

11. Abelian gauge field

Quantum electrodynamics (QED) with a single charged Dirac field (e.g. e^-) described by

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} \bar{F}^{\mu\nu} F_{\mu\nu} + \bar{\psi} (i \not{D} - m) \psi - j^\mu A_\mu$$

$\bar{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ field strength tensor of
- photon field A_μ

$j^\mu = q \bar{\psi} j^\mu \psi$ electromagnetic current (density)

$q = \text{elm charge of Dirac field } \psi$

($q = -e$ for electron field)

remark: $\bar{\psi} (i \not{D} - m) \psi - j^\mu A_\mu =$

$$= \bar{\psi} \underbrace{[i j^\mu (\partial_\mu + iq A_\mu) - m]}_\text{covariant derivative} \psi$$

\mathcal{L}_{QED} is invariant under the local $U(1)$ gauge transformation $\psi(x) \rightarrow e^{-iqA(x)} \psi(x)$

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x)$$

M/2

$$\partial_\mu \psi(x) \rightarrow \partial_\mu (e^{-iq\Lambda(x)} \psi(x)) =$$

$$= e^{-iq\Lambda(x)} (\partial_\mu \psi(x) - iq \partial_\mu \Lambda(x) \psi(x))$$

$$iq A_\mu(x) \psi(x) \rightarrow iq (A_\mu(x) + \partial_\mu \Lambda(x)) e^{-iq\Lambda(x)} \psi(x)$$

$$\Rightarrow D_\mu \psi := (\partial_\mu + iq A_\mu) \psi \rightarrow e^{-iq\Lambda(x)} D_\mu \psi$$

$$\Rightarrow \bar{\psi} i \not{D} \psi \rightarrow \bar{\psi} i \not{D} \psi \quad \text{invariant}$$

$$\bar{\psi} \psi, F_{\mu\nu} F^{\mu\nu} \quad \text{---} \quad \text{---}$$

$$\Rightarrow \mathcal{L}_{QED} \text{ invariant}$$

$U(1) = \underline{\text{abelian}} \text{ gauge group}$

$QED = \underline{\text{abelian}} \underline{\text{gauge theory}}$

still to be determined: photon propagator

consider elm. field in the presence of
a classical current density \mathcal{J}^μ ($\partial_\mu \mathcal{J}^\mu = 0$)

→ generating functional

$$Z[J] = \frac{1}{N} \int [dA^\mu] e^{i \int d^4x (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu)}$$

$$S = \int d^4x (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu)$$

$$= \int d^4x [-\frac{1}{2} A^\mu (-g_{\mu\nu} (\square - i\varepsilon) + \partial_\mu \partial_\nu) A^\nu - J_\mu A^\mu]$$

↑
partial
integration

we could try again the usual trick: shift of variables $A^\mu = A'^\mu + B^\mu$, where B^μ should fulfil

$$(g_{\mu\nu} (\square - i\varepsilon) - \partial_\mu \partial_\nu) B^\nu = J_\mu$$

→ propagator $D^{\nu S}(x) =$ Green function of the differential operator $g_{\mu\nu} (\square - i\varepsilon) - \partial_\mu \partial_\nu$:

$$(g_{\mu\nu} (\square - i\varepsilon) - \partial_\mu \partial_\nu) D^{\nu S}(x) = \delta_\mu^S \delta^{(4)}(x)$$

Fourier representation $D^{\nu\sigma}(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \tilde{D}^{\nu\sigma}(k)$

→ in momentum space, we get the equation

$$(-g_{\mu\nu} k^2 + k_\mu k_\nu) \tilde{D}^{\nu\sigma}(k) = \delta_\mu^\sigma$$

we discuss the more general problem :

find the inverse of

$$T_{\mu\nu} = A(k^2) k_\mu k_\nu + B(k^2) g_{\mu\nu}$$

ansatz for $(T^{-1})^{\nu\sigma}$:

$$(T^{-1})^{\nu\sigma} = A(k^2) k^\nu k^\sigma + B(k^2) g^{\nu\sigma}$$

$$\Rightarrow (A k_\mu k_\nu + B g_{\mu\nu}) (A k^\nu k^\sigma + B g^{\nu\sigma}) = \delta_\mu^\sigma$$

$$A A k^2 k_\mu k^\sigma + A B k_\mu k^\sigma + B A k_\mu k^\sigma$$

$$+ B B g_\mu^\sigma = \delta_\mu^\sigma$$

$$\Rightarrow B = \frac{1}{\beta}, \quad \alpha A k^2 + \alpha B + \beta A = 0$$

$$A (\alpha k^2 + \beta) = -\frac{\alpha}{\beta} \Rightarrow A = -\frac{\alpha}{\beta} \frac{1}{\alpha k^2 + \beta^2}$$

$$\Rightarrow (T^{-1})^{\nu\sigma} = -\frac{\alpha k^\nu k^\sigma}{\beta(\alpha k^2 + \beta)} + \frac{-g^{\nu\sigma}}{\beta}$$

$\Rightarrow T^{-1}$ exists if $\alpha k^2 + \beta \neq 0$ and $\beta \neq 0$

in the case of the photon field we have:

$$\alpha = 1, \quad \beta = -k^2 \Rightarrow \alpha k^2 + \beta = 0$$

\Rightarrow inverse of $(-g_{\mu\nu} k^2 + k_\mu k_\nu)$ does
not exist

$$\text{indeed: } (-g_{\mu\nu} k^2 + k_\mu k_\nu) k^\nu = 0 \quad (k^\nu \neq 0)$$

can already be seen in the space-time representation:

$$(g_{\mu\nu} \square - \partial_\mu \partial_\nu) \partial^\nu \Lambda = \square \partial_\mu \Lambda - \partial_\mu \square \Lambda = 0$$

way out: gauge fixing term in the Lagrangian
(Fermi's trick)

$$\mathcal{L} \rightarrow \mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \underbrace{\frac{\xi}{2} (\partial_\mu A^\mu)^2}_{\text{gauge fixing term}} - \mathcal{J}_\mu A^\mu$$

gauge parameter

the action can now be written in the form

$$S = \int d^4x \left\{ \frac{1}{2} A^\mu [g_{\mu\nu} \square - (1-\xi) \partial_\mu \partial_\nu] A^\nu - \mathcal{J}_\mu A^\mu \right\}$$

$$\text{the equation } [g_{\mu\nu} \square - (1-\xi) \partial_\mu \partial_\nu] B^\nu = \mathcal{J}_\mu$$

can now be inverted:

$$T_{\mu\nu} = -R^2 g_{\mu\nu} + (1-\xi) R_{\mu\nu}$$

$$\alpha = 1 - \xi, \quad \beta = -R^2$$

$$\Rightarrow (T^{-1})^{\mu\nu} = -\frac{1}{R^2} \left[g^{\mu\nu} - (1 - \frac{1}{\xi}) \frac{R^2 R^2}{R^2} \right]$$

11/7

Feynman gauge ($\xi = 1$): $(T^{-1})^{\mu\rho} = -\frac{g^{\mu\rho}}{k^2}$

$$D^{\nu\rho}(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \underbrace{\frac{-1}{k^2 + i\varepsilon} \left[g^{\nu\rho} - (1 - \frac{1}{\xi}) \frac{k^\nu k^\rho}{k^2} \right]}_{\tilde{D}^{\nu\rho}(k)}$$

$$S_{\text{eff}} = \int d^4x \left\{ \frac{1}{2} A^\mu [g_{\mu\nu}(\square - i\varepsilon) - (1 - \xi) \partial_\mu \partial_\nu] A^\nu - \bar{J}_\mu A^\mu \right\}$$

$$\begin{aligned} A_\mu &= A'_\mu + B_\mu \\ &\stackrel{!}{=} \int d^4x \left\{ \frac{1}{2} A'^\mu [g_{\mu\nu}(\square - i\varepsilon) - (1 - \xi) \partial_\mu \partial_\nu] A'^\nu \right. \\ &\quad + A'^\mu [g_{\mu\nu}(\square - i\varepsilon) - (1 - \xi) \partial_\mu \partial_\nu] B^\nu - \bar{J}_\mu A'^\mu \\ &\quad \left. + \frac{1}{2} B^\mu [g_{\mu\nu}(\square - i\varepsilon) - (1 - \xi) \partial_\mu \partial_\nu] B^\nu - \bar{J}_\mu B^\mu \right\} \end{aligned}$$

$$[g_{\mu\nu}(\square - i\varepsilon) - (1 - \xi) \partial_\mu \partial_\nu] B^\nu = \bar{J}_\mu$$

i.e. $B^\mu(x) = \int d^4y D^{\mu\nu}(x-y) \bar{J}_\nu(y)$

→ terms linear in A'^μ disappear

M8

$$\Rightarrow S_{\text{eff}} = \int d^4x \left\{ \frac{1}{2} A'^\mu [g_{\mu\nu} (\square - i\varepsilon) - (1-\xi) \partial_\mu \partial_\nu] A'^\nu - \frac{1}{2} \bar{J}_\mu B^\mu \right\}$$

$$= \int d^4x \frac{1}{2} A'^\mu [g_{\mu\nu} (\square - i\varepsilon) - (1-\xi) \partial_\mu \partial_\nu] A'^\nu - \frac{1}{2} \int d^4x d^4y D^{\mu\nu}(x-y) \bar{J}_\mu(x) J_\nu(y)$$

$$\Rightarrow Z[J] = e^{-\frac{i}{2} \int d^4x d^4y D^{\mu\nu}(x-y) \bar{J}_\mu(x) J_\nu(y)}$$

remark: $Z[J]$ is independent of the gauge parameter ξ for a conserved current ($\partial^\mu J_\mu = 0$): to see this, we compute

$$\int d^4y D^{\mu\nu}(x-y) \bar{J}_\nu(y)$$

and convince ourselves that this expression is independent of ξ if $\partial^\mu J_\mu = 0$

11/9

$$J_\nu(y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipy} \tilde{J}_\nu(p)$$

$$\partial^\nu J_\nu(y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipy} (-ip^\nu) \tilde{J}_\nu(p)$$

$$\partial^\nu J_\nu = 0 \Leftrightarrow p^\nu \tilde{J}_\nu(p) = 0$$

$$\Rightarrow \int d^4 y D^{\mu\nu}(x-y) J_\nu(y) =$$

$$= \int d^4 y \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-y)} \frac{-1}{k^2 + i\varepsilon} [g^{\mu\nu} - (1-\frac{1}{\xi}) \frac{k^\mu k^\nu}{k^2}]$$

$$\times \int \frac{d^4 p}{(2\pi)^4} e^{-ipy} \tilde{J}_\nu(p)$$

$$= \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \frac{-1}{k^2 + i\varepsilon} [g^{\mu\nu} - (1-\frac{1}{\xi}) \frac{k^\mu k^\nu}{k^2}] \tilde{J}_\nu(k)$$

$$= \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \frac{-g^{\mu\nu}}{k^2 + i\varepsilon} \tilde{J}_\nu(k)$$

$$k^\nu \tilde{J}_\nu(k) = 0$$

11/10

$$\Rightarrow \int d^4x d^4y D^{\mu\nu}(x-y) J_\mu(x) J_\nu(y)$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{-\tilde{J}^\mu(k)}{k^2 + i\varepsilon} \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \tilde{J}^\nu(p)$$

$$= - \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}^\mu(k) \tilde{J}_\mu(-k)}{k^2 + i\varepsilon}$$

$$J^\mu(x) \text{ real} \Rightarrow \tilde{J}^\mu(-k) = \tilde{J}^\mu(k)^*$$

$$\Rightarrow \int d^4x d^4y D^{\mu\nu}(x-y) J_\mu(x) J_\nu(y)$$

$$= - \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}^\mu(k) \tilde{J}_\mu(k)^*}{k^2 + i\varepsilon}$$

$$\Rightarrow \mathbb{E}[J] = \langle 0_{\text{out}} | 0_{\text{in}} \rangle_J = e^{\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}^\mu(k) \tilde{J}_\mu(k)^*}{k^2 + i\varepsilon}}$$

- path integral quantization of
an abelian gauge field

reminder : $S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$

$$= \frac{1}{2} \int d^4x A^\mu \underbrace{(g_{\mu\nu} \square - \partial_\mu \partial_\nu)}_{\text{not invertible}} A^\nu$$

→ usual "trick" (shift of integration variables) not applicable

reason : $S[A_\mu + \partial_\mu \lambda] = S[A_\mu]$

(action invariant under abelian gauge transformations)

in the expression

$$\langle O | T O[A] | O \rangle = \frac{\int [dA] e^{iS[A]} O[A]}{\int [dA] e^{iS[A]}}$$

↑
gauge invariant
observable

integration performed also over gauge-equivalent field configurations

goal: we want to factorize out the redundant part of the integration measure

→ Faddeev - Popov Quantization

possible procedure:

$$\Delta[A, \omega] \int [d\lambda] S(\partial_\mu A^\mu + \square \lambda - \omega) = 1$$

this equation defines $\Delta[A, \omega]$

remark: $\square \rightarrow \square + i\varepsilon$ tacitly assumed to ensure uniqueness of the solution of

$$\partial^\mu A_\mu + (\square + i\varepsilon) \lambda - \omega = 0$$

$$\rightarrow \lambda = \frac{1}{\square + i\varepsilon} (\omega - \partial_\mu A^\mu)$$

11/13

$$\Delta[A, \omega]^{-1} = \int [d\lambda] S(\partial_\mu A + \square \lambda - \omega)$$

$$= \Delta[0, \omega - \partial_\mu A]^{-1}$$

transformation of the integration variable

$$(\text{shift}) : \lambda = \lambda' + \mu$$

↑
new variable of integration

$$\rightarrow \Delta[A, \omega]^{-1} = \int [d\lambda'] S(\partial_\mu A + \square \lambda' + \square \mu - \omega)$$

$$= \Delta[A, 0]^{-1} \quad (\text{by an appropriate choice of } \mu)$$

$\Rightarrow \Delta[A, \omega] = \Delta[0, 0]$ is independent

of A_μ, ω (this is not the case for nonabelian gauge theories)

this result can also be seen in the following way:

$$\int [d\varphi] \mathcal{S}(\varphi) = 1$$

variable transformation $\varphi = \partial \cdot A + (\square + i\varepsilon) \lambda - \omega$

$$\rightarrow \text{unique mapping } \lambda = \frac{1}{\square + i\varepsilon} (\varphi + \omega - \partial \cdot A)$$

$$\int [d\lambda] \left(\det \frac{\delta \varphi(x)}{\delta \lambda(y)} \right) \mathcal{S}(\partial \cdot A + (\square + i\varepsilon) \lambda - \omega) = 1$$

$$\frac{\delta \varphi(x)}{\delta \lambda(y)} = (\square + i\varepsilon) \delta(x-y)$$

$$\Rightarrow \Delta[A, \omega] = \det(\square + i\varepsilon) \underset{\text{independent}}{\underline{\text{of }}} A, \omega$$

11/15

$$\rightarrow \langle 0 | T \mathcal{O}[A] | 0 \rangle$$

$$= \frac{\int [dA d\lambda] \Delta[0,0] \delta(\partial \cdot A + \square \lambda - \omega) e^{iS[A]} \mathcal{O}[A]}{\int [dA d\lambda] \Delta[0,0] \delta(\partial \cdot A + \square \lambda - \omega) e^{iS[A]}}$$

1

$$= \frac{\int [dA d\lambda] \delta(\partial \cdot A + \square \lambda - \omega) e^{iS[A]} \mathcal{O}[A]}{\int [dA d\lambda] \delta(\partial \cdot A + \square \lambda - \omega) e^{iS[A]}}$$

variable transformation $A_\mu = A'_\mu + \partial_\mu \Lambda$

(gauge transformation)

$$S[A] = S[A'] , \quad \mathcal{O}[A] = \mathcal{O}[A']$$

(gauge invariant quantities)

$\rightarrow \lambda$ disappear by an appropriate choice of Λ

$$\rightarrow \langle 0 | T \mathcal{O}[A] | 0 \rangle = \frac{\int [d\lambda] \int [dA] \delta(\partial \cdot A - \omega) e^{iS[A]} \mathcal{O}[A]}{\int [d\lambda] \int [dA] \delta(\partial \cdot A - \omega) e^{iS[A]}}$$

desired factorization $\xrightarrow{\hspace{1cm}}$

11/16

numerator and denominator independent

of $\omega \rightarrow$ integrate these expressions over a weight function

$$\langle O | T O[A] | O \rangle =$$

$$= \frac{\int [d\omega] e^{-\frac{i\varepsilon}{2} \int d^4x \omega(x)^2} \int [dA] \delta(\partial \cdot A - \omega) e^{iS[A]} O[A]}{\int [d\omega] e^{-\frac{i\varepsilon}{2} \int d^4x \omega(x)^2} \int [dA] \delta(\partial \cdot A - \omega) e^{iS[A]}}$$

$$= \frac{\int [dA] e^{i[S[A] - \int d^4x \frac{\varepsilon}{2} (\partial \cdot A)^2]}}{\int [dA] e^{i[S[A] - \int d^4x \frac{\varepsilon}{2} (\partial \cdot A)^2]}}$$

$$\rightarrow S_{\text{eff}}[A] = S[A] - \frac{\varepsilon}{2} \int d^4x (\partial \cdot A)^2$$