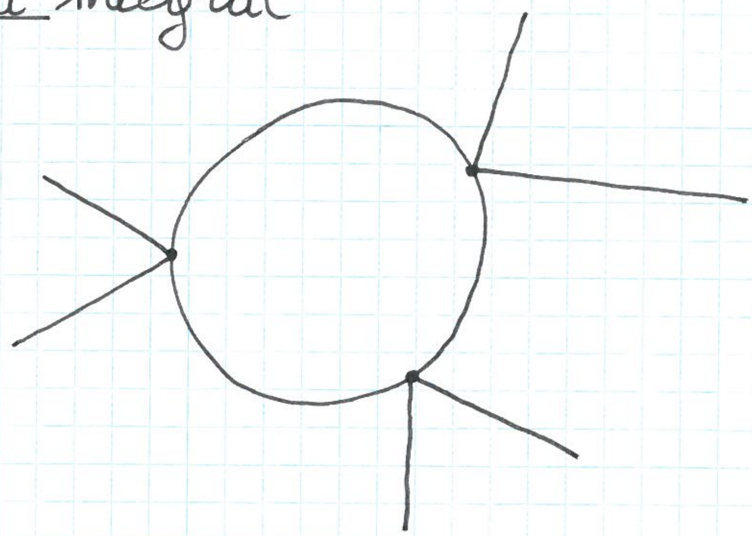
6. Renormalizability

our one-loop calculation showed: two-point and four-point function requires renormalization

six-point function (at one-loop) determined by convergent integral



$$\sim \int^{\Lambda} d^dk \, \frac{1}{k^6} \sim \Lambda^{d-6} \qquad d-6=-2 \text{ for } d=4$$

Λ ... UV-cutoff

generally: Γ_n finite for $n \geq 6$ at one-loop

higher orders (two-loop, ...): field-, mass- and renormalization of λ sufficient to obtain finite results for all observable quantities

$$\phi = \sqrt{Z}\,\phi_{ph}\,, \quad m^2 = m_{ph}^2 - \delta m^2\,, \quad \lambda = Z_\lambda \lambda_{ph}$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\, \partial^\mu \phi - m^2 \phi^2\right) - \frac{\lambda}{4!}\phi^4$$

$$= \frac{1}{2}\left(\partial_\mu \phi_{ph}\, \partial^\mu \phi_{ph} - m_{ph}^2 \phi_{ph}^2\right) - \frac{\lambda_{ph}}{4!}\phi_{ph}^4$$

$$+ \frac{1}{2}\underbrace{(Z-1)}_{A}\,\partial_\mu \phi_{ph}\, \partial^\mu \phi_{ph}$$

$$- \frac{1}{2}\underbrace{\left[(Z-1)\, m_{ph}^2 - Z\,\delta m^2\right]}_{B}\phi_{ph}^2$$

$$- \underbrace{(Z^2 Z_\lambda - 1)\,\lambda_{ph}}_{C}\,\frac{1}{4!}\,\phi_{ph}^4$$

"physical" perturbation theory (using m_{ph}, λ_{ph})

with <u>counterterms</u> $\frac{1}{2}A\,\partial_\mu \phi_{ph}\,\partial^\mu \phi_{ph} - \frac{1}{2}B\,\phi_{ph}^2 - \frac{C}{4!}\phi_{ph}^4$

coefficients A, B, C determined <u>iteratively</u> (order by order in perturbation theory)

by <u>three renormalization conditions</u>:

Fourier transform of two-point function has

(1) a pole at $p^2 = m_{ph}^2$

(2) with residue equal to one

(3) scattering amplitude evaluated at s_0, t_0 should be equal to $-\lambda_{ph}$

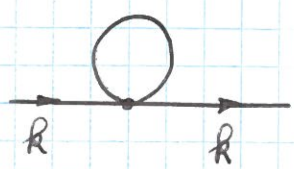
example: one-loop calculation in "physical" perturbation theory

two-loop function:

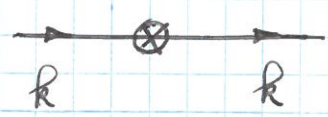
$$\langle 0 | T \phi_{ph}(x) \phi_{ph}(0) | 0 \rangle =$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{-ikx} \left\{ \frac{1}{i} \frac{1}{m_{ph}^2 - k^2 - i\varepsilon} \right.$$

$$+ \frac{\lambda_{ph} \Delta(0)}{2} \frac{1}{(m_{ph}^2 - k^2 - i\varepsilon)^2}$$



$$\left. + (-iAk^2 + iB) \frac{1}{(m_{ph}^2 - k^2 - i\varepsilon)^2} \right\}$$



$$\Rightarrow A = 0 + \mathcal{O}(\lambda_{ph}^2), \quad B = \frac{i\lambda_{ph}}{2} \Delta(0) + \mathcal{O}(\lambda_{ph}^2)$$

four-point function

→ scattering amplitude

$$\mathcal{M}(s,t) = -\lambda_{ph} + \frac{\lambda_{ph}^2}{2}\left[B(s,m_{ph}^2) + B(t,m_{ph}^2) + B(u,m_{ph}^2)\right]$$

$$- C$$

$$\Rightarrow \quad C = \frac{\lambda_{ph}^2}{2}\left[\mathrm{Re}\, B(s_0,m_{ph}^2) + B(t_0,m_{ph}^2) + B(u_0,m_{ph}^2)\right]$$

$$+ \mathcal{O}(\lambda_{ph}^3)$$

<u>all</u> observables made finite by this procedure (in <u>all orders</u> of the perturbative expansion)

φ^4-theory is an example of a <u>renormalizable</u> QFT (determined by a <u>finite</u> number of parameters)

general proof of the renormalizability of φ^4-theory is beyond the scope of this course

→ here we just analyze the general structure of the divergences occuring at <u>higher orders</u>

remember: $(2\pi)^d \delta^{(d)}(k_1 + \ldots + k_n)\, \Gamma_n(k_1, \ldots, k_n) =$

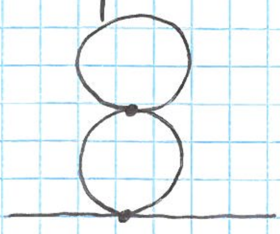
$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-i\lambda}{4!}\right)^k \int d^d y_1 \ldots d^d y_k$$

$$\left\langle\!\left\langle \tilde{\varphi}(k_1) \ldots \tilde{\varphi}(k_n)\, \varphi(y_1)^4 \ldots \varphi(y_k)^4 \right\rangle\!\right\rangle_c$$

study <u>one-particle irreducible</u> diagram with n external momenta and k vertices

<u>one-particle irreducible graph</u>: does <u>not</u> fall apart if one of the internal lines is cut

examples:



one-particle irreducible

<u>not</u> one-particle irred.

number of <u>internal lines</u>: $I = \frac{1}{2}(4k - n)$

number of <u>loops</u>: $L = I - (k-1)$

overall energy-momentum conservation

$\Rightarrow L = k - \frac{n}{2} + 1$

degree of divergence

one-particle irred. graph (euclidean):

$$\sim \int^{\Lambda} d^d l_1 \dots d^d l_L \prod_{i=1}^{I} \frac{1}{m^2 + q_i^2} \sim \begin{cases} \Lambda^{Ld-2I} & \text{for } Ld \neq 2I \\ \ln \Lambda & \text{for } Ld = 2I \end{cases}$$

$$q_i = q_i(l_1, \dots, l_L ; \underbrace{k_1, \dots, k_n}_{\text{external momenta}})$$

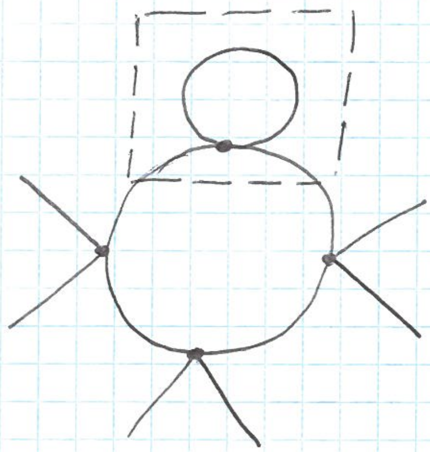
superficial degree of divergence:

$$\omega = Ld - 2I = \left(k - \frac{n}{2} + 1\right) d - 4k + n$$

$$= d + (d-4)k - \frac{1}{2}(d-2)\,n$$

$d = 4$: $\quad \omega = 4 - n$

$\omega < 0$ <u>necessary condition</u> for convergence of integral (but <u>not</u> sufficient) counterexample:



$\omega = -2$

but diagram divergent because of <u>divergent subgraph</u>

$\omega_{sub} = 4 - 2 = 2$ (quadr. div.)

<u>Weinberg's theorem</u>: integral convergent if the <u>graph</u> as well as all of its <u>subgraphs</u> posses a negative superficial degree of divergence;

divergences occur <u>only</u> if the graph or some of its subgraphs have $\omega \geq 0$

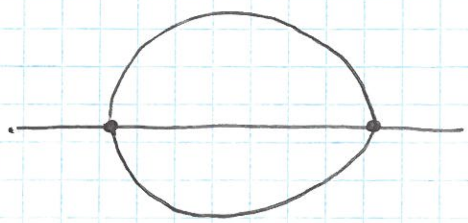
in our case: $\omega = 4 - n$

$n = 0 \rightarrow$ vacuum bubbles do not contribute to perturbation series

$n = 2$, $n = 4$ <u>only</u> possibility for divergent graphs and subgraphs we had seen: <u>divergent</u> one-loop contributions to two- and four-point functions had the same structure as the tree terms φ^2 (mass term) and φ^4 (interaction term) $\rightarrow$ divergences could be absorbed by renormalization of bare parameters m, λ

<u>generalization</u> of this property holds true also at higher orders: divergences only associated with local terms $\partial_\mu\varphi\,\partial^\mu\varphi$, φ^2, φ^4

remark: investigation of general case rather complicated, mainly due to <u>overlapping divergences</u>

example: 

$\rightarrow$ φ^4 - theory is <u>renormalizable</u>