10. Fermionic path integral

in the case of bosonic variables, the Gaussian mean-value could be interpreted as the limit of (ordinary) integrals of the form

$$\ll F(\varphi) \gg = \frac{\int d\varphi_1 \dots d\varphi_N \, e^{-\frac{1}{2}\varphi^T A \varphi} \, F(\varphi)}{\int d\varphi_1 \dots d\varphi_N \, e^{-\frac{1}{2}\varphi^T A \varphi}}$$

$$\varphi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{bmatrix} \quad , \quad A^T = A$$

$$\ll \varphi_{k_1} \dots \varphi_{k_n} \gg = \ll \varphi_{k_{\sigma(1)}} \dots \varphi_{k_{\sigma(n)}} \gg \qquad \sigma \in S_n$$

$$= \begin{cases} 0 & \text{if } n \text{ odd} \\ \sum_{\text{pairings}} \ll \varphi_{k_1'} \varphi_{k_2'} \gg \dots \ll \varphi_{k_{n-1}'} \varphi_{k_n} \gg & \text{if } n \text{ even} \end{cases}$$

fermionic variables

anticommuting objects $\chi_1, \dots, \chi_N$

$$\{\chi_k, \chi_\ell\} = 0$$

elements of a Grossmann algebra

Gaussian mean value of Grassmann variables:

$$\ll \chi_{k_1} \dots \chi_{k_n} \gg = (-1)^\sigma \ll \chi_{k_{\sigma(1)}} \dots \chi_{k_{\sigma(n)}} \gg \quad \sigma \in S_n$$

$$(-1)^\sigma = \begin{cases} 1 & \text{if } \sigma \text{ even permutation of } 1, \dots, n \\ -1 & \text{-"- -"- odd -"- -"- -"-} \end{cases}$$

$$\ll \chi_{k_1} \dots \chi_{k_n} \gg = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sum\limits_{\text{pairings}} (-1)^\sigma \ll \chi_{k_1'} \chi_{k_2'} \gg \dots \ll \chi_{k_{n-1}'} \chi_{k_n'} \gg \end{cases}$$

$$(-1)^\sigma = \begin{cases} +1 & \text{if } k_1', \dots, k_n' \text{ even permutation of} \\ -1 & \text{-"- -"- odd -"- -"-} \end{cases} k_1, \dots, k_n$$

Gaussian mean value of fermionic variables cannot be represented by ordinary integrals but by "integrals"

$$\int \underbrace{d\chi_1 \dots d\chi_N}_{[d\chi]} f(\chi_1, \dots, \chi_N)$$

over the Grassmann variables $\chi_1, \dots, \chi_N$, which are not defined by a limiting process

(like the Riemann- or Lebesgue-integral) but are defined implicitly in terms of their properties (essential property: integration measure is <u>translation invariant</u> $\Rightarrow \int d\chi = 0$, $\int d\chi\, \chi = \text{const.} (=1)$)

↑ single variable

we consider the integral

$$I = \int [d\chi] \exp\left(-\frac{1}{2} D^{kl} \chi_k \chi_l + f^k \chi_k\right)$$

$D^{kl} = -D^{lk}$ is an anti-symmetric matrix, the f^k are also Grassmann variables:

$$\{\chi_k, f^l\} = \{f^k, f^l\} = 0$$

change of variables $\chi_k = \chi_k' + c_k \qquad [d\chi] = [d\chi']$

$$I = \int [d\chi'] \exp\left(-\frac{1}{2} D^{kl} (\chi_k' + c_k)(\chi_l' + c_l) + f^k (\chi_k' + c_k)\right)$$

$$= \exp\left(-\frac{1}{2} D^{kl} c_k c_l + f^k c_k\right)$$

$$\times \int [d\chi'] \exp\left(-\frac{1}{2} D^{kl} \chi_k' \chi_l' - \frac{1}{2} D^{kl} \chi_k' c_l - \frac{1}{2} D^{kl} c_k \chi_l' + f^k \chi_k'\right)$$

we choose $c_k = -D^{-1}_{kl} f^l \Rightarrow$ terms linear in χ' disappear

$$\Rightarrow \quad I = \exp\left(-\frac{1}{2} D^{-1}_{kl} f^k f^l\right) \underbrace{\int [d\chi] \exp\left(-\tfrac{1}{2} D^{kl} \chi_k \chi_l\right)}_{I_0}$$

Gaussian mean value defined by

$$\ll F(\chi) \gg = \frac{\int [d\chi] \exp\left(-\frac{1}{2} D^{kl} \chi_k \chi_l\right) F(\chi)}{\int [d\chi] \exp\left(-\frac{1}{2} D^{kl} \chi_k \chi_l\right)}$$

$$\Rightarrow \quad \ll \exp(f^k \chi_k) \gg = \exp\left(-\frac{1}{2} D^{-1}_{kl} f^k f^l\right)$$

expansion of exponential on both sides $\Rightarrow$

$$1 + \ll f^k \chi_k \gg + \frac{1}{2!} \ll \underbrace{f^k \chi_k \, f^l \chi_l}_{- f^k f^l \chi_k \chi_l} \gg + \ldots$$

$$= 1 - \frac{1}{2} D^{-1}_{kl} f^k f^l + \ldots$$

$$\Rightarrow \quad \ll \chi_k \gg = 0$$

$$\Rightarrow \quad \ll \chi_k \chi_l \gg = D^{-1}_{kl}$$

$\ldots \Rightarrow$ pairing rule (see above)

special case:

$$D = \begin{pmatrix} 0 & -B^T \\ B & 0 \end{pmatrix}, \quad \chi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \quad f = \begin{pmatrix} \bar{\eta} \\ -\eta \end{pmatrix}$$

$$-\frac{1}{2} D^{kl} \chi_k \chi_l + f^k \chi_k = -\frac{1}{2} \chi^T D \chi + f^T \chi =$$

$$= -\frac{1}{2} \quad (\psi^T, \bar{\psi}^T) \begin{pmatrix} 0 & -B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

$$+ (\bar{\eta}^T, -\eta^T) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

$$= -\frac{1}{2} \quad (\psi^T, \bar{\psi}^T) \begin{pmatrix} -B^T \bar{\psi} \\ B \psi \end{pmatrix} + \bar{\eta}^T \psi - \eta^T \bar{\psi}$$

$$= +\frac{1}{2} \psi_\alpha B^{\beta\alpha} \bar{\psi}_\beta - \frac{1}{2} \bar{\psi}_\alpha B^{\alpha\beta} \psi_\beta + \bar{\eta}^\alpha \psi_\alpha + \bar{\psi}_\alpha \eta^\alpha$$

$$= -\bar{\psi}_\alpha B^{\alpha\beta} \psi_\beta + \bar{\eta}^\alpha \psi_\alpha + \bar{\psi}_\alpha \eta^\alpha$$

$\psi_\alpha, \bar{\psi}_\alpha$ treated as independent variables; <u>no</u> (anti-) symmetry condition for $B^{\alpha\beta}$

$$D^{-1} = \begin{pmatrix} 0 & B^{-1} \\ -(B^T)^{-1} & 0 \end{pmatrix}$$

$$\Rightarrow \; -\frac{1}{2} f^k D^{-1}_{kl} f^l = -\frac{1}{2} (\bar{\eta}^T, -\eta^T) \begin{pmatrix} 0 & B^{-1} \\ -(B^T)^{-1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\eta} \\ -\eta \end{pmatrix}$$

$$= -\frac{1}{2} (\bar{\eta}^T, -\eta^T) \begin{pmatrix} -B^{-1}\eta \\ -(B^T)^{-1}\bar{\eta} \end{pmatrix}$$

$$= \frac{1}{2} \left[\bar{\eta}^T B^{-1} \eta - \eta^T (B^T)^{-1} \bar{\eta} \right]$$

$$= \frac{1}{2} \left[\bar{\eta}^\alpha B^{-1}_{\alpha\beta} \eta^\beta - \eta^\beta B^{-1}_{\alpha\beta} \bar{\eta}^\alpha \right]$$

$$= \bar{\eta}^\alpha B^{-1}_{\alpha\beta} \eta^\beta$$

$$\Rightarrow \; I = \int [d\psi \, d\bar{\psi}] \exp\left(-\bar{\psi}_\alpha B^{\alpha\beta} \psi_\beta + \bar{\eta}^\alpha \psi_\alpha + \bar{\psi}_\alpha \eta^\alpha \right)$$

$$= I_0 \exp\left(\bar{\eta}^\alpha B^{-1}_{\alpha\beta} \eta^\beta \right)$$

$$\Rightarrow \; \ll \psi_\alpha \bar{\psi}_\beta \gg = B^{-1}_{\alpha\beta}$$

$$\ll \psi_\alpha \psi_\beta \gg = \ll \bar{\psi}_\alpha \bar{\psi}_\beta \gg = 0$$

generating functional of the free Dirac field

$$Z[\eta, \bar{\eta}] =$$

field operators

$$= \langle 0 | T \exp\left\{ i \int d^4x \left[\bar{\eta}(x) \Psi(x) + \bar{\Psi}(x) \eta(x) \right] \right\} | 0 \rangle$$

fermionic sources

$$= \frac{1}{\mathcal{N}} \int [d\psi \, d\bar{\psi}] \exp\left\{ i \int d^4x \left[\bar{\psi}(x)(i\not{\partial} - m + i\varepsilon)\psi(x) + \bar{\eta}(x)\psi(x) + \right.\right.$$

$$\left.\left. + \bar{\psi}(x)\eta(x) \right] \right\}$$

$\psi, \bar{\psi}, \eta, \bar{\eta}$ Grassmann fields

$[d\psi \, d\bar{\psi}]$ translation-invariant measure

normalization $Z[0, 0] = 1$

computation of $Z[\eta, \bar{\eta}]$:

shift $\psi = \psi' + \chi$

$\bar{\psi} = \bar{\psi}' + \bar{\chi}$

$$S = \int d^4x \left[\bar{\psi} \left(i\not{\partial} - m + i\varepsilon \right) \psi + \bar{\eta}\psi + \bar{\psi}\eta \right]$$

$$= \int d^4x \left[\left(\bar{\psi}' + \bar{\chi} \right) \left(i\not{\partial} - m + i\varepsilon \right) \left(\psi' + \chi \right) \right.$$

$$\left. + \bar{\eta} \left(\psi' + \chi \right) + \left(\bar{\psi}' + \bar{\chi} \right) \eta \right]$$

$$= \int d^4x \left[\bar{\psi}' \left(i\not{\partial} - m + i\varepsilon \right) \psi' \right. \qquad \sim \bar{\psi}' \ldots \psi'$$

$$+ \bar{\chi} \left(i\not{\partial} - m + i\varepsilon \right) \psi' + \bar{\eta}\psi' \qquad \sim \psi'$$

$$+ \bar{\psi}' \left(i\not{\partial} - m + i\varepsilon \right) \chi + \bar{\psi}'\eta \qquad \sim \bar{\psi}'$$

$$\left. + \underbrace{\bar{\chi} \left(i\not{\partial} - m + i\varepsilon \right) \chi + \bar{\eta}\chi + \bar{\chi}\eta}_{\text{terms without } \psi' \text{ or } \bar{\psi}'} \right]$$

terms linear in ψ' or $\bar{\psi}'$ should disappear

→ appropriate choice of χ and $\bar{\chi}$:

$$\left(i\not{\partial} - m + i\varepsilon \right) \chi + \eta = 0 \quad \Rightarrow \quad \chi = \frac{1}{m - i\not{\partial} - i\varepsilon} \eta$$

$$\bar{\chi} \left(i\overleftarrow{\not{\partial}} - m + i\varepsilon \right) + \bar{\eta} = 0 \qquad \text{(after partial integration)}$$

$$\Rightarrow \quad S = \int d^4x \; [\, \bar{\psi}' (i\not{\partial} - m + i\varepsilon)\, \psi' + \bar{\eta}\, \chi \,]$$

$$= \int d^4x \; [\, \bar{\psi}' (i\not{\partial} - m + i\varepsilon)\, \psi' + \bar{\eta} \frac{1}{m - i\not{\partial} - i\varepsilon}\, \eta \,]$$

$$\Rightarrow \quad Z[\eta, \bar{\eta}\,] = \exp\left\{ \int d^4x \; \bar{\eta}(x) \frac{i}{m - i\not{\partial} - i\varepsilon}\, \eta(x) \right\}$$

$$(m - i\not{\partial}_x - i\varepsilon)\, S(x-y) = \delta^{(4)}(x-y)$$

$$\Rightarrow \quad (m - i\not{\partial}_x - i\varepsilon) \int d^4y \, S(x-y)\, \eta(y) = \eta(x)$$

$$\Rightarrow \quad \frac{1}{m - i\not{\partial}_x - i\varepsilon}\, \eta(x) = \int d^4y \;\; S(x-y)\, \eta(y)$$

$$\Rightarrow \quad Z[\eta, \bar{\eta}\,] = e^{\, i \int d^4x\, d^4y \; \bar{\eta}(x)\; S(x-y)\, \eta(y)}$$

$\Rightarrow$ pairing rules shown on p. 9/11