Our one-loop calculation showed: two-point and four-point function requires renormalization. Six-point function (at one-loop) determined by convergent integral.

\[ \Lambda \propto \int \frac{d^d k}{k^6} \sim \Lambda^{d-6} \quad d-6 = -2 \quad \text{for} \quad d=4 \]

\[ \Lambda \quad \text{UV-cutoff} \]

Generally: \( \Gamma_n \) finite for \( n \geq 6 \) at one-loop. Higher orders (two-loop, \ldots): field, mass, and renormalization of a sufficient to obtain finite results for all observable quantities.
\[
\phi = \sqrt{Z} \phi_{ph}, \quad m^2 = m_{ph}^2 - 8m^2, \quad \lambda = Z \lambda_{ph}
\]

\[
\mathcal{L} = \frac{1}{2} \left( \partial_{\mu} \phi \partial^{\mu} \phi - m^2 \phi^2 \right) - \frac{\lambda}{4!} \phi^4
\]

\[
= \frac{1}{2} \left( \partial_{\mu} \phi_{ph} \partial^{\mu} \phi_{ph} - m_{ph}^2 \phi_{ph}^2 \right) - \frac{\lambda_{ph}}{4!} \phi_{ph}^4
\]

\[
+ \frac{1}{2} \left( (Z - 1) \partial_{\mu} \phi_{ph} \partial^{\mu} \phi_{ph} \right)
\]

\[
A
\]

\[
- \frac{1}{2} \left[ (Z - 1) m_{ph}^2 - 8m^2 \right] \phi_{ph}^2
\]

\[
B
\]

\[
- (Z^2 Z_a - 1) \lambda_{ph} \frac{1}{4!} \phi_{ph}^4
\]

\[
C
\]

"physical" perturbation theory (using m_{ph}, \lambda_{ph})

with counterterms \( \frac{1}{2} A \partial_{\mu} \phi_{ph} \partial^{\mu} \phi_{ph} - \frac{1}{2} B \phi_{ph}^2 - \frac{\lambda_{ph}}{4!} \phi_{ph}^4 \)

coefficients A, B, C determined iteratively

(order by order in perturbation theory)

by three renormalization conditions:
Fourier transform of two-point function has
(1) a pole at \( p^2 = m_{ph}^2 \)
(2) with residue equal to one
(3) scattering amplitude evaluated at \( s_0 \), to
should be equal to \( \lambda_{ph} \)

Example: one-loop calculation in "physical"
perturbation theory

two-loop function:

\[
\langle 0 | T \phi_{ph}(x) \phi_{ph}(0) | 0 \rangle =
\]

\[
= \int \frac{d^4 k}{(2\pi)^4} \ e^{-i k x} \left\{ \frac{1}{i} \frac{1}{m_{ph}^2 - k^2 - i\varepsilon} + \frac{\lambda_{ph} \Delta(0)}{2} \frac{1}{(m_{ph}^2 - k^2 - i\varepsilon)^2} + \left( -i A \frac{R^2}{2} + i B \right) \frac{1}{(m_{ph}^2 - k^2 - i\varepsilon)^2} \right\}
\]

\[\Rightarrow A = 0 + O(\lambda_{ph}^2), \quad B = \frac{i\lambda_{ph}^2}{2} \Delta(0) + O(\lambda_{ph}^2)\]
four-point function

\[ M(0, t) = -\alpha_{ph} + \frac{\alpha_{ph}^2}{2} \left[ B(0, m_{ph}^2) + B(t, m_{ph}^2) + B(u_0, m_{ph}^2) \right] - C \]

\[ \Rightarrow C = \frac{\alpha_{ph}^2}{2} \left[ \text{Re} B(0, m_{ph}^2) + B(t, m_{ph}^2) + B(u_0, m_{ph}^2) \right] + O(\alpha_{ph}^3) \]

all observables made finite by this procedure (in all orders of the perturbative expansion)

\( \phi^4 \) - theory is an example of a renormalizable QFT (determined by a finite number of parameters)

general proof of the renormalizability of \( \phi^4 \) - theory is beyond the scope of this course

\( \rightarrow \) here we just analyze the general structure of the divergences occurring at higher orders
\[ \text{remember: } (2\pi)^d \delta^{(d)}(\mathbf{R}_1 + \ldots + \mathbf{R}_n) \prod_{i=1}^n (\mathbf{R}_i) = \]
\[ = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{-i\mathbf{v}}{4!} \right)^k \int d\mathbf{y}_1 \ldots d\mathbf{y}_n \]
\[ \left\langle \bar{\psi}(\mathbf{R}_1) \ldots \bar{\psi}(\mathbf{R}_n) \psi(\mathbf{y}_1)^* \ldots \psi(\mathbf{y}_n)^* \right\rangle_c \]

study one-particle irreducible diagram with \( n \) external momenta and \( k \) vertices

one-particle irreducible graph: does not fall apart if one of the internal lines is cut

examples:

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{example1} \\
\includegraphics[width=0.3\textwidth]{example2}
\end{array}
\]

one-particle irreducible

not one-particle irreducible

number of internal lines: \( I = \frac{1}{2} (4k - n) \)

number of loops: \( L = I - (k - 1) \)

\[ \Rightarrow L = k - \frac{n}{2} + 1 \]

overall energy-momentum conservation
degree of divergence

one-particle irred. graph (euclidean):

\[ \Lambda \sim \int d^{d}l_1 \cdots d^{d}l_n \prod_{i=1}^{n} \frac{1}{m^2 + q_i^2} \sim \Lambda^{Ld-2I} \quad \text{for } Ld \neq 2I \]

\[ \ln \Lambda \quad \text{for } Ld = 2I \]

\[ q_i = q_i (l_1, \ldots, l_n; k_1, \ldots, k_n) \]

external momenta

superficial degree of divergence:

\[ \omega = Ld - 2I = (R - \frac{n}{2} + 1)d - 4R + n \]

\[ = d + (d-4)R - \frac{1}{2}(d-2)n \]

\[ d = 4 : \quad \omega = 4 - n \]

\[ \omega < 0 \text{ necessary condition for convergence of integral (but not sufficient)} \]
w = -2
but diagram divergent
because of divergent subgraph

\[ \omega_{\text{sub}} = 4 - 2 = 2 \quad \text{(quadr. div.)} \]

**Weinberg's theorem**: integral convergent if the graph as well as all of its subgraphs possess a negative superficial degree of divergence;

divergences occur only if the graph or some of its subgraphs have \( \omega \geq 0 \)

in our case: \( \omega = 4 - n \)

\( n = 0 \rightarrow \) vacuum bubbles do not contribute to perturbation series

\( n = 2, \ n = 4 \) only possibility for divergent graphs and subgraphs
we had seen: divergent one-loop contributions to two- and four-point functions had the same structure as the tree terms $\phi^2$ (mass term) and $\phi^4$ (interaction term) → divergences could be absorbed by renormalization of bare parameters $m, \alpha$

generalization of this property holds true also at higher orders: divergences only associated with local terms $\partial^2 \phi \partial \phi, \phi^2, \phi^4$

remark: investigation of general case rather complicated, mainly due to overlapping divergences

example:

$\phi^4$ theory is renormalizable