</u>, the state vectors are <u>time-dependent</u> and the operators are <u>time-independent</u> (unless in the case of an <u>explicitly</u>

time-dependent observable)

in the case of constant H, A, Heisenberg picture and Schrödinger picture are related by

$$\psi_S(t) = e^{-iHt}\,\psi_H \qquad (\psi_S(0) = \psi_H)$$

and

$$A_S = A_H(0)$$

which leaves the expectation value of A unchanged:

$$\langle \psi_S(t) \,|\, A_S \,|\, \psi_S(t) \rangle$$

in the Schrödinger picture, the equation of motion is given by the Schrödinger equation

$$i \frac{d}{dt} \psi_S(t) = H \psi_S(t)$$

the form of this equation remains valid also in the case of a time-dependent Hamiltonian $H(t)$, however, the solution of the Schrödinger equation <u>cannot</u> be written in exponential form but as

$$\psi_S(t) = U(t)\, \psi_S(0)$$

with a <u>unitary</u> evolution operator $U(t)$ which fulfils the differential equation

$$i \frac{d}{dt} U(t) = H(t)\, U(t),$$

$$U(0) = \mathbb{1} \qquad \text{(initial condition)}$$

we can write this also as an integral equation:

$$U(t) = \mathbb{1} - i \int_0^t d\tau\, H(\tau)\, U(\tau)$$

this integral equation can be solved by iteration; the first step leads to

$$U(t) = \mathbb{1} - i \int_0^t d\tau_1\, H(\tau_1) \left(\mathbb{1} - i \int_0^{\tau_1} d\tau_2\, H(\tau_2)\, U(\tau_2) \right)$$

$$= \mathbb{1} - i \int_0^t d\tau_1\, H(\tau_1) + (-i)^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2\, H(\tau_1)\, H(\tau_2)\, U(\tau_2)$$

$$\Rightarrow \quad U(t) = \mathbb{1} - i \int_0^t d\tau_1 \, H(t_1)$$

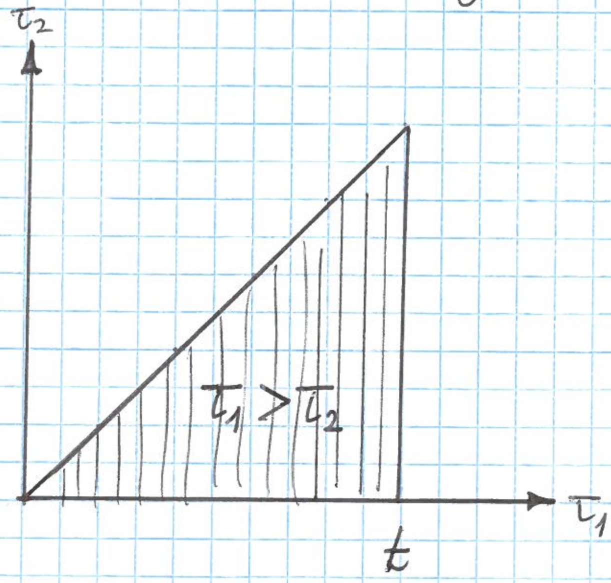
$$+ (-i)^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \; H(\tau_1)\, H(\tau_2)$$

$$+ \ldots$$

$$+ (-i)^n \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n \; H(\tau_1) \ldots H(\tau_n)$$

$$+ \ldots$$

$$= \sum_{n=0}^{\infty} (-i)^n \int_0^t d\tau_1 \ldots \int_0^{\tau_{n-1}} d\tau_n \; H(\tau_1) \ldots H(\tau_n)$$



$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \; H(\tau_1)\, H(\tau_2) = \frac{1}{2!} \int_0^t d\tau_1 \int_0^t d\tau_2 \; T\big(H(\tau_1)\, H(\tau_2)\big)$$

general case:

$$\int_0^t d\tau_1 \ldots \int_0^{\tau_{n-1}} d\tau_n \; H(\tau_1) \ldots H(\tau_n) =$$

$$= \frac{1}{n!} \int_0^t d\tau_1 \ldots \int_0^t d\tau_n \; T\big(H(\tau_1) \ldots H(\tau_n)\big)$$

$$\Rightarrow \; U(t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t d\tau_1 \ldots \int_0^t d\tau_n \; T\big(H(\tau_1) \ldots H(\tau_n)\big)$$

$$= T e^{-i \int_0^t d\tau \, H(\tau)}$$

<u>interaction picture</u>

we consider the situation where the Hamiltonian takes the form

$$H(t) = H_0 + H_1(t) \qquad \text{(Schrödinger picture)}$$

<u>examples</u>:

1) harmonic oscillator in the presence of an external time-dep. force:

$$H(t) = \underbrace{P^2/2m + m\omega^2 Q^2/2}_{H_0} \; \underbrace{- f(t)\, Q}_{H_1(t)}$$

2) (self-interacting) scalar field in the presence of an external source term

$$H = \underbrace{\int d^3x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right]}_{H_0} \underbrace{- \int d^3x\, f(t,\vec{x})\,\phi}_{H_1(t)}$$

the time evolution of the operators in the IP is given by

$$A_{IP}(t) = e^{iH_0 t} A_{IP}(0)\, e^{-iH_0 t}$$

the full time evolution of the expectation value of the observable A is given by

$$\langle \psi_H | U(t)^\dagger A_{IP}(t)\, U(t) | \psi_H \rangle$$

$$= \langle \psi_H | U(t)^\dagger e^{iH_0 t} A_{IP}(0)\, e^{-iH_0 t} U(t) | \psi_H \rangle$$

with an appropriately chosen unitary operator $U(t)$ (initial condition $U(0) = \mathbb{1}$)

the time evolution of $U(t)$ can be read off from the Schrödinger equation:

$$i \frac{d}{dt}\left[e^{-iH_0 t} U(t)\right] = \left[H_0 + H_1(t)\right] e^{-iH_0 t} U(t)$$

$$H_0 e^{-iH_0 t} U(t) + e^{-iH_0 t} i \frac{d}{dt} U(t) =$$

$$= H_0 e^{-iH_0 t} U(t) + H_1(t) e^{-iH_0 t} U(t)$$

$$\Rightarrow \quad i \frac{d}{dt} U(t) = \underbrace{e^{iH_0 t} H_1(t) e^{-iH_0 t}}_{H_{1,IP}(t)} U(t)$$

$$\Rightarrow \quad U(t) = T\, e^{-i \int_0^t d\tau\, H_{1,IP}(\tau)}$$

the time evolution in the Schrödinger picture and the one in the interaction picture are related by

$$|\psi_S(t)\rangle = e^{-iH_0 t} \overbrace{U(t) \underbrace{|\psi_S(0)\rangle}_{=|\psi_H\rangle}}^{|\psi_{IP}(t)\rangle}$$

or $$|\psi_{IP}(t)\rangle = e^{iH_0 t}\,|\psi_S(t)\rangle$$

Heisenberg picture and interaction picture are related by

$$|\psi_{IP}(t)\rangle = U(t)\,|\psi_H\rangle$$

$$\Updownarrow$$

$$|\psi_H\rangle = \underbrace{U(t)^{-1}}_{U(t)^{\dagger}}\,|\psi_{IP}(t)\rangle$$

we assume that the time-dependent term $H_1(t)$ vanishes for $t \to \pm\infty$ (the external force $f(t)$ tends to zero both in the remote past and the far future)

suppose that the motion starts in the ground state $|0\rangle$ of the unperturbed system ($H_0|0\rangle = 0$), i.e.

$$|\psi_S(-\infty)\rangle = \lim_{t\to-\infty} e^{-iH_0 t}\,|\psi_{IP}(t)\rangle = |0\rangle$$

being equivalent to

$$\lim_{t \to -\infty} |\psi_{IP}(t)\rangle = |0\rangle$$

or $$U(-\infty)\underbrace{|0, in\rangle}_{\text{Heisenberg state}} = |0\rangle$$

$$\Updownarrow$$

$$|0, in\rangle = T e^{i\int_0^{-\infty} d\tau\, H_{1,IP}(\tau)} |0\rangle$$

$$= T e^{-i\int_{-\infty}^{0} d\tau\, H_{1,IP}(\tau)} |0\rangle$$

in the course of time, the state vector moves away from the ground state,

$$|\psi_{IP}(t)\rangle = T e^{-i\int_{-\infty}^{t} d\tau\, H_{1,IP}(\tau)} |0\rangle,$$

until, for sufficiently large times, it again remaines put, because $H_1(t)$ tends to zero for $H_1 \equiv 0$, the vector $|\psi(+\infty)\rangle$ would coincide with the ground state of H_0, but in the presence of $H_1(t)$ it does in general not represent an eigenstate of H_0; the perturbation violates time-translation invariance, so that the energy fails to be conserved: the vector $|\psi(+\infty)\rangle$ does not coincide with the ground state, but may be represented as a superposition of energy eigenstates of H_0

the probability amplitude that the system is found in the ground state at $t \to \infty$ after the perturbation described by $H_1(t)$ is given by

$$\langle 0 | T e^{-i\int_{-\infty}^{+\infty} d\tau \, H_{1,IP}(\tau)} | 0 \rangle$$

using

$$|0, out\rangle = T\, e^{i\int_0^\infty d\tau\, H_{1,IP}(\tau)} |0\rangle ,$$

we may also write

$$\langle 0|\, T\, e^{-i\int_{-\infty}^{+\infty} d\tau\, H_{1,IP}(\tau)} |0\rangle$$

$$= \langle 0, out \,|\, 0, in\rangle_{H_1(t)}$$

which is referred to as the <u>vacuum-to-vacuum transition amplitude</u>

in our field theoretic model, the operator $H_{1,IP}(t)$ is the space integral of $-f(x)\,\phi(x)$, where $\phi(x)$ evolves according to H_0

$$\Rightarrow \langle 0\, out\,|\,0\, in\rangle_f = \langle 0|\, T\, e^{i\int d^dx\, f(x)\,\phi(x)} |0\rangle$$

Perturbation series in QFT

$$S[\varphi] = \underset{\substack{\uparrow \\ \text{bilinear in } \varphi}}{S_0[\varphi]} + S_{int}[\varphi]$$

Heisenberg op. — vacuum of full (interacting) theory

$$Z[f] = \langle 0 | T e^{i\int d^dx\, f(x)\, \phi_H(x)} | 0 \rangle$$

$$= \frac{\int [d\varphi]\, e^{iS[\varphi]}\, e^{i\int d^dx\, f(x)\varphi(x)}}{\int [d\varphi]\, e^{iS[\varphi]}} \quad \leftarrow \text{path integral representation}$$

$$= \frac{\int [d\varphi]\, e^{iS_0[\varphi]}\, e^{iS_{int}[\varphi]}\, e^{if\cdot\varphi}}{\int [d\varphi]\, e^{iS_0[\varphi]}} \times \frac{\int [d\varphi]\, e^{iS_0[\varphi]}}{\int [d\varphi]\, e^{iS_0[\varphi]}\, e^{iS_{int}[\varphi]}}$$

$$f\cdot\varphi := \int d^dx\, f(x)\,\varphi(x)$$

$$\Rightarrow \quad Z[f] = \frac{\langle\!\langle e^{iS_{int}[\varphi]}\, e^{if\cdot\varphi} \rangle\!\rangle}{\langle\!\langle e^{iS_{int}[\varphi]} \rangle\!\rangle}$$

Gaussian mean value $\quad \langle\!\langle \mathcal{F}[\varphi] \rangle\!\rangle := \dfrac{\int [d\varphi]\, e^{iS_0[\varphi]}\, \mathcal{F}[\varphi]}{\int [d\varphi]\, e^{iS_0[\varphi]}}$

perturbative expansion:

$$e^{iS_{int}[\varphi]} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(S_{int}[\varphi]\right)^n$$

translating back to <u>operator language</u>:

$$Z[f] = \langle 0 | T e^{i \int d^dx\, f(x)\, \phi_H(x)} | 0 \rangle$$

$$= \frac{(0 | T\, e^{i S_{int}[\phi_{IP}]}\, e^{i \int d^dx\, f(x)\, \phi_{IP}(x)} | 0)}{(0 | T e^{i S_{int}[\phi_{IP}]} | 0)}$$

$\uparrow$ vacuum of <u>free</u> theory (perturbative vacuum)

$$T e^{i S_{int}[\phi_{IP}]} = T e^{i \int d^dx\, \mathcal{L}_{int, IP}} = U(+\infty, -\infty)$$

$\rightarrow$ n-point function

$$\langle 0 | T\, \phi_H(x_1) \dots \phi_H(x_n) | 0 \rangle =$$

$$= \frac{(0 | T e^{i S_{int}[\phi_{IP}]}\, \phi_{IP}(x_1) \dots \phi_{IP}(x_n) | 0)}{(0 | T e^{i S_{int}[\phi_{IP}]} | 0)}$$

<u>remark:</u> $\langle 0 | T e^{i S_{int}[\phi_{IP}]} | 0 \rangle = e^{iL}$

is a phase factor (L is IR divergent) which <u>cancels</u> all diagrams with disconnected <u>vacuum bubbles</u>

$Z[f]$ related to $Z_0[f]$ by

$$Z[f] = \frac{1}{\mathcal{N}}\, e^{i S_{int}\left[\frac{1}{i}\frac{\delta}{\delta f}\right]}\, Z_0[f]$$

(perturbative) definition of functional integral of interacting theory in terms of

$$Z_0[f] = e^{+\frac{i}{2}\int d^dx\, d^dy\, f(x)\,\Delta(x-y)\, f(y)}$$

<u>remark:</u> $\mathcal{N}$ determined by normalization condition

$Z[0] = 1 \Rightarrow$

$$\mathcal{N} = e^{i S_{int}\left[\frac{1}{i}\frac{\delta}{\delta f}\right]}\, Z_0[f]\Big|_{f=0}$$