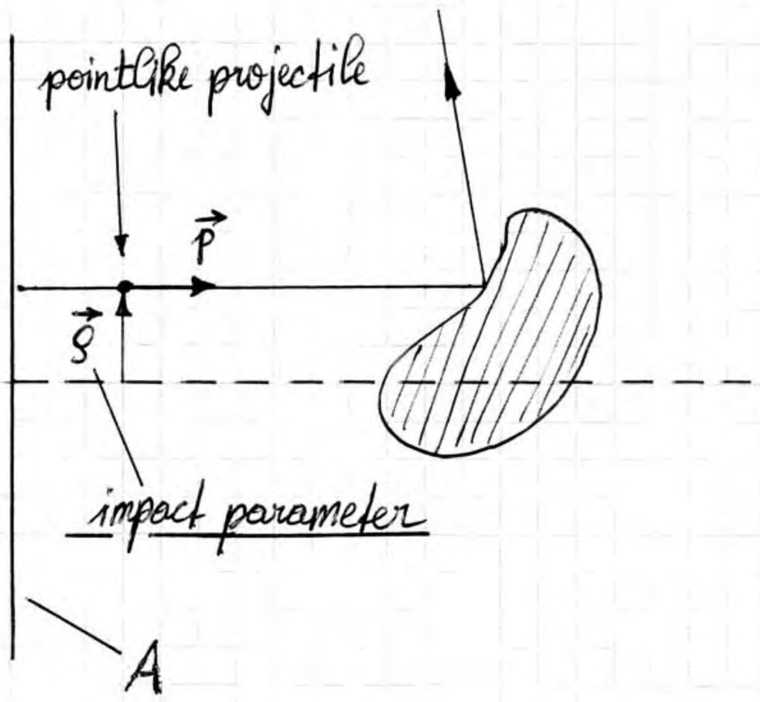
8. Cross section

idea:



we repeat this experiment N times with randomly distributed impact parameter $\vec{\varrho}$ (area A larger than dimensions of the object)

N_{sc} = number of scattered particles

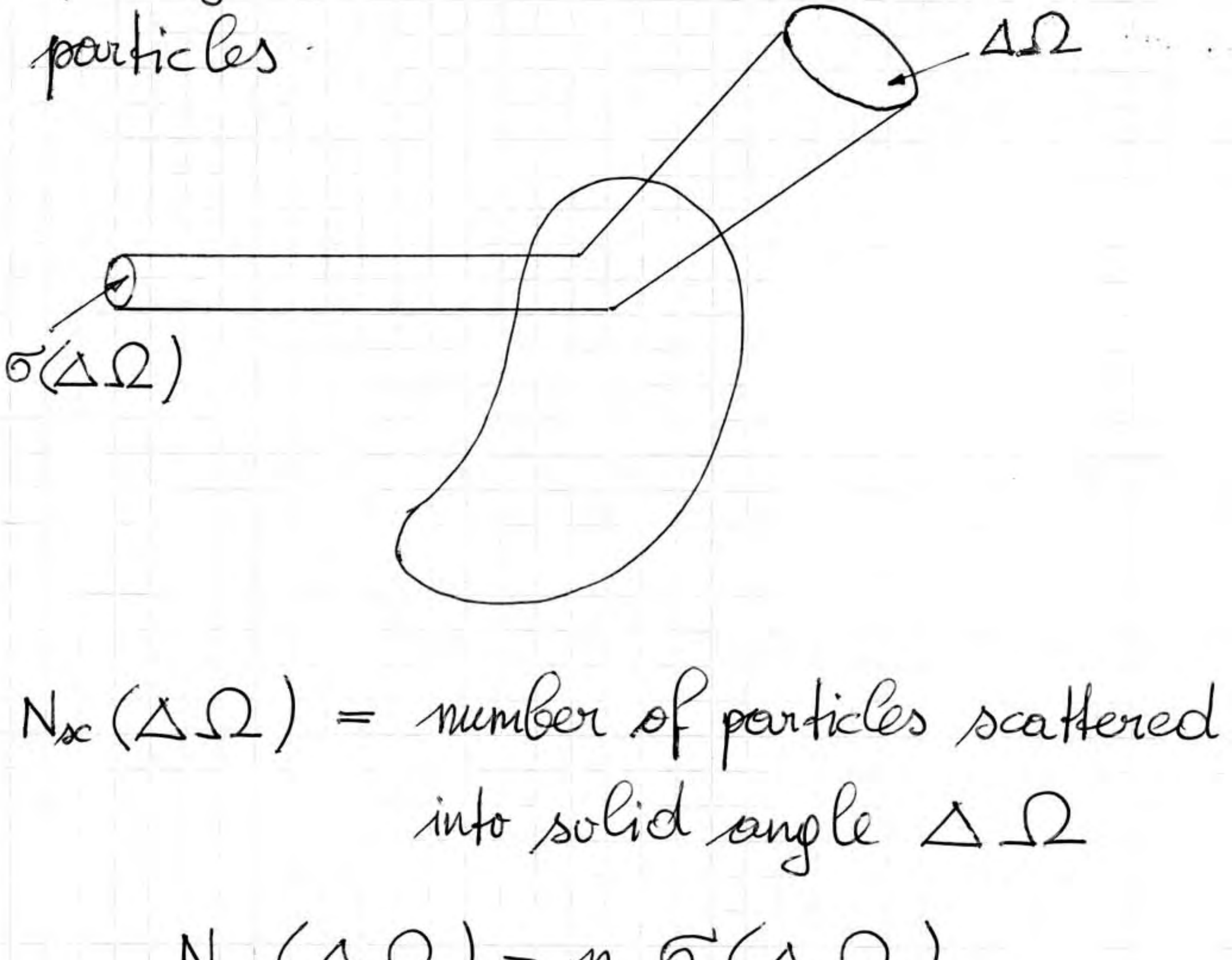
$$\Rightarrow \quad \frac{N_{sc}}{N} = \frac{\sigma}{A}$$

σ = cross section (in this case geometric cross section of object $\perp \vec{p}$)

$$N_{sc} = \underbrace{\frac{N}{A}}_{n}\, \sigma$$

n = number of incoming particles per area

more detailed information from observation of angular distribution of scattered particles.



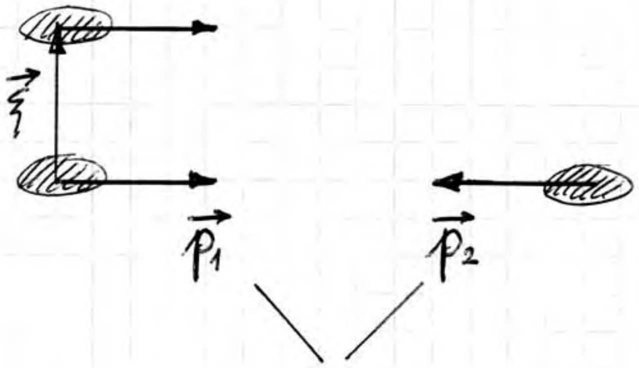
$N_{sc}(\Delta\Omega)$ = number of particles scattered into solid angle $\Delta\Omega$

$$N_{sc}(\Delta\Omega) = n\, \sigma(\Delta\Omega)$$

infinitesimal: $\sigma(d\Omega) = \underbrace{\frac{d\sigma}{d\Omega}}\, d\Omega$

<u>differential cross section</u>

concept of cross section in particle physics:



mean values of (three-) momenta

$|\phi\rangle$ $\qquad\qquad$ $|\chi\rangle$

$|\psi \text{ in}\rangle = |\phi\rangle \otimes |\chi\rangle$

particles <u>distinguishable</u>

(identical particles → see later)

S matrix element $\langle k_1, \ldots, k_n \text{ out} | \psi \text{ in}\rangle =$

$$= \int d\mu(p)\, d\mu(q) \underbrace{\langle k_1, \ldots, k_n \text{ out}}_{n \text{ distinguishable particles}} | p, q \text{ in}\rangle \phi(p) \chi(q)$$

(identical particles → see later); spin indices suppressed

$$\langle k_1, \dots, k_n \text{ out} \mid p, q \text{ in} \rangle =$$

$$= i\,(2\pi)^4\, \delta^{(4)}\Big(p + q - \sum_{i=1}^{n} k_i\Big)\; \mathcal{M}(p, q \to k_1, \dots, k_n)$$

$$w(|\psi\rangle \to B) = \int_B d\mu(k_1) \dots d\mu(k_n)\, |\langle k_1, \dots, k_n \text{ out} \mid \psi \text{ in} \rangle|^2$$

region in n-particle "phase-space"

$$= \int_B d\mu(k_1) \dots d\mu(k_n)\, \langle \psi \text{ in} \mid k_1, \dots, k_n \text{ out} \rangle \langle k_1, \dots, k_n \text{ out} \mid \psi \text{ in} \rangle$$

$= \langle \psi \text{ in} \mid P_B \mid \psi \text{ in} \rangle =$ expectation value of P_B in the state $|\psi \text{ in}\rangle$

$$P_B := \int d\mu(k_1) \dots d\mu(k_n)\, |k_1, \dots, k_n \text{ out}\rangle \langle k_1, \dots, k_n \text{ out}|$$

projector on region B of n-particle phase-space ($\to$ generalization in the case of some identical particles in the final state now obvious)

$$N_{sc}\,(|\psi\rangle \to B) = \underbrace{\frac{N}{A}}_{n} \underbrace{\int_A d^2_{\xi}\, w\,(|\phi_{\vec{\xi}}\rangle|\chi\rangle \to B)}_{=:\,\sigma\,(|\psi\rangle \to B)}$$

$$\phi_{\vec{\xi}}\,(p) = e^{-i\vec{p}\cdot\vec{\xi}}\,\phi\,(p)$$

$$\sigma\,(|\psi\rangle \to B) = \int d^2_{\xi}\; w\,(|\phi_{\vec{\xi}}\rangle|\chi\rangle \to B)$$

$$= \int_{A\to\mathbb{R}^2} d^2_{\xi} \int_B d\mu(k_1)\ldots d\mu(k_n) \int d\mu(p)\,d\mu(q)\,\langle k_1,\ldots,k_n\,\text{out}\,|\,p,q\,\text{in}\rangle$$

$$e^{-i\vec{p}\cdot\vec{\xi}}\,\phi(p)\,\chi(q) \int d\mu(p')\,d\mu(q')\;\langle k_1,\ldots,k_n\,\text{out}\,|\,p',q'\,\text{in}\rangle^*$$

$$e^{+i\vec{p}'\cdot\vec{\xi}}\,\phi(p')^*\;\chi(q')^*$$

$$= \int_B d\mu(k_1)\ldots d\mu(k_n) \int d\mu(p)\,d\mu(q)\;d\mu(p')\,d\mu(q')$$

$$(2\pi)^2\,\delta^{(2)}(\underbrace{\vec{p}_\perp - \vec{p}_\perp{}'}_{\perp\,\vec{p}_{1,2}})\;\phi(p)\;\phi(p')^*\;\chi(q)\;\chi(q')^*$$

$$(2\pi)^4\,\delta^{(4)}(p+q-\sum_{i=1}^{n} k_i)\;\mathcal{M}(p,q \to k_1,\ldots,k_n)$$

$$(2\pi)^4\;\delta^{(4)}(p'+q'-\sum_{i=1}^{n} k_i)\;\mathcal{M}(p',q' \to k_1,\ldots,k_n)^*$$

$$= (2\pi)^{10} \int d\mu(k_1)\dots d\mu(k_n) \int d\mu(p)\, d\mu(q)$$

$$\delta^{(4)}\Big(p+q-\sum_{i=1}^{n} k_i\Big) \underbrace{\int d\mu(p')\, d\mu(q')}_{\text{six integrations}}$$

$$\underbrace{\delta^{(4)}(p'+q'-p-q)\ \delta^{(2)}(\vec{p}_\perp - \vec{p}_\perp{}')}_{\text{six } \delta\text{-functions}}$$

$$\mathcal{M}(p,q \to k_1,\dots,k_n)\ \mathcal{M}(p',q' \to k_1,\dots,k_n)^*$$

$$\phi(p)\ \phi(p')^*\ \chi(q)\ \chi(q')^*$$

$\vec{p}_1 = |\vec{p}_1|\, \vec{e}_z$:

$$\delta^{(4)}(p'+q'-p-q)\ \delta^{(2)}(\vec{p}_\perp - \vec{p}_\perp{}')$$

$$= \delta\Big(\sqrt{\vec{p}\,'^2+m_1^2} + \sqrt{\vec{q}\,'^2+m_2^2} - \sqrt{\vec{p}^{\,2}+m_1^2} - \sqrt{\vec{q}^{\,2}+m_2^2}\Big)$$

$$\delta(p_x'+q_x'-p_x-q_x)\ \delta(p_y'+q_y'-p_y-q_y)$$

$$\delta(p_z'+q_z'-p_z-q_z)\ \delta(p_x-p_x')\ \delta(p_y-p_y')$$

$$= \delta^{(2)}(\vec{p}_\perp - \vec{p}_\perp{}') \; \delta^{(2)}(\vec{q}_\perp - \vec{q}_\perp{}')$$

$$\delta(p_z' + q_z' - p_z - q_z)$$

$$\delta\Big(\sqrt{p_z'^2 + \vec{p}_\perp{}^2 + m_1^2} + \sqrt{q_z'^2 + \vec{q}_\perp{}^2 + m_2^2} -$$

$$- \sqrt{p_z^2 + \vec{p}_\perp{}^2 + m_1^2} - \sqrt{q_z^2 + \vec{q}_\perp{}^2 + m_2^2}\Big) = (*)$$

$\rightarrow$ system of equations

$$p_z' + q_z' = p_z + q_z$$

$$\sqrt{p_z'^2 + M_1^2} + \sqrt{q_z'^2 + M_2^2} = \sqrt{p_z^2 + M_1^2} + \sqrt{q_z^2 + M_2^2}$$

where $M_1^2 = \vec{p}_\perp{}^2 + m_1^2$, $M_2^2 = \vec{q}_\perp{}^2 + m_2^2$

relevant solution: $p_z' = p_z$, $q_z' = q_z$

other solution possible, but contribution vanishes when multiplied with wave functions in momentum space (concentrated around $\vec{p}_1$ and $\vec{p}_2$, respectively) we have an expression of the form

$$\delta[f(x,y)]\,\delta[g(x,y)]$$

where $f(a,b)=0$, $g(a,b)=0$

$\overset{ex.}{\Rightarrow}$ $\delta[f(x,y)]\,\delta[g(x,y)] =$

$$= \frac{\delta(x-a)\,\delta(x-b)}{\left|\det\begin{pmatrix} f_{,x} & g_{,x} \\ f_{,y} & g_{,y}\end{pmatrix}_{\substack{x=a\\ y=b}}\right|} + \dots$$

contributions related to further solutions of $f(x,y)=g(x,y)=0$

in our case, we obtain:

$$\delta(p_z' + q_z' - p_z - q_z)\,\delta\Big(\sqrt{p_z'^2+M_1^2} + \sqrt{q_z'^2+M_2^2} - \sqrt{p_z^2+M_1^2} - \sqrt{q_z^2+M_2^2}\Big)$$

$$= \frac{\delta(p_z'-p_z)\,\delta(q_z'-q_z)}{\left|\frac{q_z}{q^0} - \frac{p_z}{p^0}\right|} + \dots$$

$$\Rightarrow \quad (*) = \frac{\delta^{(3)}(\vec{p}\,' - \vec{p}) \; \delta^{(3)}(\vec{q}\,' - \vec{q})}{\left| \frac{p_\parallel}{p^0} - \frac{q_\parallel}{q^0} \right|}$$

$\parallel$ means: $\parallel$ to $\vec{p}_1$

$$\Rightarrow \quad \sigma(|\psi\rangle \to B) = (2\pi)^{10} \int_B d\mu(k_1) \ldots d\mu(k_n)$$

$$\int d\mu(p)\, d\mu(q) \; \delta^{(4)}(p + q - \sum_{i=1}^n k_i)$$

$$\int d\mu(p')\, d\mu(q') \; \frac{\delta^{(3)}(\vec{p}\,' - \vec{p}) \; \delta^{(3)}(\vec{q}\,' - \vec{q})}{\left| \frac{p_\parallel}{p^0} - \frac{q_\parallel}{q^0} \right|}$$

$$\mathcal{M}(p, q \to k_1, \ldots, k_n) \;\; \mathcal{M}(p', q' \to k_1, \ldots, k_n)^*$$

$$\phi(p)\, \phi(p')^* \, \chi(q) \, \chi(q')^* =$$

$$= \int_B d\mu(k_1) \ldots d\mu(k_n) \int d\mu(p)\, d\mu(q) \; (2\pi)^4 \delta^{(4)}(p + q - \sum_{i=1}^n k_i)$$

$$\frac{1}{2p^0} \, \frac{1}{2q^0} \, \frac{1}{\left| \frac{p_\parallel}{p^0} - \frac{q_\parallel}{q^0} \right|} \, |\mathcal{M}(p, q \to k_1, \ldots, k_n)|^2 \, |\phi(p)|^2 \, |\chi(q)|^2$$

<u>if</u> $|\phi(p)|^2$ sufficiently <u>concentrated</u> around $\vec{p}_1$

and $|\chi(q)|^2$ — " — — " — — " — $\vec{p}_2$

(relative to variations of $|\mathcal{M}(p,q \to k_1,\dots,k_n)|^2$)

$\longrightarrow \quad \sigma(|\psi\rangle \to B) \simeq$

$$\simeq \int_B d\mu(k_1)\dots d\mu(k_n)\ (2\pi)^4\, \delta^{(4)}\Big(p_1+p_2-\sum_{i=1}^{n} k_i\Big)$$

$$\frac{1}{4\,p_1^0\,p_2^0}\ \frac{1}{\underbrace{\frac{|\vec{p}_1|}{p_1^0}+\frac{|\vec{p}_2|}{p_2^0}}_{|\vec{v}_1-\vec{v}_2|}}\ |\mathcal{M}(p_1,p_2 \to k_1,\dots,k_n)|^2$$

$$\underbrace{\int d\mu(p)\ |\phi(p)|^2}_{1}\ \ \underbrace{\int d\mu(q)\ |\chi(q)|^2}_{1}$$

<u>independent</u> of the exact form of ϕ and χ

$$\sigma(p_1,p_2 \to B) = \int_B d\mu(k_1)\dots d\mu(k_n)\ (2\pi)^4\, \delta^{(4)}\Big(p_1+p_2-\sum_{i=1}^{n} k_i\Big)\ \frac{|\mathcal{M}|^2}{4\,(|\vec{p}_1|\,p_2^0+|\vec{p}_2|\,p_1^0)}$$

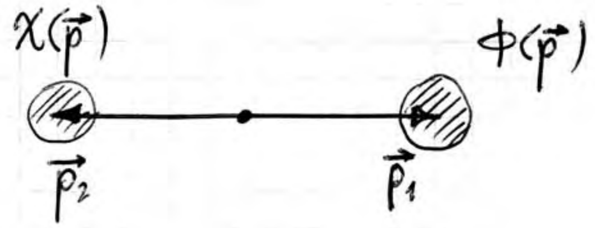
remark: $|\vec{p}_1|\,p_2^0+|\vec{p}_2|\,p_1^0 = \sqrt{(p_1\cdot p_2)^2-m_1^2m_2^2}$ Lorentz invariant

<u>identical</u> particles in initial state

$\phi(p)$ one-particle (momentum space) wave function concentrated around $\vec{p} = \vec{p}_1$ $\left(\int d\mu(p)\, |\phi(p)|^2 = 1 \right)$

analogously: $\chi(q)$ one-particle wave function concentrated around $\vec{q} = \vec{p}_2$ $\left(\int d\mu(q)\, |\chi(q)|^2 = 1 \right)$

in momentum space:



$$\langle \phi | \chi \rangle = \int d\mu(p)\, \chi(p)^* \phi(p) = 0$$

(no overlap)

$\Rightarrow |\psi \text{ in}\rangle = \frac{1}{2}\int d\mu(p)\, d\mu(q)\, |p,q \text{ in}\rangle \left[\phi(p)\,\chi(q) \pm \chi(p)\,\phi(q)\right]$

(minus sign for fermions) is a properly normalized two-particle state

$|p,q \text{ in}\rangle = \pm\, |q,p \text{ in}\rangle \Rightarrow |\psi \text{ in}\rangle = \int d\mu(p)\, d\mu(q)\, |p,q \text{ in}\rangle\, \phi(p)\,\chi(q)$

→ further calculation exactly the same as for distinguishable particles

→ result was to be expected, as no overlap of momentum-space wave functions $\phi(p)$, $\chi(p)$ (particles can be distinguished before the interaction)

identical particles in final state:

consider first the case where all n final state particles identical

$$\rightarrow P_B = \frac{1}{n!} \int_B d\mu(k_1) \ldots d\mu(k_n)\, |k_1, \ldots, k_n \text{ out}\rangle\langle k_1, \ldots, k_n \text{ out}|$$

domain of integration B symmetric with respect to permutations of $k_1, \ldots, k_n$

statistical factor $\frac{1}{n!}$ in cross section formula general case:

n_1 identical particles of type 1 (momenta $k_1^{(1)} \ldots k_{n_1}^{(1)}$)

. . . .

n_ℓ —"— —"— —"— —"— ℓ (momenta $k_1^{(\ell)}, \ldots k_{n_\ell}^{(\ell)}$)

. . . .

n_r —"— —"— —"— —"— r (momenta $k_1^{(r)}, \ldots, k_{n_r}^{(r)}$)

$$\sum_{\ell=1}^{r} n_\ell = n \rightarrow \text{statistical factor } S = \prod_{\ell=1}^{r} \frac{1}{n_\ell!}$$

domain of integration B totally symmetric with respect to permutations of $k_1^{(\ell)}, \ldots, k_{n_\ell}^{(\ell)}$ $(\ell = 1, \ldots r)$

summary: $\langle k_1, \ldots, k_n \text{ out} \mid p_1, p_2 \text{ in} \rangle =$

$$= i\,(2\pi)^4\, \delta^{(4)}\Big(p_1 + p_2 - \sum_{i=1}^{n} k_i\Big)\, \mathcal{M}(p_1, p_2 \rightarrow k_1, \ldots, k_n)$$

$$\sigma(p_1, p_2 \rightarrow B) =$$

$$= \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int_B d\mu(k_1) \ldots d\mu(k_n)\, (2\pi)^4\, \delta^{(4)}\Big(p_1 + p_2 - \sum_{i=1}^{n} k_i\Big) \times |\mathcal{M}(p_1, p_2 \rightarrow k_1, \ldots, k_n)|^2$$

application of cross section formula to scattering of identical scalars $\varphi(p_1)\varphi(p_2)\to\varphi(p_3)\varphi(p_4)$

$$(p_1+p_2)^2 = s \;\Rightarrow\; 2m^2 + 2p_1\cdot p_2 = s$$

$$\Rightarrow\; p_1\cdot p_2 = \frac{s}{2} - m^2$$

$$\Rightarrow\; (p_1\cdot p_2)^2 - m^4 = \frac{s^2}{4} - sm^2 = \frac{s}{4}(s-4m^2)$$

$$\sigma = \frac{1}{2!}\,\frac{1}{4\sqrt{\frac{s}{4}(s-4m^2)}}\int d\mu(p_3)\,d\mu(p_4)\,(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4) \times |\mathcal{M}(s,t)|^2$$

$$= \frac{1}{4\sqrt{s}\sqrt{s-4m^2}}\int \frac{d^3p_3}{(2\pi)^3\,2p_3^0}\,\frac{d^3p_4}{(2\pi)^3\,2p_4^0}\,(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4) \times |\mathcal{M}(s,t)|^2$$

$$= \frac{1}{4(2\pi)^2\sqrt{s}\sqrt{s-4m^2}}\int \frac{d^3p_3}{2p_3^0}\,d^4p_4\,\Theta(p_4^0)\,\delta(p_4^2-m^2)\,\delta^{(4)}(p_1+p_2-p_3-p_4)\,|\mathcal{M}(s,t)|^2$$

$$= \frac{1}{4(2\pi)^2\sqrt{s}\sqrt{s-4m^2}} \int \frac{d^3p_3}{2p_3^0}\, \Theta(p_1^0+p_2^0-p_3^0)$$

$$\delta\left[(p_1+p_2-p_3)^2-m^2\right] \, |\mathcal{M}(s,t)|^2$$

$$\overset{\text{CMS}}{=} \frac{1}{4(2\pi)^2\sqrt{s}\sqrt{s-4m^2}} \int \frac{d|\vec{p}_3|\,\vec{p}_3^{\,2}\,d\Omega}{2p_3^0}\, \Theta(\sqrt{s}-p_3^0)$$

$$\underbrace{\delta(s-2\sqrt{s}\,p_3^0)}_{\frac{1}{2\sqrt{s}}\delta\left(p_3^0-\frac{\sqrt{s}}{2}\right)} \, |\mathcal{M}(s,t)|^2$$

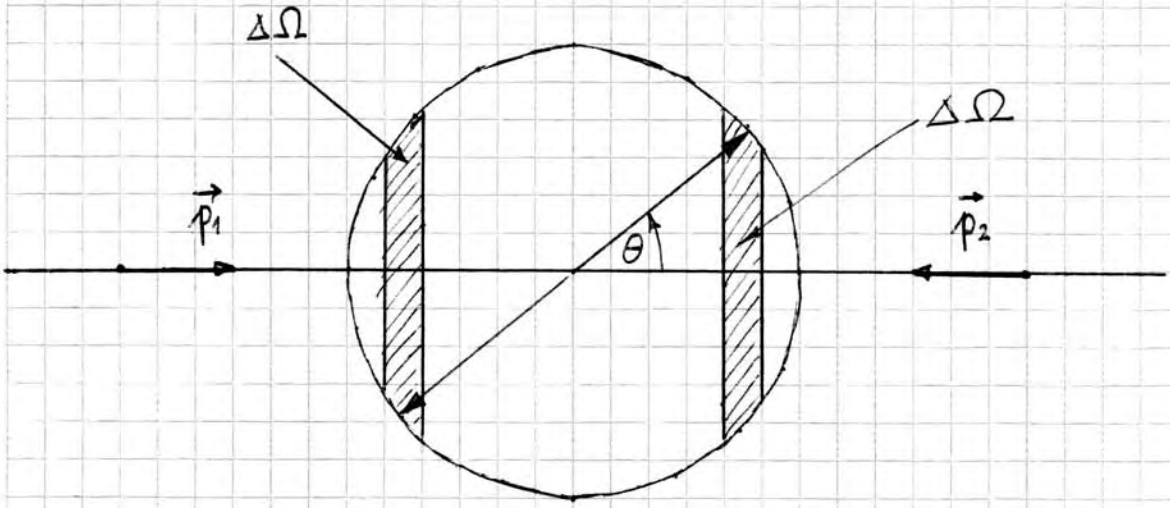
$$= \frac{1}{2}\, \frac{1}{32\pi^2 s\sqrt{s-4m^2}} \int \frac{dp_3^0\, p_3^0\, |\vec{p}_3|\, d\Omega_3}{p_3^0}\, \Theta(\sqrt{s}-p_3^0)$$

$$\times\, \delta\left(p_3^0-\frac{\sqrt{s}}{2}\right) |\mathcal{M}(s,t)|^2\, \Theta(p_3^0-m)$$

$$= \frac{1}{2}\, \frac{1}{32\pi^2 s\sqrt{s-4m^2}} \int d\Omega_3\, |\mathcal{M}(s,t)|^2 \sqrt{\frac{s}{4}-m^2}\, \Theta(\sqrt{s}-2m)$$

$$= \frac{1}{2}\, \frac{\Theta(\sqrt{s}-2m)}{64\pi^2 s} \int d\Omega_3\, |\mathcal{M}(s,t)|^2$$

differential cross section (identical particles in final state)



$$\sigma(p_1, p_2 \to \Delta\Omega) = \frac{1}{2} \frac{\Theta(\sqrt{s} - 2m)}{64\pi^2 s} \int_{\Delta\Omega + \Delta\bar{\Omega}} d\Omega \, |\mathcal{M}|^2$$

$|\mathcal{M}|^2$ symmetric under $\Theta \to \pi - \Theta$

$$\Rightarrow \sigma(p_1, p_2 \to \Delta\Omega) = \frac{\Theta(\sqrt{s} - 2m)}{64\pi^2 s} \int_{\Delta\Omega} d\Omega \, |\mathcal{M}|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \qquad (CMS)$$

$$\sigma = \int_{\text{hemisphere}} d\Omega \, \frac{d\sigma}{d\Omega} = \frac{1}{2} \int_{\text{full sphere}} d\Omega \, \frac{d\sigma}{d\Omega}$$